

Reward-Free Exploration for Reinforcement Learning

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Abstract

Exploration is widely regarded as one of the most challenging aspects of reinforcement learning (RL), with many naive approaches succumbing to exponential sample complexity. To isolate the challenges of exploration, we propose a new “reward-free RL” framework. In the exploration phase, the agent first collects trajectories from an MDP \mathcal{M} *without* a pre-specified reward function. After exploration, it is tasked with computing near-optimal policies under for \mathcal{M} for a collection of given reward functions. This framework is particularly suitable when there are many reward functions of interest, or when the reward function is shaped by an external agent to elicit desired behavior.

We give an efficient algorithm that conducts $\tilde{O}(S^2 A \text{poly}(H)/\epsilon^2)$ episodes of exploration and returns ϵ -suboptimal policies for an *arbitrary* number of reward functions. We achieve this by finding exploratory policies that visit each “significant” state with probability proportional to its maximum visitation probability under *any* possible policy. Moreover, our planning procedure can be instantiated by any black-box approximate planner, such as value iteration or natural policy gradient. We also give a nearly-matching $\Omega(S^2 A H^2/\epsilon^2)$ lower bound, demonstrating the near-optimality of our algorithm in this setting.

1 Introduction

In reinforcement learning (RL), an agent repeatedly interacts with an unknown environment with the goal of maximizing its cumulative reward. To do so, the agent must engage in *exploration*, learning to visit states in order to investigate whether they hold high reward.

Exploration is widely regarded as the most significant challenge in RL, because the agent may have to take precise sequences of actions to reach states with high reward. Here, simple randomized exploration strategies provably fail: for example, a random walk can take exponential time to reach the corner of the environment where the agent can accumulate high reward [Li, 2012]. While reinforcement learning has seen a tremendous surge of recent research activity, essentially all of the standard algorithms deployed in practice employ simple randomization or its variants, and consequently incur extremely high sample complexity.

On the other hand, sophisticated exploration strategies which deliberately incentivize the agent to visit new states are provably sample-efficient (c.f., Kearns and Singh [2002], Brafman and Tennenholtz [2002],

Azar et al. [2017], Dann et al. [2017], Jin et al. [2018]), with recent work providing a nearly-complete theoretical understanding for maximizing a single prespecified reward function Dann and Brunskill [2015], Azar et al. [2017], Zanette and Brunskill [2019], Simchowitz and Jamieson [2019]. In practice, however, reward functions are often iteratively engineered to encourage desired behavior via trial and error (e.g. in constrained RL formulations [Altman, 1999, Achiam et al., 2017, Tessler et al., 2018, Miryoosefi et al., 2019]). In such cases, repeatedly invoking the same reinforcement learning algorithm with different reward functions can be quite sample inefficient.

One solution to avoid excessive data collection in such settings is to first collect a dataset with good coverage over all possible scenarios in the environment, and then apply a “Batch-RL” algorithm. Indeed many algorithms are known for computing near optimal policies from previously collect data, provided that the dataset has good coverage [Munos and Szepesvári, 2008, Antos et al., 2008, Chen and Jiang, 2019, Agarwal et al., 2019]. However, prior work provides little guidance into how to obtain such good coverage.

In this paper, we aim to develop an end-to-end instantiation of this proposal. To this end we ask:

How can we efficiently explore an environment without using any reward information?

In particular, by exploring the environment, we aim to gather sufficient information so that we can compute the near-optimal policies for *any* reward function after-the-fact.

Our Contributions. In this paper, we present the first near-optimal upper and lower bounds which characterize the sample complexity of achieving provably sufficient coverage for Batch-RL. We do so by adopting a novel “reward-free RL” paradigm: During an exploration phase, the agent collects trajectories from an MDP \mathcal{M} *without* a pre-specified reward function. Then, in a planning phase, it is tasked with computing near-optimal policies under the transitions of \mathcal{M} for a large collection of given reward functions.

Letting S denote the number of states, A the number of actions, H the horizon, and ϵ the desired accuracy, we give an efficient algorithm which, after conducting $\tilde{O}(S^2 A \text{poly}(H)/\epsilon^2)$ episodes of exploration, collects a data set with sufficiently good coverage to enable application of standard Batch-RL solvers. Specifically, we show that when given a reward function r we can find an ϵ -suboptimal policy for the true MDP \mathcal{M} with reward r , using the dataset alone and no additional data collection. This guarantee holds for all possible reward functions simultaneously, without needing to collect more data to ensure statistical correctness as new reward functions are considered.

Our exploration phase is conceptually simple, using an existing RL algorithm as a black-box [Zanette and Brunskill, 2019], and our planning phase accommodates arbitrary Batch-RL solvers. We instantiate our result with value iteration and natural policy gradient as special cases. By decoupling exploration and planning, our work sheds light on the algorithmic mechanisms required for sample efficient reinforcement learning. We hope that this insight will be useful in the design of provably efficient algorithms for more practically relevant RL settings, such as those where function approximation is required.

In addition to our algorithmic results, we establish a nearly-matching $\Omega(S^2 AH^2/\epsilon^2)$ lower bound, demonstrating the near-optimality of our algorithm in this paradigm. Notably, this lower bound quantifies a price of “good-coverage” in the reward-free setting: while RL with a pre-specified reward has sample complexity of only $\tilde{\Theta}(SAH^2/\epsilon^2)$ [Dann and Brunskill, 2015], the reward-free sample complexity is a factor of S larger.

Technical Novelty. The main technical challenge in our work involves handling environments with states that are difficult to reach. In such cases, we cannot learn the transition operator to high accuracy uniformly over the environment, simply because we cannot reach these states to collect enough data. With

$\lambda(s)$ denoting the maximal probability of visiting state s under any policy, our key observation is that we can partition the state space into two groups: the states with $\lambda(s)$ so small that they have negligible contribution to reward optimization, and the rest. We introduce a rigorous analysis which enables us to “ignore” the difficult-to-visit states altogether and only requires that we visit the remaining states with probability proportional $\lambda(s)$. To achieve this latter guarantee, we conduct our exploration with the EULER algorithm [Zanette and Brunskill, 2019], which in our context yields refined sample complexity guarantees in terms of $\lambda(s)$. We believe that this decomposition of states into their ease of being reached may be of broader interest. Our lower bound also adopts a novel and sophisticated construction, detailed in Section 4.

Related work. For reward-free exploration in the tabular setting, we are aware of only a few prior approaches. First, when one runs a PAC-RL algorithm like RMAX with no reward function [Brafman and Tennenholtz, 2002], it does visit the entire state space and can be shown to provide a coverage guarantee. However, for RMAX in particular the resulting sample complexity is quite poor, and significantly worse than our near-optimal guarantee (See Appendix A for a detailed calculation). We expect similar behavior from other PAC algorithms, because reward-dependent exploration is typically suboptimal for the reward-free setting.

Second, one can extract the exploration component of recent results for RL with function approximation [Du et al., 2019, Misra et al., 2019]. Specifically, the former employs a model based approach where a model is iteratively refined by planning to visit unexplored states, while the latter uses model free dynamic programming to identify and reach all states. While these papers address a more difficult setting, it is relatively straightforward to specialize their results to the tabular setting. In this case, both methods guarantee coverage, but they have suboptimal sample complexity and require that all states can be visited with significant probability. In contrast, our approach requires no visitation probability assumptions and achieves the optimal sample complexity.

The last point of comparison is a recent result of Hazan et al. [2018], that gives an efficient algorithm for finding a certain exploratory policy. They use a Frank-Wolfe style algorithm to find a policy whose state occupancy measure has maximum entropy. One can show that an exact optimizer for their objective has a similar coverage property to our exploratory policy, but the Frank-Wolfe style algorithm can only guarantee an approximate optimizer. They do not analyze how the optimization error enters in the coverage guarantee, but we are able to show that setting the error to $O(1/S)$ suffices (see Appendix B). Unfortunately, this implies that their sample complexity scales with S^5 , which is much worse than ours. More generally, their result is not end-to-end in that they do not show how to use their policy for planning, and they do not establish a final sample complexity bound, both of which we do here.

Finally, the main source of motivation for our work is recent and classical results on batch reinforcement learning [Munos and Szepesvári, 2008, Antos et al., 2008, Chen and Jiang, 2019, Agarwal et al., 2019], a setting where the goal is to find a near optimal policy, given an *a priori* dataset collected by some logging policy that satisfies certain coverage properties. In this paper, we show how to find such a logging policy for the tabular setting, which enables straightforward application of these batch RL results. As an example, we show how to apply both value iteration and natural policy gradient to optimize the policy given any reward function. More generally, these works typically also consider the function approximation setting, and we believe our modular approach will facilitate development of provably efficient algorithms for these challenging settings.

2 Preliminaries

We consider the setting of a tabular episodic Markov decision process, $\text{MDP}(\mathcal{S}, \mathcal{A}, H, \mathbb{P}, r)$, where \mathcal{S} is the set of states with $|\mathcal{S}| = S$, \mathcal{A} is the set of actions with $|\mathcal{A}| = A$, H is the number of steps in each episode, \mathbb{P} is the time-dependent transition matrix so that $\mathbb{P}_h(\cdot|s, a)$ gives the distribution over the next state if action a is taken from state s at step $h \in [H]$, and $r_h : \mathcal{S} \times \mathcal{A} \rightarrow [0, 1]$ is the deterministic reward function at step h .¹ Note that we are assuming that rewards are in $[0, 1]$ for normalization.

In each episode of a standard MDP, an initial state s_1 is picked from an unknown initial distribution $\mathbb{P}_1(\cdot)$. Then, at each step $h \in [H]$, the agent observes state $s_h \in \mathcal{S}$, picks an action $a_h \in \mathcal{A}$, receives reward $r_h(s_h, a_h)$, and then transitions to the next state s_{h+1} , which is drawn from the distribution $\mathbb{P}_h(\cdot|s_h, a_h)$. The episode ends after the H^{th} reward is collected.

A (non-stationary, stochastic) policy π is a collection of H functions $\{\pi_h : \mathcal{S} \rightarrow \Delta_{\mathcal{A}}\}_{h \in [H]}$, where $\Delta_{\mathcal{A}}$ is the probability simplex over action set \mathcal{A} . As notation, we use $\pi(\cdot|s)$ to denote the action distribution for policy π in state s . We use $V_h^\pi : \mathcal{S} \rightarrow \mathbb{R}$ to denote the value function at step h under policy π , which gives the expected sum of remaining rewards received under policy π , starting from $s_h = s$, until the end of the episode. That is,

$$V_h^\pi(s) := \mathbb{E}_\pi \left[\sum_{h'=h}^H r_{h'}(s_{h'}, a_{h'}) | s_h = s \right].$$

Accordingly, we also define $Q_h^\pi : \mathcal{S} \times \mathcal{A} \rightarrow \mathbb{R}$ to denote action-value function at step h , so that $Q_h^\pi(s, a)$ gives the expected sum of remaining rewards received under policy π , starting from $s_h = s, a_h = a$, until the end of the episode. Formally:

$$Q_h^\pi(s, a) := \mathbb{E}_\pi \left[\sum_{h'=h}^H r_{h'}(s_{h'}, a_{h'}) | s_h = s, a_h = a \right].$$

Since the state and action spaces, and the horizon, are all finite, there always exists (see, e.g., Azar et al. [2017]) an optimal policy π^* which gives the optimal value $V_h^*(s) = \sup_\pi V_h^\pi(s)$ for all $s \in \mathcal{S}$ and $h \in [H]$. As notation, define $[\mathbb{P}_h V_{h+1}](s, a) := \mathbb{E}_{s' \sim \mathbb{P}(\cdot|s, a)} V_{h+1}(s')$. Recall the Bellman equation

$$V_h^\pi(s) = Q_h^\pi(s, \pi_h(s)), \quad Q_h^\pi(s, a) = (r_h + \mathbb{P}_h V_{h+1}^\pi)(s, a) \quad (1)$$

and the Bellman optimality equation:

$$V_h^*(s) = \max_{a \in \mathcal{A}} Q_h^*(s, a), \quad Q_h^*(s, a) := (r_h + \mathbb{P}_h V_{h+1}^*)(s, a). \quad (2)$$

where we define $V_{H+1}^\pi(s) = V_{H+1}^*(s) = 0$ for any $s \in \mathcal{S}$.

The RL objective is to find an ϵ -optimal policy π , satisfying

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_1^*(s_1) - V_1^\pi(s_1)] \leq \epsilon$$

¹While we study deterministic reward functions for notational simplicity, our results generalize to randomized reward functions.

Protocol 1 Reward-Free Exploration

for $k = 1$ **to** K **do**
 learner decides a policy π_k
 environment samples the initial state $s_0 \sim \mathbb{P}_1$.
 for $h = 1$ **to** H **do**
 learner selects action $a_h \sim \pi_h(\cdot|s_h)$
 environment transitions to $s_{h+1} \sim \mathbb{P}_h(\cdot|s_h, a_h)$
 learner observes the next state s_{h+1}

Reward-free Exploration. In the reward-free setting, we would like to design algorithms that efficiently explore the state space without the guidance of reward information. Formally, the agent interacts with the environment through Protocol 1—a reward-free version of the MDP, where the agent can transit as usual but does not collect any rewards. Over the course of K episodes following Protocol 1, the agent collects a dataset of visited states, actions, and transitions $\mathcal{D} = \{s_h^{(k)}, a_h^{(k)}\}_{(k,h) \in [K] \times [H]}$, which is the outcome of the *exploration phase*.

The effectiveness of the exploration strategy is evaluated in the next phase—the *planning phase*—in which the agent is no longer allowed to interact with the MDP. In this phase, the agent is given a reward function $r(\cdot, \cdot)$ that can be potentially adversarially designed, and the **objective** here is to compute a near optimal policy for this reward function using the dataset \mathcal{D} . Performance is measured in terms of how many episodes K are required in the exploration phase so that the agent can reliably achieve the objective above. As notation, we use $V(\cdot; r)$ to emphasize that the value function depends on the reward r .

We remark that providing the reward function after the exploration phase (as opposed to before) makes the setting more challenging, and so our algorithm applies to the easier setting. We also note that our results address the setting where the reward is observed through interaction with the environment, as learning the reward is typically not the statistical barrier to efficient RL. Indeed, a provably effective reward-free exploration strategy must visit all “significant” state-action pairs (see Definition 3.2) sufficiently many times anyway, and this experience is sufficient to learn the reward function.

3 Main Results

We are now ready to state our main theorem. It asserts that our algorithm, which we will describe in the subsequent sections, is a reward-free exploration algorithm with sample complexity $\tilde{O}(H^5 S^2 A / \epsilon^2)$, ignoring lower order terms. In other words, after this many episodes interacting with the MDP via Protocol 1, our algorithm can compute ϵ -optimal policies for arbitrarily many reward functions. The theorem demonstrates that the sample complexity of reward-free exploration is at most $\tilde{O}(H^5 S^2 A / \epsilon^2)$, which we will show to be near-optimal with our lower bound in the next section.

Theorem 3.1. *There exists an absolute constant $c > 0$ and a reward-free exploration algorithm such that, for any $p \in (0, 1)$, with probability at least $1 - p$, the algorithm outputs ϵ -optimal policies for an arbitrary number of adaptively chosen reward functions. The number of episodes collected in the exploration phase is bounded by*

$$c \cdot \left[\frac{H^5 S^2 A \iota}{\epsilon^2} + \frac{S^4 A H^7 \iota^3}{\epsilon} \right], \quad (3)$$

where $\iota := \log(SAH/(p\epsilon))$.

Algorithm 2 Reward-free RL-Explore

```
1: Input: iteration number  $N_0, N$ .
2: set policy class  $\Psi \leftarrow \emptyset$ , and dataset  $\mathcal{D} \leftarrow \emptyset$ .
3: for all  $(s, h) \in \mathcal{S} \times [H]$  do
4:    $r_{h'}(s', a') \leftarrow \mathbb{1}[s' = s \text{ and } h' = h]$  for all  $(s', a', h') \in \mathcal{S} \times \mathcal{A} \times [H]$ .
5:    $\Phi^{(s,h)} \leftarrow \text{EULER}(r, N_0)$ .
6:    $\pi_h(\cdot|s) \leftarrow \text{Uniform}(\mathcal{A})$  for all  $\pi \in \Phi^{(s,h)}$ .
7:    $\Psi \leftarrow \Psi \cup \Phi^{(s,h)}$ .
8: for  $n = 1 \dots N$  do
9:   sample policy  $\pi \sim \text{Uniform}(\Psi)$ .
10:  play  $\mathcal{M}$  using policy  $\pi$ , and observe the trajectory  $z_n = (s_1, a_1, \dots, s_H, a_H, s_{H+1})$ .
11:   $\mathcal{D} \leftarrow \mathcal{D} \cup \{z_n\}$ 
12: Return: dataset  $\mathcal{D}$ .
```

We emphasize that the correctness guarantee here is quite strong: the dataset \mathcal{D} collected by the algorithm is such that any number of adaptively chosen reward functions can be optimized with no further data collection. In contrast, if we naïvely deployed a reward-sensitive RL algorithm, we would have to collect additional trajectories for each reward function, which could be quite sample inefficient. We emphasize that requiring near-optimal policies for many reward functions is quite common in applications, especially when we design reward functions by trial and error to elicit specific behaviors.

Algorithm overview. Our algorithm proceeds with following high level steps:

1. learn a set of policies Ψ which allow us to visit all “significant” states with reasonable probability.
2. collect a sufficient amount of data by executing policies in Ψ .
3. compute the empirical transition matrix $\hat{\mathbb{P}}$ using the collected data.
4. for each reward function r , find a near-optimal policy by invoking a planning algorithm with transitions $\hat{\mathbb{P}}$ and reward r .

The first two steps are performed in the exploration phase, while the latter two steps are performed in the planning phase. In Section 3.1 and Section 3.2, we will present our formal algorithms and the corresponding theoretical guarantees for two phases separately. One important feature of our algorithm is that we can use existing approximate MDP solvers or batch-RL algorithms in the last step. We demonstrate with two examples, namely Value Iteration (VI) and Natural Policy Gradient (NPG), in Section 3.3.

3.1 Exploration Phase

The goal of exploration is to visit all possible states so that the agent can gather sufficient information in order to find the optimal policy eventually. However, rather different from the bandit setting where agent can select an arbitrary arm to pull, it is possible that certain state in the MDP is very difficult to reach no matter what policy the agent is taking. Therefore, we first introduce the concept of the state being “significant”. See Figure 1 for illustrations.

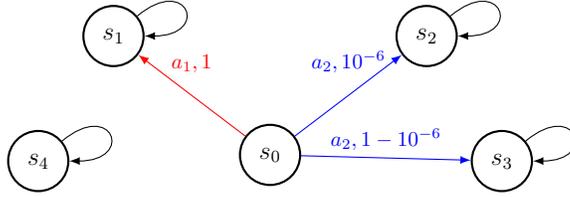


Figure 1: Illustration of significant states (Definition 3.2) v.s. insignificant states. In this toy example we have 5 states, where s_0 is the initial state. Only from state s_0 the agent can transit to other states and the other states are absorbing whatever action the agent takes. For state s_0 , we use blue arrows to represent transition if action a_1 is taken and red ones if action a_2 is taken. The numbers on the arrows following the actions are the transition probability. In this example, s_4 is insignificant, because it can never be reached. For $\delta = 10^{-5}$, s_2 is also δ -insignificant, because the best policy to reach s_2 is by taking action a_2 at initial state s_0 , which gives the maximum probability 10^{-6} to reach s_2 . The remaining states s_1, s_3 are all δ -significant.

Definition 3.2. A state s in step h is δ -**significant** if there exists a policy π , so that the probability to reach s following policy π is greater than δ . In symbol:

$$\max_{\pi} P_h^{\pi}(s) \geq \delta$$

Intuitively, with limited budget of samples and runtime, one can be only hopefully to visit all significant states. On the other hand, since insignificant states can be rarely visited no matter what policy is used, they will not significantly change the value from the initial states. Thus, for the sake of finding near-optimal policies, it is sufficient to visit all significant states with proper significance level ϵ . Indeed, Algorithm 2 is able to provide such a guarantee as follows.

Theorem 3.3. *There exists absolute constant $c > 0$ such that for any $\epsilon > 0$ and $p \in (0, 1)$, if we set $N_0 \geq cS^2AH^4v_0^3/\delta$ where $v_0 := \log(SAH/(p\delta))$, then with probability at least $1 - p$, that Algorithm 2 will returns a dataset \mathcal{D} consisting of N trajectories $\{z_n\}_{n=1}^N$, which are i.i.d sampled from a distribution μ satisfying:*

$$\forall \delta\text{-significant}(s, h), \quad \max_{a, \pi} \frac{P_h^{\pi}(s, a)}{\mu_h(s, a)} \leq 2SAH. \quad (4)$$

Theorem 3.3 claims that using Algorithm 2, we can collect data from a underlying distribution μ , which ensures that for policy π , the ratio $P_h^{\pi}(s, a)/\mu_h(s, a)$ will be upper bounded for any significant state and action. That is, all significant state and action will be visited by distribution μ with reasonable amount of probability. Notice as δ becomes smaller, there will be more significant states and the condition (4) becomes stronger. As a result we need to take larger N_0 . As we will see later, the δ we take eventually will be $\epsilon/(2SH^2)$, where ϵ is the suboptimality of the policy we find in the planning phase.

Algorithm 2 can be decompose into two parts, where Line 3-7 learns a set of exploration policies Ψ and Line 8-11 simply collects data by uniformly executing policies in Ψ . Therefore, the key mechanism lies in how to learn the set of exploration policies Ψ . Our strategy is to first learn the best policies that maximize the probability to research each state s at step h individually, and then combine them.

Concretely, for each state s at step h , algorithm 2 first create a reward function r that is always zero except for the state s at step h . Then we can simulate a standard MDP by properly feeding this designed reward r when an agent interact with the environment using protocol 1. It is easy to verify that the optimal policy for the MDP with this reward r is precisely the policy that maximizes the probability to reach (s, h) .

Algorithm 3 Reward-free RL-Plan

- 1: **Input:** a dataset of transition \mathcal{D} , reward function r , accuracy ϵ .
 - 2: **for** all $(s, a, s', h) \in \mathcal{S} \times \mathcal{A} \times \mathcal{S} \times [H]$ **do**
 - 3: $N_h(s, a, s') \leftarrow \sum_{(s_h, a_h, s_{h+1}) \in \mathcal{D}} \mathbb{1}[s_h = s, a_h = a, s_{h+1} = s']$.
 - 4: $N_h(s, a) \leftarrow \sum_{s'} N_h(s, a, s')$.
 - 5: $\hat{\mathbb{P}}_h(s'|s, a) = N_h(s, a, s')/N_h(s, a)$.
 - 6: $\hat{\pi} \leftarrow \text{APPROXIMATE-MDP-SOLVER}(\hat{\mathbb{P}}, r, \epsilon)$.
 - 7: **Return:** policy $\hat{\pi}$.
-

Thus, any RL algorithms with PAC or regret guarantees Azar et al. [2017], Jin et al. [2018] can be used here to approximately find this optimal policy. In particular, we use EULER algorithm Zanette and Brunskill [2019], whose theoretical guarantee in our setting is presented as follows ²

Lemma 3.4. *There exists absolute constant $c > 0$ such that for any $N_0 > 0$ and $p \in (0, 1)$, with probability at least $1 - p$, if we run EULER algorithm for N_0 episodes, it will output a policy set Φ with $|\Phi| = N_0$ that satisfies:*

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} \left[V_1^*(s_1) - \frac{1}{N_0} \sum_{\pi \in \Phi} V_1^\pi(s_1) \right] \leq c \cdot \left\{ \sqrt{\frac{SAH\iota_0 \cdot \mathbb{E}_{s_1 \sim \mathbb{P}_1} V_1^*(s_1)}{N_0}} + \frac{S^2 AH^4 \iota_0^3}{N_0} \right\}$$

where $\iota_0 = \log(SAHN_0/p)$.

We comment that one unique feature of EULER algorithm is that its suboptimality scales with the value of the optimal policy $\mathbb{E}_{s_1 \sim \mathbb{P}_1} V_1^*(s_1)$. This is key in obtaining a sharp result, and is especially helpful in dealing with those states that are still significant but their maximum reaching probability is low. Finally, since the best policy to reach (s, h) is only meaningful at steps before h , algorithm 2 then alter the policy for state s at step h to be Uniform(\mathcal{A}) to ensure good probability of choosing all actions for this state.

3.2 Planning Phase

In planning phase, the agent is given the reward function r , and aims to find a near-optimal policy based on r and dataset \mathcal{D} collected in the exploration phase. Algorithm 3 proceeds with two steps. Line 2-5 use counts based on dataset \mathcal{D} to estimate the empirical transition matrix $\hat{\mathbb{P}}$. Then, algorithm 3 calls a approximate MDP solver. Subroutine APPROXIMATE-MDP-SOLVER($\hat{\mathbb{P}}, r, \epsilon$) can be any algorithm that finds ϵ -suboptimal policy $\hat{\pi}$ for MDP with known transition matrix and reward (they are $\hat{\mathbb{P}}, r$ in this case). See Section 3.3 for examples of such approximate MDP solvers.

Now we are ready to state the guarantee for Algorithm 3, which asserts that as long as the number of data collected in the exploration phase is sufficiently large, the output policy $\hat{\pi}$ is not only a near-optimal policy for the estimated MDP with transition $\hat{\mathbb{P}}$, but also a near-optimal policy for the true MDP.

Theorem 3.5. *There exists absolute constant $c > 0$, for any $\epsilon > 0$, $p \in (0, 1)$, assume dataset \mathcal{D} has N i.i.d. samples from distribution μ which satisfies Eq.(4) with $\delta = \epsilon / (2SH^2)$, and $N \geq cH^5 S^2 A\iota / \epsilon^2$, then*

²In Zanette and Brunskill [2019], EULER is studied under stationary setting, where \mathbb{P} and r does not depend on h . A stationary MDP can simulate a non-stationary MDP by augmenting state s to (s, h) . Therefore, the effective number of states becomes SH when we apply the results in Zanette and Brunskill [2019].

with probability at least $1 - p$, for any reward function r simultaneously, the output policy $\hat{\pi}$ of Algorithm 3 is 3ϵ -suboptimal. That is:

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_1^*(s_1; r) - V_1^{\hat{\pi}}(s_1; r)] \leq 3\epsilon$$

The mechanism behind Theorem 3.5 is that: by sample sufficient number of exploring data, we ensure that the empirical transition $\hat{\mathbb{P}}$ and the true transition \mathbb{P} are close so that the near-optimal policy for the estimated MDP with transition $\hat{\mathbb{P}}$ is also near optimal for the true MDP. We note that the closeness of $\hat{\mathbb{P}}$ and \mathbb{P} can not be established in the usual sense of the TV-distance (or other distributional distance) between $\hat{\mathbb{P}}_h(\cdot|s, a)$ and $\mathbb{P}_h(\cdot|s, a)$ is small for any (s, a, h) , due to the existence of insignificant states. The key observation is that, nevertheless, we can establish the closeness of $\hat{\mathbb{P}}$ and \mathbb{P} in the sense that for any policy π , the value functions starting from initial states are close. That is, the difference in policy evaluations of two MDPs is small, which is summarized in the following lemma.

Lemma 3.6. *Under the preconditions of Theorem 3.5, with probability at least $1 - p$, for any reward function r and any policy π , we have:*

$$|\mathbb{E}_{s_1 \sim \mathbb{P}_1} [\hat{V}_1^\pi(s_1; r) - V_1^\pi(s_1; r)]| \leq \epsilon \quad (5)$$

where \hat{V} is the value function of MDP with the transition $\hat{\mathbb{P}}$.

The establishment of Lemma 3.6 is a natural consequence of the followings: (1) the total contribution from all insignificant states is small; (2) $\hat{\mathbb{P}}$ is reasonably accurate for all significant states; and (3) a new sharp concentration inequality (see Lemma C.2 in Appendix). With Lemma 3.6, now we are ready to prove Theorem 3.5.

Proof of Theorem 3.5. We denote the optimal policy of $\text{MDP}(\mathbb{P}, r)$ and $\text{MDP}(\hat{\mathbb{P}}, r)$ by π^* and $\hat{\pi}^*$ respectively. The theorem is a direct consequence of the following decomposition

$$\begin{aligned} & \mathbb{E}_{s_1 \sim \mathbb{P}_1} \{V_1^{\pi^*}(s_1; r) - V_1^{\hat{\pi}^*}(s_1; r)\} \\ & \leq \underbrace{|\mathbb{E}_{s_1 \sim \mathbb{P}_1} \{V_1^{\pi^*}(s_1; r) - \hat{V}_1^{\pi^*}(s_1; r)\}|}_{\text{Evaluation error 1}} + \underbrace{|\mathbb{E}_{s_1 \sim \mathbb{P}_1} \{\hat{V}_1^{\pi^*}(s_1; r) - \hat{V}_1^{\hat{\pi}^*}(s_1; r)\}|}_{\leq 0 \text{ by definition}} \\ & \quad + \underbrace{|\mathbb{E}_{s_1 \sim \mathbb{P}_1} \{\hat{V}_1^{\hat{\pi}^*}(s_1; r) - \hat{V}_1^{\hat{\pi}}(s_1; r)\}|}_{\text{Optimization error}} + \underbrace{|\mathbb{E}_{s_1 \sim \mathbb{P}_1} \{\hat{V}_1^{\hat{\pi}}(s_1; r) - V_1^{\hat{\pi}}(s_1; r)\}|}_{\text{Evaluation error 2}} \end{aligned}$$

where evaluation errors are bounded by ϵ by Lemma 3.6 and optimization error is bounded by ϵ by assumption. \square

3.3 Approximate MDP Solvers

Approximate MDP solvers aim to find a near-optimal policy when the exact transition matrix \mathbb{P} and reward r are known. The simplest way to achieve this is by **Value Iteration** (VI) algorithm, which solves the Bellman optimality equation Eq.(2) in a dynamical programming fashion. Then the greedy policy induced by the result Q^* gives precisely the optimal policy without error.

Another popular approach frequently used in practice is the **Natural Policy Gradient** (NPG) algorithm as shown in Algorithm 4. In each iteration, the algorithm first evaluates the value of policy $\pi^{(t)}$ using Bellman equation Eq.(1). Then it updates the policy by first scale it with the exponential of learning η times value $Q^{\pi^{(t)}}$, and then performs a normalization. For completeness, we provides its guarantee here. Similar analysis also appears in Agarwal et al. [2019].

Algorithm 4 Natural Policy Gradient (NPG)

- 1: **Input:** transition matrix \mathbb{P} , reward function r , stepsize η , iteration number T .
 - 2: initialize $\pi_h^{(0)}(\cdot|s) \leftarrow \text{Uniform}(\mathcal{A})$ for all (s, h)
 - 3: **for** $t = 0, \dots, T - 1$ **do**
 - 4: evaluate $Q_h^{\pi^{(t)}}(s, a)$ using Bellman equation Eq.(1) for all (s, a, h) .
 - 5: update $\pi_h^{\pi^{(t+1)}}(a|s) \propto \pi_h^{\pi^{(t)}}(a|s) \cdot \exp(\eta Q_h^{\pi^{(t)}}(s, a))$ for all (s, a, h) .
 - 6: **Return:** policy $\pi^{(T)}$.
-

Proposition 3.7. *for any learning rate η and iteration number T , the output policy $\pi^{(T)}$ of Algorithm 4 satisfies the following:*

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_1^*(s_1) - V_1^{\pi^{(T)}}(s_1)] \leq \frac{H \log A}{\eta T} + \eta H^2$$

Therefore, it is easy to verify, by choosing $\eta = \sqrt{\log A / HT}$ and $T = 4H^3 \log A / \epsilon^2$, the policy $\pi^{(T)}$ returned by NPG is ϵ -optimal.

4 Lower Bound

In this section, we establish that $\Omega(H^2 S^2 A / \epsilon^2)$ trajectories are necessary to satisfy the guarantee from Theorem 3.1.

Theorem 4.1. *Let $C > 0$ be a universal constant. Then for $A \geq 2$, $S \geq C \log_2 A$, $H \geq C \log_2 S$, and any $\epsilon \leq \min\{1/4, H/48\}$, any reward-free exploration algorithm Alg which satisfies the guarantee of Theorem 3.1 with $p = 1/2$ and accuracy parameter ϵ must collect $\Omega(S^2 AH^2 / \epsilon^2)$ trajectories in expectation. This is true even if Alg can return randomized or history-dependent (non-Markov) policies, and holds even if the rewards and transitions are identical across stages h .*

In particular, Theorem 4.1 shows that our upper bound (Theorem 3.1) is tight in S, A, ϵ , up to logarithmic factors and lower-order terms. Note that lower bound holds against querying an *unlimited* number of reward vectors. It is left as an open question whether such a lower bound holds when the algorithm is only required to ensure correctness over a smaller number of reward vectors pre-determined in advance. In what follows, we sketch a proof of Theorem 4.1; a formal proof is given in Appendix D.

4.1 Reward Free Exploration at a Single State

The core of our construction is a simple instance with a single initial state $x_1 = 0$ and $2n$ absorbing states $s \in [2n]$; the transition from states $0 \rightarrow s$ is described by a vector $q \in \mathbb{R}^{[2n] \times [A]}$, where $q(s, a)$ is the transition probability to state s if action a is taken at state 0. We shall also restrict to vectors q are close to uniform, i.e.,

$$\forall s, a, \quad \left| q(s, a) - \frac{1}{2n} \right| \leq \frac{\epsilon}{2n} \quad (6)$$

The learner is then tasked with learning near optimal policies for reward vectors r_ν parametrized by $\nu \in [0, 1]^{2n}$, which assigns a state-dependent but action-independent reward $\nu(s)$ to states $s \in [2n]$, and no reward to $x_1 = 0$. The blue (“left”) transitions or red (“right”) transition in Figure 2 mirror this construction,

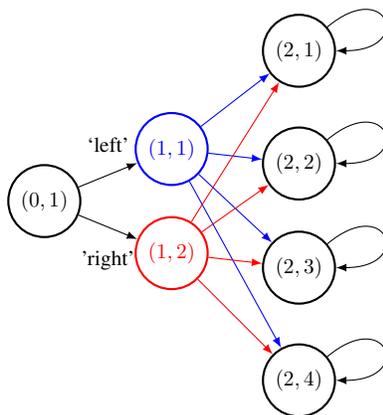


Figure 2: The “left” (blue) instance and “right” (red) instance embed two copies of the instance from Lemma 4.2. In each copy, the agent begins in stage $s = 0$, and moves to states $s \in [2n]$, $n = 2$. Different actions correspond to different probability distributions over next states $s \in [2n]$. States $s \in [2n]$ are absorbing, and rewards are action-independent. Lemma 4.2 shows that this construction requires the learner to learn $\Omega(n)$ bits about the transition probabilities $p(\cdot|0, a)$. By embedding this construction into a large MDP, this construction forces the learner to learn the transition probabilities at $n = 2$ states, $\{(x, \log_2 n) : x \in [n]\}$. The learner can deterministically access these states by appropriate choice of “left” and “right” actions.

which we formalize in Definition D.1. We show that reward-free exploration essentially forces the learner to learn the probability vectors $q(\cdot, a)$ in total-variation distance for each $a \in [A]$, yielding an $\Omega(nA/\epsilon^2)$ lower bound for this construction. A formal statement of the following Lemma is given in Lemma D.2 in the appendix.

Lemma 4.2 (Informal). *Suppose $S \geq C \log_2(A)$ for a universal constant $C > 0$. Suppose Alg, when faced with the instances described above (with q satisfying Eq. (6)) successfully returns ϵ -suboptimal policies for exponentially many reward vectors with total failure probability $1/2$. Then Alg requires $\Omega(SA/\epsilon^2)$ trajectories in expectation.*

Proof Sketch. Unfortunately, we cannot show a direct reduction from estimating q in total variation to learning near optimal-policies. Instead, by selecting appropriate reward vectors r_ν , the algorithm can decode a packing of $\exp(\Omega(n))$ transition vectors $q(\cdot, a)$ for each action $a \in [A]$. By a variant of Fano’s inequality, this leads to the same $\Omega(nA/\epsilon^2)$ lower bound that would be obtained by a direct reduction. \square

Lemma 4.2 differs from existing $\Omega(SA/\epsilon^2)$ lower bounds in that the only quantities unknown to the learner are the transition probabilities associated with the single state 0. This is in contrast to most existing lower bounds where the learner needs to collect transition information at multiple states. In particular, here the factor of S arises because the transition is to $\Theta(S)$ states, while in most constructions this factor arises because transitions from $\Theta(S)$ states must be estimated.

4.2 Lower Bound for Multiple States

To obtain an $\Omega(S^2 H^2 A/\epsilon^2)$ lower bound, we embed $n = \Omega(S)$ instances from above as the second-to-last layer of a binary tree of depth $1 + \log_2 n$. All n such instances share the same $2n$ -terminal leaves (assume n

is a power of 2). We index states by pairs (x, ℓ) , where ℓ denotes the layer. From the binary tree construction, there are at most $4n$ states, so $n = \Omega(S)$. We assume that the MDP begins in stage $(0, 1)$, and for layers $\ell < \log_2 n$, action 1 always moves “left” in the tree, and actions $2, \dots, A$ always moves “right” in the tree. Moreover, the leaf-states are all absorbing. The construction is given in Figure 2.

The only part unknown to the learner are the transition vectors $\{q_x\}_{x \in [n]}$, where $q_x(s, a)$ describes the probability of transitioning to leaf $(s, 1 + \log_2 n)$ when taking action a from state $(x, \log_2 n)$. We now index rewards by $(x, \nu) \in [n] \times [0, 1]^{2n}$, where $r_{x, \nu}$ places action-independent reward 1 on state $(x, \log_2 n)$, action-independent reward $\nu(s)$ on states $(s, 1 + \log_2 n)$, and reward 0 everywhere.

Assume that the transitions q_x satisfy the near-uniformity condition of (6) for $\epsilon = 1/4H$. Then, for reward $r_{x, \nu}$, the high reward of 1 at $(x, \log_2 n)$ forces any near-optimal policy to visit $(x, \log_2 n)$ and subsequently play near optimal actions at this state. However, playing optimally at $(x, \log_2 n)$ under reward $r_{x, \nu}$ for all ν is equivalent to reward-free learning of a single instance of the construction from Lemma 4.2. By varying $x \in [n]$ for the reward vectors $r_{x, \nu}$, the learner is forced to learn n such instances, yielding the $\Omega(n \cdot nA/\epsilon^2) = \Omega(S^2 A/\epsilon^2)$ lower bound. This can be improved to $\Omega(H^2 S^2 A/\epsilon^2)$ by using the absorbing states to create a chain of $\Omega(H)$ rewards.

5 Conclusion

In this paper, we propose a new “reward-free RL” framework, comprising of two phases. In the exploration phase, the learner first collects trajectories from an MDP \mathcal{M} *without* receiving any reward information. After the exploration phase, the learner is no longer allowed to interact with the MDP and she is instead tasked with computing near-optimal policies under for \mathcal{M} for a collection of given reward functions. This framework is particularly suitable when there are many reward functions of interest, or when we are interested in learning the transition operator directly.

This paper provides an efficient algorithm that conducts $\tilde{O}(S^2 A \text{poly}(H)/\epsilon^2)$ episodes of exploration and returns ϵ -suboptimal policies for an *arbitrary* number of adaptively chosen reward functions. Our planning procedure can be instantiated by any black-box approximate planner, such as value iteration or natural policy gradient. We also give a nearly-matching $\Omega(S^2 A H^2/\epsilon^2)$ lower bound, demonstrating the near-optimality of our algorithm in this setting.

We close with some directions for future work. On the technical level, an interesting direction is to understand the sample complexity for reward-free RL with a pre-specified reward function that is unobserved during the exploration phase. Our lower bound proofs requires the agent to be able to optimize all possible reward functions, so it does not directly apply to this potentially easier setting. Can we use $\tilde{O}(S A \text{poly}(H)/\epsilon^2)$ samples in the exploration phase to achieve this goal?

Another interesting direction is to design reward-free RL algorithms for settings with function approximation. We believe our work highlights and introduces some mechanisms that may be useful in the function approximation setting, such as the concept of significant states (Definition 3.2) and the coverage guarantee (4). How do we generalize these concepts to the function approximation setting?

We hope to pursue these directions in future work.

References

Joshua Achiam, David Held, Aviv Tamar, and Pieter Abbeel. Constrained policy optimization. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 22–31. JMLR. org, 2017.

- Alekh Agarwal, Sham M Kakade, Jason D Lee, and Gaurav Mahajan. Optimality and approximation with policy gradient methods in markov decision processes. *arXiv preprint arXiv:1908.00261*, 2019.
- Eitan Altman. *Constrained Markov decision processes*, volume 7. CRC Press, 1999.
- András Antos, Csaba Szepesvári, and Rémi Munos. Learning near-optimal policies with bellman-residual minimization based fitted policy iteration and a single sample path. *Machine Learning*, 71(1):89–129, 2008.
- Mohammad Gheshlaghi Azar, Ian Osband, and Rémi Munos. Minimax regret bounds for reinforcement learning. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 263–272. JMLR. org, 2017.
- Ronen I Brafman and Moshe Tennenholtz. R-max-a general polynomial time algorithm for near-optimal reinforcement learning. *Journal of Machine Learning Research*, 3(Oct):213–231, 2002.
- Qi Cai, Zhuoran Yang, Chi Jin, and Zhaoran Wang. Provably efficient exploration in policy optimization. *arXiv preprint arXiv:1912.05830*, 2019.
- Jinglin Chen and Nan Jiang. Information-theoretic considerations in batch reinforcement learning. *arXiv preprint arXiv:1905.00360*, 2019.
- Xi Chen, Adityanand Guntuboyina, and Yuchen Zhang. On bayes risk lower bounds. *The Journal of Machine Learning Research*, 17(1):7687–7744, 2016.
- Christoph Dann and Emma Brunskill. Sample complexity of episodic fixed-horizon reinforcement learning. In *Advances in Neural Information Processing Systems*, pages 2818–2826, 2015.
- Christoph Dann, Tor Lattimore, and Emma Brunskill. Unifying pac and regret: Uniform pac bounds for episodic reinforcement learning. In *Advances in Neural Information Processing Systems*, pages 5713–5723, 2017.
- Simon S Du, Akshay Krishnamurthy, Nan Jiang, Alekh Agarwal, Miroslav Dudík, and John Langford. Provably efficient rl with rich observations via latent state decoding. *arXiv preprint arXiv:1901.09018*, 2019.
- Elad Hazan, Sham M Kakade, Karan Singh, and Abby Van Soest. Provably efficient maximum entropy exploration. *arXiv preprint arXiv:1812.02690*, 2018.
- Chi Jin, Zeyuan Allen-Zhu, Sebastien Bubeck, and Michael I Jordan. Is q-learning provably efficient? In *Advances in Neural Information Processing Systems*, pages 4863–4873, 2018.
- Sham Kakade and John Langford. Approximately optimal approximate reinforcement learning. In *ICML*, volume 2, pages 267–274, 2002.
- Sham Machandranath Kakade. *On the sample complexity of reinforcement learning*. PhD thesis, University of London, London, England, 2003.
- Emilie Kaufmann, Olivier Cappé, and Aurélien Garivier. On the complexity of best-arm identification in multi-armed bandit models. *The Journal of Machine Learning Research*, 17(1):1–42, 2016.

- Michael Kearns and Satinder Singh. Near-optimal reinforcement learning in polynomial time. *Machine learning*, 49(2-3):209–232, 2002.
- Lihong Li. Sample complexity bounds of exploration. In *Reinforcement Learning*, pages 175–204. Springer, 2012.
- Sobhan Miryoosefi, Kianté Brantley, Hal Daumé III, Miroslav Dudik, and Robert Schapire. Reinforcement learning with convex constraints. *arXiv preprint arXiv:1906.09323*, 2019.
- Dipendra Misra, Mikael Henaff, Akshay Krishnamurthy, and John Langford. Kinematic state abstraction and provably efficient rich-observation reinforcement learning. *arXiv preprint arXiv:1911.05815*, 2019.
- Rémi Munos and Csaba Szepesvári. Finite-time bounds for fitted value iteration. *Journal of Machine Learning Research*, 9(May):815–857, 2008.
- Max Simchowitz and Kevin G Jamieson. Non-asymptotic gap-dependent regret bounds for tabular mdps. In *Advances in Neural Information Processing Systems*, pages 1151–1160, 2019.
- Chen Tessler, Daniel J Mankowitz, and Shie Mannor. Reward constrained policy optimization. *arXiv preprint arXiv:1805.11074*, 2018.
- Andrea Zanette and Emma Brunskill. Tighter problem-dependent regret bounds in reinforcement learning without domain knowledge using value function bounds. *arXiv preprint arXiv:1901.00210*, 2019.

A The ZERORMAX algorithm

RMAX is a well-known PAC exploration algorithm Brafman and Tenenholz [2002]. Here, we show that a modified version of RMAX, which we call ZERORMAX, addresses the reward-free exploration setting. The difference between ZERORMAX and RMAX is that we set the reward in “known” states to 0 instead of the true reward, which explains the name. We briefly describe the algorithm and derive the PAC bound relying heavily on prior arguments. Details about RMAX and its analysis can be found in prior work Brafman and Tenenholz [2002], Kakade [2003].

Following the reward-free exploration framework proposed in Section 2, the ZERORMAX algorithm first collects samples without knowledge about reward (exploration) and then computes a policy for each configuration of reward function (planning). We define set of known states \mathcal{K} to be

$$\mathcal{K} := \{(s, h) : \forall a \in \mathcal{A}, N_h(s, a) \geq m\}$$

where $N_h(s, a)$ counts how many times s has been visited and a was taken in the h -th step and m is a parameter to be specified later. The set \mathcal{K} contains states that we have visited enough times to estimate the corresponding transition kernel, and is typically referred to as the “known set” in the literature. For (s, h) not in \mathcal{K} , we call them “unknown.”

Now ZERORMAX explores as follows. In each episode $i \in [N]$, the agent has a known set \mathcal{K}_i and

1. builds an empirical MDP $\hat{\mathcal{M}}_{i, \mathcal{K}_i}$ with parameters

$$\mathbb{P}_h(\cdot | s, a) = \begin{cases} \hat{\mathbb{P}}_{h,i}(\cdot | s, a) & \text{if } (s, h) \in \mathcal{K}_i \\ \mathbb{1}\{s' = s\} & \text{otherwise} \end{cases} \quad r_h(s, a) = \begin{cases} 0 & \text{if } (s, h) \in \mathcal{K}_i \\ 1 & \text{otherwise} \end{cases} \quad (7)$$

where $\hat{\mathbb{P}}_{h,i}$ is the empirical estimation of \mathbb{P}_h in the i -th episode.

2. computes $\pi_i = \pi_{\hat{\mathcal{M}}_{i, \mathcal{K}_i}}^*$ on $\hat{\mathcal{M}}_{i, \mathcal{K}_i}$ by value iteration.
3. samples a trajectory from the environment following π_i .
4. constructs \mathcal{K}_{i+1} for the next episode

For the planning phase, we first sample an index $i \in [N]$ uniformly and construct the MDP $\hat{\mathcal{M}}_{i, \mathcal{K}_i}$. Then given reward function, we can just perform value iteration on $\hat{\mathcal{M}}_{i, \mathcal{K}_i}$, which gives us a near optimal policy.

A.1 Analysis

A central concept for analyzing the sample complexity of ZERORMAX is the escape probability, which is the probability of visiting the unknown states. Formally,

$$p_{\mathcal{K}}^{\pi} = \mathbb{P}_{\mathcal{M}, \pi} \{ \exists (s_h, h) \text{ s.t. } (s_h, h) \notin \mathcal{K} \}$$

The above definition also depends on the corresponding MDP \mathcal{M} . Since we only care about the escape probability w.r.t the true MDP \mathcal{M} , we will omit this dependence. The key observation is that there cannot be too many episodes where the escape probability is large. The intuition is that, if the escape probability is big, then the agent will soon visit an unknown states. However, the agent can visit unknown states at most mSA times in total.

	\mathcal{M}	$\mathcal{M}_{\mathcal{K}}$	$\hat{\mathcal{M}}_{\mathcal{K}}$
Known (\mathcal{K})	$= \mathcal{M}$	$= \mathcal{M}$	$\approx \mathcal{M}$
Unknown	$= \mathcal{M}$	self loop	self loop

Table 1: A comparison between the three MDPs involved.

Lemma A.1 (Lemma 8.5.2 in Kakade [2003]). *Let π_i be the policy followed in the i^{th} episode and \mathcal{K}_i be corresponding set of known states. Then with probability $1 - p$, there can be at most $\mathcal{O}\left(\frac{mSA}{\varepsilon} \log \frac{SANH}{p}\right)$ episodes where $p_{\mathcal{K}_i}^{\pi_i} > \varepsilon$.*

As a result, we have the following corollary.

Corollary A.2. *If we sample i uniformly from 1 to K , then with probability $1 - p - \mathcal{O}\left(\frac{mSA}{\varepsilon N} \log \frac{SANH}{p}\right)$, we have $p_{\mathcal{K}_i}^{\pi_i} \leq \varepsilon$.*

In what follows, we focus on a single “good” episode i where $p_{\mathcal{K}_i}^{\pi_i} \leq \varepsilon$. Since we focus on a single episode, let us denote \mathcal{K}_i by \mathcal{K} and π_i by $\pi_{\hat{\mathcal{M}}_{\mathcal{K}}}^*$. There are three MDPs of interest, with important details presented in Table 1.

\mathcal{M} is the true MDP of interest, that we will use to measure the performance of the policy we find in the planning phase. $\hat{\mathcal{M}}_{\mathcal{K}}$ is the MDP we use for computing policies in both exploration and planning phases. The final MDP, $\mathcal{M}_{\mathcal{K}}$ is an intermediate MDP which agrees with \mathcal{M} on the known set but follows self-loops in the unknown states. Our plan is to prove with high probability, the value of any policy π on \mathcal{M} and $\hat{\mathcal{M}}_{\mathcal{K}}$ are close, which implies the desired sample complexity result using the same argument as in Theorem 3.5.

The first step is to prove that for any policy π , the values on $\mathcal{M}_{\mathcal{K}}$ and $\hat{\mathcal{M}}_{\mathcal{K}}$ are similar.

Lemma A.3. *With probability $1 - p$, for any policy π and reward function r ,*

$$\left| \mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_{1, \hat{\mathcal{M}}_{\mathcal{K}}}^{\pi}(s_1; r) - V_{1, \mathcal{M}_{\mathcal{K}}}^{\pi}(s_1; r)] \right| \leq \mathcal{O} \left(H^2 \sqrt{\frac{S}{m} \log \frac{SANH}{p}} \right).$$

Proof. We apply Lemma C.1 to $\mathcal{M}_{\mathcal{K}}$ and $\hat{\mathcal{M}}_{\mathcal{K}}$, since the reward function is the same and the transition kernel is the same for unknown states,

$$\begin{aligned} \left| \mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_{1, \hat{\mathcal{M}}_{\mathcal{K}}}^{\pi}(s_1; r) - V_{1, \mathcal{M}_{\mathcal{K}}}^{\pi}(s_1; r)] \right| &\leq \mathbb{E}_{M_{\mathcal{K}}, \pi} \left\{ \sum_{h=1}^H \mathbb{1} \{ (s_h, h) \in \mathcal{K} \} \mid \left(\mathbb{P}_h - \hat{\mathbb{P}}_h \right) V_{h+1, \hat{\mathcal{M}}_{\mathcal{K}}}^{\pi}(s_h, a_h) \right\} \\ &\leq \mathcal{O} \left(H^2 \sqrt{\frac{S}{m} \log \frac{SANH}{p}} \right). \quad \square \end{aligned}$$

The second step is to prove that for any policy π , the values on $\mathcal{M}_{\mathcal{K}}$ and \mathcal{M} are similar, which is less straightforward.

Lemma A.4. *With probability $1 - p$ and i is a “good” episode, for any policy π ,*

$$\left| \mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_{1, \hat{\mathcal{M}}_{\mathcal{K}}}^{\pi}(s_1; r) - V_{1, \mathcal{M}_{\mathcal{K}}}^{\pi}(s_1; r)] \right| \leq H^3 \varepsilon + \mathcal{O} \left(H^4 \sqrt{\frac{S}{m} \log \frac{SANH}{p}} \right).$$

Proof. Notice that for any policy π , if we can upper bound the escape probability, then \mathcal{M}_K and \mathcal{M} must be similar for this policy. Fortunately, this is actually the case, due to our setting of the reward function in the exploration phase, following (7). Then by definition for any s ,

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} V_{\mathcal{M}_K}^\pi(s_1) \geq p_K^\pi, \quad \text{and} \quad H p_K^\pi \geq \mathbb{E}_{s_1 \sim \mathbb{P}_1} V_{\mathcal{M}_K}^\pi(s_1).$$

and using Lemma A.3,

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} V_{\mathcal{M}_K}^\pi(s_1) \geq p_K^\pi - \mathcal{O}\left(H^2 \sqrt{\frac{S}{m} \log \frac{SANH}{p}}\right)$$

However, since we are considering a good episode, we know that for the optimal policy on $\hat{\mathcal{M}}_K$, $\pi_{\hat{\mathcal{M}}_K}^*$, we have $p_K^{\pi_{\hat{\mathcal{M}}_K}^*} \leq \varepsilon$. Therefore,

$$\begin{aligned} H\varepsilon + \mathcal{O}\left(H^2 \sqrt{\frac{S}{m} \log \frac{SANH}{p}}\right) &\geq H p_K^{\pi_{\hat{\mathcal{M}}_K}^*} + \mathcal{O}\left(H^2 \sqrt{\frac{S}{m} \log \frac{SANH}{p}}\right) \\ &\geq \mathbb{E}_{s_1 \sim \mathbb{P}_1} V_{\mathcal{M}_K}^{\pi_{\hat{\mathcal{M}}_K}^*}(s_1) + \mathcal{O}\left(H^2 \sqrt{\frac{S}{m} \log \frac{SANH}{p}}\right) \geq \mathbb{E}_{s_1 \sim \mathbb{P}_1} V_{\hat{\mathcal{M}}_K}^{\pi_{\hat{\mathcal{M}}_K}^*}(s_1) \geq \mathbb{E}_{s_1 \sim \mathbb{P}_1} V_{\mathcal{M}_K}^\pi(s_1) \\ &\geq p_K^\pi - \mathcal{O}\left(H^2 \sqrt{\frac{S}{m} \log \frac{SANH}{p}}\right) \end{aligned}$$

and as a result

$$p_K^\pi \leq H\varepsilon + \mathcal{O}\left(H^2 \sqrt{\frac{S}{m} \log \frac{SANH}{p}}\right).$$

Now notice \mathcal{M}_K and \mathcal{M} are only different on unknown states, which will not influence the agent unless the agent escapes from K . Using Lemma C.1 on \mathcal{M}_K and \mathcal{M} we have

$$\left| \mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_{1, \hat{\mathcal{M}}_K}^\pi(s_1; r) - V_{1, \mathcal{M}_K}^\pi(s_1; r)] \right| \leq H^3 \varepsilon + \mathcal{O}\left(H^4 \sqrt{\frac{S}{m} \log \frac{SANH}{p}}\right). \quad \square$$

Finally we can put everything together. Again following the argument in Theorem 3.5, we have

Theorem A.5. *With probability $1 - 2p - \mathcal{O}\left(\frac{mSA}{\varepsilon K} \log \frac{SANH}{p}\right)$, given any reward function, the ZERORMAX algorithm can output a policy π such that*

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_{1, \mathcal{M}}^*(s_1) - V_{1, \mathcal{M}}^\pi(s_1)] \leq H^3 \varepsilon + \mathcal{O}\left(H^4 \sqrt{\frac{S}{m} \log \frac{SANH}{p}}\right).$$

Now we can set the parameters m and ε . To make $\mathbb{E}_{s_1 \sim \mathbb{P}_1} [V_{1, \mathcal{M}}^*(s_1) - V_{1, \mathcal{M}}^\pi(s_1)] \leq \epsilon$, we need $m \geq \Omega\left(\frac{SH^8}{\epsilon^2} \log \frac{SAKH}{p}\right)$ and $\varepsilon \leq \mathcal{O}(\epsilon/H^3)$. This means we must set

$$N \geq \Omega\left(\frac{H^{11} S^2 A}{\epsilon^3 p} \left(\log \frac{SANH}{p}\right)^2\right)$$

or equivalently,

$$N \geq \Omega \left(\frac{H^{11} S^2 A}{\epsilon^3 p} \left(\log \frac{SAH}{p\epsilon} \right)^2 \right)$$

This sample complexity is quite poor because it scales with ϵ^{-3} and polynomially, rather than logarithmically, with $1/p$.

B MaxEnt Exploration

Another approach for reward-free exploration was studied in Hazan et al. [2018]. They consider the infinite horizon discounted setting with discount factor γ , and they show that with $\tilde{O}(\frac{S^2 A}{\epsilon^3 (1-\gamma)^2})$ trajectories of length $\tilde{O}(\frac{\log S}{\epsilon^{-1} \log(1/\gamma)})$, they can find a policy $\hat{\pi}$ such that

$$\frac{1}{S} \sum_s \log(d_{\hat{\pi}}(s)) \geq \max_{\pi} \frac{1}{S} \sum_s \log(d_{\pi}(s)) - \epsilon$$

where $d_{\pi}(s) = (1 - \gamma) \sum_{t=1}^{\infty} \gamma^t d_{t,\pi}(s)$ and $d_{t,\pi}(s) = \mathbb{P}[s_t = s \mid \pi]$.

For reward free exploration, we want to use this guarantee to establish a condition similar to the conclusion of Theorem 3.3. For the sake of contradiction, suppose there exists some policy $\tilde{\pi}$ and some state \tilde{s} such that

$$\frac{d_{\tilde{\pi}}(\tilde{s})}{d_{\hat{\pi}}(\tilde{s})} > 4S.$$

We want to show that the non-Markovian mixture policy $(1 - \alpha)\hat{\pi} + \alpha\tilde{\pi}$ for some $\alpha > 0$ demonstrates that $\hat{\pi}$ violates its near-optimality guarantee for the optimization problem. To do this, we lower bound the difference in objective values between the mixture policy and $\hat{\pi}$:

$$\begin{aligned} \frac{1}{S} \sum_s \log((1 - \alpha)d_{\hat{\pi}}(x) + \alpha d_{\tilde{\pi}}(s)) - \log(d_{\hat{\pi}}(s)) &= \frac{1}{S} \sum_s \log \left(1 - \alpha \frac{d_{\tilde{\pi}}(s) - d_{\hat{\pi}}(s)}{d_{\hat{\pi}}(s)} \right) \\ &\geq \frac{S-1}{S} \log(1 - \alpha) + \frac{1}{S} \log(1 + \alpha(4S - 1)) \\ &\geq \frac{S-1}{S} \frac{-\alpha}{1 - \alpha} + \frac{1}{S} \frac{\alpha(4S - 1)}{1 + \alpha(4S - 1)} \\ &= \frac{\alpha}{S} \left(\frac{4S}{1 + \alpha(4S - 1)} - \frac{1}{1 + \alpha(4S - 1)} - \frac{(S - 1)}{1 - \alpha} \right). \end{aligned}$$

Here we are using that $\log(1 - x_1 + x_2)$ is monotonically increasing in x_2 so we use the lower bound of $4S$ on \tilde{s} and the trivial lower bound of 0 on all of the other states. We also use that $\log(1 + x) \geq \frac{x}{1+x}$, which holds for any $x > -1$. The expression inside the parenthesis can be simplified to

$$\frac{3S + S\alpha - 4S^2\alpha}{(1 - \alpha)(1 + \alpha(4S - 1))}.$$

At this point we can see that if $\alpha \geq 1/S$ then this expression is negative, so the mixture policy with large α does not yield any improvement in objective. On the other hand, for any $\alpha < 1/S$ then this inner expression

is $\Theta(S)$. So if we set $\alpha = \Theta(1/S)$ the overall improvement in objective is $\Omega(1/S)$. This means that if we want establish the guarantee in Theorem 3.3, we must set $\varepsilon = 1/S$, at which point the overall sample complexity scales with S^5 , which is quite poor.

Note that this calculation shows that $O(S^5)$ samples is sufficient for the maximum entropy approach to find a suitable exploratory policy, but we do not claim that it is necessary for this method. A sharper analysis may be possible, but we are not aware of any such results.

C Proof for Main Results

In this section, we present proofs for results in Section 3.

C.1 Exploration Phase

We begin with the proof of Lemma 3.4, which is a simple modification of the Theorem 1 in Zanette and Brunskill [2019].

Proof of Lemma 3.4. WLOG, we can assume s_1 is fixed. This is because for s_1 stochastic from \mathbb{P}_1 , we can simply add an artificial step before the first step of MDP, which always starts from the same state s_0 , has only one action, and the transition to s_1 satisfies \mathbb{P}_1 . This creates a new MDP with fixed initial state with length $H + 1$, which is equivalent to the original MDP.

We use an alternative upper-bound for equation (156) in Zanette and Brunskill [2019], which gives:

$$\begin{aligned} & \frac{1}{N_0 H} \sum_{k=1}^{N_0} \mathbb{E}_{\pi_k} \left[\left(\sum_{h=1}^H r(s_h, a_h) - V_1^{\pi_k}(s_1) \right)^2 \middle| s_1 \right] \\ & \leq \frac{2}{N_0 H} \sum_{k=1}^{N_0} \mathbb{E}_{\pi_k} \left[\left(\sum_{h=1}^H r(s_h, a_h) \right)^2 + (V_1^{\pi_k}(s_1))^2 \middle| s_1 \right] \\ & \stackrel{(i)}{\leq} \frac{2}{N_0 H} \sum_{k=1}^{N_0} \mathbb{E}_{\pi_k} \left[\sum_{h=1}^H r(s_h, a_h) + V_1^{\pi_k}(s_1) \middle| s_1 \right] \\ & \leq \frac{4}{N_0 H} \sum_{k=1}^{N_0} V_1^{\pi_k}(s_1) \leq \frac{4}{H} V_1^*(s_1) \end{aligned}$$

where π_k is the policy used in EULER in the k -th episode. Step (i) is because using the reward function designed in Line 4 in Algorithm 2, we have all reward equal to zero except one state. Therefore, we have $\sum_{h=1}^H r(s_h, a_h) \leq 1$ and $V_1^{\pi}(s_1) \leq 1$. Therefore, we have replace the upper bound \mathcal{G}^2 in (156) of Zanette and Brunskill [2019] by $4V_1^*(s_1)$.

This allows us also replace the \mathcal{G}^2 in Theorem 1 of Zanette and Brunskill [2019] by $4V_1^*(s_1)$, which gives the regret of algorithm (note Zanette and Brunskill [2019] is for stationary MDP, while our paper is for non-stationary MDP, thus S in Zanette and Brunskill [2019] need to be replaced by SH in our paper due to state augmentation, which creates new states as (s, h)):

$$\sum_{k=1}^{N_0} [V_1^*(s_1) - V^{\pi_k}(s_1)] \leq \tilde{O}(\sqrt{V_1^*(s_1)SAT} + S^2 AH^4)$$

Finally, plug in $T = N_0 H$, we finish the proof. □

Now we can prove the main result in this section.

Proof of Theorem 3.3. In the following we can fix a state (s, h) and consider the corresponding policy given by EULER. Remember in our setting (Line 4 in Algorithm 2),

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} V_1^*(s_1) = \max_{\pi} P_h^{\pi}(s)$$

Therefore the regret guarantee Lemma 3.4 implies

$$\max_{\pi} P_h^{\pi}(s) - \frac{1}{N_0} \sum_{\pi \in \Phi(s, h)} P_h^{\pi}(s) \leq c_0 \sqrt{\frac{SAH\iota_0 \cdot \max_{\pi} P_h^{\pi}(s)}{N_0}} + \frac{S^2 AH^4 \iota_0^3}{N_0}$$

for some absolute constant c_0 . Therefore, in order to make the following true

$$\max_{\pi} P_h^{\pi}(s) - \frac{1}{N_0} \sum_{\pi \in \Phi(s, h)} P_h^{\pi}(s) \leq \frac{1}{2} \max_{\pi} P_h^{\pi}(s)$$

We simply need to choose N_0 large enough so that:

$$\begin{aligned} \sqrt{\frac{SAH\iota_0 \cdot \max_{\pi} P_h^{\pi}(s)}{N_0}} &\leq c_1 \cdot \max_{\pi} P_h^{\pi}(s) \\ \frac{S^2 AH^4 \iota_0^3}{N_0} &\leq c_1 \cdot \max_{\pi} P_h^{\pi}(s) \end{aligned}$$

for a sufficient small absolute constant c_1 . Combining with the fact that for δ -significant (s, h) , $\max_{\pi} P_h^{\pi}(s) \geq \delta$, we know choosing $N_0 = \mathcal{O}(S^2 AH^4 \iota_0^3 / \delta)$ is sufficient. As a result, we have

$$\max_{\pi} \frac{P_h^{\pi}(s)}{\frac{1}{N_0} \sum_{\pi \in \Phi(s, h)} P_h^{\pi}(s)} \leq 2$$

Since Algorithm 2 sets all policy in $\Phi(s, h)$ to choose action uniformly randomly at (s, h) , this implies

$$\max_{\pi, a} \frac{P_h^{\pi}(s, a)}{\frac{1}{N_0} \sum_{\pi \in \Phi(s, h)} P_h^{\pi}(s, a)} \leq 2A$$

Finally, we can apply the same argument for all δ -significant (s, h) , and let $\Psi = \cup \{\Phi(s, h)\}_{(s, h)}$ which gives:

$$\forall \delta\text{-significant } (s, h), \quad \max_{\pi, a} \frac{P_h^{\pi}(s, a)}{\frac{1}{N_0 SH} \sum_{\pi \in \Psi} P_h^{\pi}(s, a)} \leq 2SAH.$$

This finishes the proof. □

C.2 Planning Phase

The following lemma (E.15 in Dann et al. [2017]) will be useful to characterize the difference between $V_h^{\pi}(s; r)$ and $\hat{V}_h^{\pi}(s; r)$.

Lemma C.1 (Lemma E.15 in Dann et al. [2017]). *For any two MDPs \mathcal{M}' and \mathcal{M}'' with rewards r' and r'' and transition probabilities \mathbb{P}' and \mathbb{P}'' , the difference in values V' , V'' with respect to the same policy π can be written as*

$$V'_h(s) - V''_h(s) = \mathbb{E}_{\mathcal{M}'', \pi} \left[\sum_{i=h}^H [r'_i(s_i, a_i) - r''_i(s_i, a_i) + (\mathbb{P}'_i - \mathbb{P}''_i)V'_{i+1}(s_i, a_i)] \middle| s_h = s \right]$$

With this decomposition in mind, we can prove Lemma 3.6.

Proof of Lemma 3.6. In this section, we always use \mathbb{E} to denote the expectation under the true MDP \mathcal{M} . Using Lemma C.1 on \mathcal{M} (the true MDP) and $\hat{\mathcal{M}}$ (the empirical version), we have

$$|\mathbb{E}_{s_1 \sim \mathbb{P}_1} \{\hat{V}_1^\pi(s_1; r) - V_1^\pi(s_1; r)\}| \leq |\mathbb{E}_\pi \sum_{h=1}^H (\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s_h, a_h)| \leq \mathbb{E}_\pi \sum_{h=1}^H |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s_h, a_h)|$$

Let $\mathcal{S}_h^\delta := \{s : \max_\pi P_h^\pi(s) \geq \delta\}$ be the set of δ -significant states in the h -th step. We further have:

$$\mathbb{E}_\pi |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s_h, a_h)| \leq \underbrace{\sum_{a, s \in \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s, a)| P_h^\pi(s, a)}_{\xi_h} + \underbrace{\sum_{a, s \notin \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s, a)| P_h^\pi(s, a)}_{\zeta_h}$$

By definition of insignificant state, we have:

$$\zeta_h \leq H \sum_{a, s \notin \mathcal{S}_h^\delta} P_h^\pi(s, a) = H \sum_{s \notin \mathcal{S}_h^\delta} P_h^\pi(s) \leq H \sum_{s \notin \mathcal{S}_h^\delta} \delta \leq HS\delta. \quad (8)$$

On the other hand, by Cauchy-Schwartz inequality, we have:

$$\xi_h \leq \left[\sum_{a, s \in \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s, a)|^2 P_h^\pi(s, a) \right]^{\frac{1}{2}} = \left[\sum_{a, s \in \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s, a)|^2 P_h^\pi(s) \pi_h(a|s) \right]^{\frac{1}{2}}$$

We note since \hat{V}_{h+1}^π only depends on π at $h+1, \dots, H$ steps, it does not depend on π_h . Therefore, we have:

$$\begin{aligned} \sum_{a, s \in \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s, a)|^2 P_h^\pi(s) \pi_h(a|s) &\leq \max_{\pi'_h} \sum_{a, s \in \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s, a)|^2 P_h^\pi(s) \pi'_h(a|s) \\ &= \max_{\nu: \mathcal{S} \rightarrow \mathcal{A}} \sum_{a, s \in \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h) \hat{V}_{h+1}^\pi(s, a)|^2 P_h^\pi(s) \mathbb{1}\{a = \nu(s)\} \end{aligned}$$

where the last step is because the maximization over π'_h achieves at deterministic policies.

Recall that by preconditions, we have 4 holds for $\delta = \epsilon/(2SH^2)$. That is, for any $s \in \mathcal{S}_h^\delta$ we always have

$$\max_{\tilde{\pi}} \frac{P_h^{\tilde{\pi}}(s, a)}{\mu_h(s, a)} \leq 2SAH$$

Therefore, for any (s, a) pair, we can design a policy π' so that $\pi'_{h'} = \pi_{h'}$ for all $h' < h$, and $\pi'_h(s) = a$. This will give that

$$P_h^\pi(s) = P_h^{\pi'}(s) = P_h^{\pi'}(s, a) \leq 2SAH\mu_h(s, a)$$

which gives:

$$\begin{aligned} & \sum_{a, s \in \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)\hat{V}_{h+1}^\pi(s, a)|^2 P_h^\pi(s) \mathbf{1}\{a = \nu(s)\} \\ & \leq 2SAH \sum_{a, s \in \mathcal{S}_h^\delta} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)\hat{V}_{h+1}^\pi(s, a)|^2 \mu_h(s) \mathbf{1}\{a = \nu(s)\} \\ & \leq 2SAH \sum_{s, a} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)\hat{V}_{h+1}^\pi(s, a)|^2 \mu_h(s) \mathbf{1}\{a = \nu(s)\} \\ & = 2SAH \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)\hat{V}_{h+1}^\pi(s, a)|^2 \mathbf{1}\{a = \nu(s)\} \end{aligned}$$

By Lemma C.2, we have:

$$\mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)\hat{V}_{h+1}^\pi(s, a)|^2 \mathbf{1}\{a = \nu(s)\} \leq \mathcal{O}\left(\frac{H^2 S}{N} \log\left(\frac{AHN}{p}\right)\right)$$

Therefore, combine all equations above, we have

$$|\mathbb{E}_{s_1 \sim \mathbb{P}_1} \{\hat{V}_1^\pi(s_1; r) - V_1^\pi(s_1; r)\}| \leq \mathcal{O}\left(\sqrt{\frac{H^5 S^2 A}{N} \log\left(\frac{AHN}{p}\right)}\right) + H^2 S \delta$$

Recall our choice $\delta = \epsilon/(2SH^2)$ and $N \geq c \frac{H^5 S^2 A}{\epsilon^2} \log\left(\frac{SAH}{p\epsilon}\right)$ for sufficiently large absolute constant c , which finishes the proof. \square

Lemma C.2. *Suppose $\hat{\mathbb{P}}$ is the empirical transition matrix formed by sampling according to μ distribution for N samples, then with probability at least $1 - p$, we have for any $h \in [H]$:*

$$\max_{G: \mathcal{S} \rightarrow [0, H]} \max_{\nu: \mathcal{S} \rightarrow \mathcal{A}} \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbf{1}\{a = \nu(s)\} \leq \mathcal{O}\left(\frac{H^2 S}{N} \log\left(\frac{AHN}{p}\right)\right)$$

Proof. Define random variable

$$X_i = (\hat{\mathbb{P}}_h G(s_i, a_i) - G(s'_i))^2 - (\mathbb{P}_h G(s_i, a_i) - G(s'_i))^2$$

where $(s_i, a_i, s'_i) \sim \mu_h \times \mathbb{P}_h(\cdot | s_i, a_i)$ is the i -th sample in level h we collect.

Also we define

$$Y_i = X_i \mathbf{1}\{a_i = \nu(s_i)\}.$$

To simplify the notation, when some property of Y_i holds for any i , we just use the notation Y to describe a generic Y_i .

We first state some properties of the random variables Y_i , which are justified at the end of the proof.

- (Expectation) $\mathbb{E}Y = \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbf{1}\{a = \nu(s)\}$
- (Empirical risk minimization) $\sum_{i=1}^N Y_i \leq 0$

- (Self-bounded) $\text{Var}\{Y\} \leq 4H^2\mathbb{E}Y$

Given these three properties, now we are ready to apply Bernstein's inequality to $(\sum_{i=1}^N Y_i)/N$. Since we are taking maximum over ν and $G(s)$ and $\hat{\mathbb{P}}$ is random, we need to cover all the possible values of $\hat{\mathbb{P}}G(s, a)\mathbb{1}\{a = \nu(s)\}$ and $\mathbb{P}G(s, a)\mathbb{1}\{a = \nu(s)\}$ to ε accuracy to make Bernstein's inequality hold. For ν , there are A^S deterministic policies in total. Given a fixed ν , $\hat{\mathbb{P}}G(s, a)\mathbb{1}\{a = \nu(s)\}$ and $\mathbb{P}G(s, a)\mathbb{1}\{a = \nu(s)\}$ can be covered by $(H/\varepsilon)^{2S}$ values by boundedness condition because for $a \neq \nu(s)$ they are always 0. The overall approximation error will be at most $12H\varepsilon$ by boundedness condition.

As a result, with probability at least $1 - p/H$, for any ν , $G(s)$ and $\hat{\mathbb{P}}$,

$$\begin{aligned} \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbb{1}\{a = \nu(s)\} &= \mathbb{E}Y \leq \mathbb{E}Y - \frac{1}{N} \sum_{i=1}^N Y_i \\ &\leq \sqrt{\frac{2\text{Var}\{Y\} \log((\frac{H}{\varepsilon})^{2S} \cdot A^S \cdot \frac{H}{p})}{N}} + \frac{H^2 \log((\frac{H}{\varepsilon})^{2S} \cdot A^S \cdot \frac{H}{p})}{3N} + 12H\varepsilon \\ &\leq \sqrt{\frac{2\text{Var}\{Y\} [2S \log(\frac{HA}{\varepsilon}) + \log \frac{H}{p}]}{N}} + \frac{H^2 [2S \log(\frac{HA}{\varepsilon}) + \log \frac{H}{p}]}{3N} + 12H\varepsilon \end{aligned}$$

We can simply choose $\varepsilon = HS/36N$ and thus

$$\begin{aligned} &\mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbb{1}\{a = \nu(s)\} \\ &\leq \sqrt{8H^2 \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbb{1}\{a = \nu(s)\}} \frac{2S \log(\frac{36AN}{S}) + \log \frac{H}{p}}{N} + \frac{H^2 [2S \log(\frac{36AN}{S}) + \log \frac{H}{p} + S]}{3N} \end{aligned}$$

Solving this quadratic formula we get

$$\mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbb{1}\{a = \nu(s)\} \leq \mathcal{O}\left(\frac{H^2 S}{N} \log\left(\frac{ANH}{p}\right)\right)$$

Since the above upper bound holds for arbitrary ν , $G(s)$ and \mathbb{P}_h ,

$$\max_{G: S \rightarrow [0, H]} \max_{\nu: S \rightarrow \mathcal{A}} \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbb{1}\{a = \nu(s)\} \leq \mathcal{O}\left(\frac{H^2 S}{N} \log\left(\frac{AHN}{p}\right)\right)$$

Taking union bound w.r.t. h , the claim holds for any h with probability $1 - p$.

Finally we give the proofs for the claimed three properties of Y_i . We begin with the expectation property:

$$\begin{aligned} \mathbb{E}Y &= \mathbb{E}_{s, a \sim \mu_h} \mathbb{E}_{s' \sim \mathbb{P}_h(\cdot | s, a)} \{\mathbb{1}\{a = \nu(s)\} [(\hat{\mathbb{P}}_h G(s, a) - G(s'))^2 - (\mathbb{P}_h G(s, a) - G(s'))^2]\} \\ &\stackrel{(i)}{=} 2\mathbb{E}_{s, a \sim \mu_h} \mathbb{E}_{s' \sim \mathbb{P}_h(\cdot | s, a)} \{\mathbb{1}\{a = \nu(s)\} (\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a) (\mathbb{P}_h G(s, a) - G(s'))\} \\ &\quad + \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbb{1}\{a = \nu(s)\} \\ &\stackrel{(ii)}{=} \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbb{1}\{a = \nu(s)\} \end{aligned}$$

where (i) is by $b^2 - d^2 = (b - d + d)^2 - d^2 = (b - d)^2 + 2b(d - b)$ with $b = \hat{\mathbb{P}}_h G(s, a) - G(s')$ and $d = \mathbb{P}_h G(s, a) - G(s')$ and (ii) is because $\mathbb{E}_{s' \sim \mathbb{P}_h(\cdot | s, a)} \{G(s')\} = \mathbb{P}_h G(s, a)$.

The empirical risk minimization property is true because the evaluation rule is essentially minimizing the empirical Bellman error for each (s, a) pair separately. Mathematically,

$$\hat{\mathbb{P}}_h G(s, a) = \arg \max_g \sum_{i=1}^N \mathbf{1}\{s_i = s, a_i = a\} (g - G(s'))^2$$

The self-bounded property is because

$$\begin{aligned} \text{Var}\{Y\} &\leq \mathbb{E}(Y)^2 \\ &\stackrel{(i)}{=} \mathbb{E}\{\mathbf{1}\{a = \nu(s)\} [(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)]^2 [(\hat{\mathbb{P}}_h + \mathbb{P}_h)G(s, a) - 2G(s')]^2\} \\ &\leq 4H^2 \mathbb{E}_{\mu_h} |(\hat{\mathbb{P}}_h - \mathbb{P}_h)G(s, a)|^2 \mathbf{1}\{a = \nu(s)\} \\ &= 4H^2 \mathbb{E}Y \end{aligned}$$

where (i) by $b^2 - d^2 = (b + d)(b - d)$ with $b = \hat{\mathbb{P}}_h G(s, a) - G(s')$ and $d = \mathbb{P}_h G(s, a) - G(s')$. \square

C.3 Proof of Theorem 3.1

Putting everything together we can prove the main theorem.

Proof of Theorem 3.1. We only need to choose the parameter δ and N_0 . From the proof of Lemma 3.6 we can see, we need $\delta = \epsilon/(2SH^2)$ and thus $N_0 \geq cS^3 AH^6 \iota^3/\epsilon$. Since we need N_0 episodes for each (s, h) , the total number episodes required for finding Ψ is $\mathcal{O}(cS^4 AH^7 \iota^3/\epsilon)$, which gives the second term in (3). The proof is completed by combining Theorem 3.5, which gives the first term in (3). \square

C.4 Approximate MDP Solvers

The convergence of NPG is well studied in Agarwal et al. [2019] (tabular & infinite horizon) and Cai et al. [2019] (linear approximation). However, the episodic setting has some unique characters (For example, we not every state can be arrive at the first step and the corresponding analysis in Agarwal et al. [2019] does not apply). Therefore the guarantee given in Proposition 3.7 is different.

Since we only need to prove the guarantee on the true MDP, we will not distinguish true MDP \mathcal{M} and estimated MDP $\hat{\mathcal{M}}$ here. Remember the NPG is defined by

$$\pi_h^{(0)}(a|s) = 1/A$$

and

$$\pi_h^{(t+1)}(a|s) = \pi_h^{(t)}(a|s) \exp\{\eta(Q_h^{(t)}(s, a) - V_h^{(t)}(s))\} / Z_h^{(t)}(s)$$

where $Q_h^{(t)}(s, a) := Q_h^{\pi^{(t)}}(s, a)$ is computed following the value iteration procedure. Similarly we define $V_h^{(t)}(s) := V_h^{\pi^{(t)}}(s)$. The normalization constant can be written explicitly as

$$Z_h^{(t)}(s) := \sum_{a \in \mathcal{A}} \pi_h^{(t)}(a|s) \exp\{\eta[Q_h^{(t)}(s, a) - V_h^{(t)}(s)]\}$$

Notice the definition of the normalization constant is not unique. Here we choose the form that makes the following proof simpler but different choice will essentially gives exactly the same algorithm.

We begin with a lemma showing that the value function monotonically increases.

Lemma C.3 (Lemma 5.8 in Agarwal et al. [2019]). *Following the NPG iterations,*

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} \{V_1^{(t+1)}(s_1; r) - V_1^{(t)}(s_1; r)\} \geq \frac{1}{\eta} \sum_{h=1}^H \mathbb{E}_{s_h \sim M, \pi^{(t+1)}} \{\log Z_h^{(t)}(s_h)\} \geq 0$$

Proof. By performance difference lemma Kakade and Langford [2002],

$$\begin{aligned} & \mathbb{E}_{s_1 \sim \mathbb{P}_1} \{V_1^{(t+1)}(s_1; r) - V_1^{(t)}(s_1; r)\} \\ &= \sum_{h=1}^H \mathbb{E}_{\pi^{(t+1)}} \left\{ \sum_{a \in \mathcal{A}} \pi_h^{(t+1)}(a|s) [Q_h^{(t)}(s, a) - V_h^{(t)}(s)] \right\} \\ &= \frac{1}{\eta} \sum_{h=1}^H \mathbb{E}_{\pi^{(t+1)}} \left\{ \sum_{a \in \mathcal{A}} \pi_h^{(t+1)}(a|s_h) \log \frac{\pi_h^{(t+1)}(a|s_h) Z_h^{(t)}(s_h)}{\pi_h^{(t)}(a|s_h)} \right\} \\ &= \frac{1}{\eta} \sum_{h=1}^H \mathbb{E}_{\pi^{(t+1)}} \{ \text{KL}(\pi_h^{(t+1)}(s_h) || \pi_h^{(t)}(s_h)) + \log Z_h^{(t)}(s_h) \} \\ &\geq \frac{1}{\eta} \sum_{h=1}^H \mathbb{E}_{\pi^{(t+1)}} \{ \log Z_h^{(t)}(s_h) \} \\ &\stackrel{(i)}{\geq} 0 \end{aligned}$$

where (i) is by

$$\begin{aligned} \log Z_h^{(t)}(s) &= \log \left\{ \sum_{a \in \mathcal{A}} \pi_h^{(t)}(a|s) \exp \{ \eta [Q_h^{(t)}(s, a) - V_h^{(t)}(s)] \} \right\} \\ &\geq \eta \sum_{a \in \mathcal{A}} \pi_h^{(t)}(a|s) [Q_h^{(t)}(s, a) - V_h^{(t)}(s)] \\ &= 0 \end{aligned}$$

because $V_h^{(t)}(s) = \sum_{a \in \mathcal{A}} \pi_h^{(t)}(a|s) Q_h^{(t)}(s, a)$ by definition. \square

Equipped with the monotone property, we can simply prove an upper bound for the cumulative regret, which immediately implies the convergence rate for the last iteration.

Proof of Proposition 3.7. Again by performance difference lemma,

$$\begin{aligned} & \mathbb{E}_{s_1 \sim \mathbb{P}_1} \{V_1^*(s_1; r) - V_1^{(t)}(s_1; r)\} \\ &= \sum_{h=1}^H \mathbb{E}_{\pi^*} \left\{ \sum_{a \in \mathcal{A}} \pi_h^*(a|s) [Q_h^{(t)}(s, a) - V_h^{(t)}(s)] \right\} \\ &= \frac{1}{\eta} \sum_{h=1}^H \mathbb{E}_{\pi^*} \left\{ \sum_{a \in \mathcal{A}} \pi_h^*(a|s_h) \log \frac{\pi_h^{(t+1)}(a|s_h) Z_h^{(t)}(s_h)}{\pi_h^{(t)}(a|s_h)} \right\} \\ &= \frac{1}{\eta} \sum_{h=1}^H \mathbb{E}_{\pi^*} \{ \text{KL}(\pi_h^*(s_h) || \pi_h^{(t)}(s_h)) - \text{KL}(\pi_h^*(s_h) || \pi_h^{(t+1)}(s_h)) + \log Z_h^{(t)}(s_h) \} \end{aligned}$$

Now we can upper bound the regret of $\pi^{(T-1)}$ by upper bound the cumulative regret using Lemma C.3

$$\begin{aligned}
& \mathbb{E}_{s_1 \sim \mathbb{P}_1} \{V_1^*(s_1; r) - V_1^{(T-1)}(s_1; r)\} \\
& \leq \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E}_{s_1 \sim \mathbb{P}_1} \{V_1^*(s_1; r) - V_1^{(t)}(s_1; r)\} \\
& \leq \frac{1}{\eta T} \sum_{t=0}^{T-1} \sum_{h=1}^H \mathbb{E}_{\pi^*} \{\text{KL}(\pi_h^*(s_h) \parallel \pi_h^{(t)}(s_h)) - \text{KL}(\pi_h^*(s_h) \parallel \pi_h^{(t+1)}(s_h)) + \log Z_h^{(t)}(s_h)\} \\
& \leq \frac{1}{\eta T} \sum_{h=1}^H \mathbb{E}_{\pi^*} \{\text{KL}(\pi_h^*(s_h) \parallel \pi_h^{(0)}(s_h))\} + \frac{1}{\eta T} \sum_{t=0}^{T-1} \mathbb{E}_{\pi^*} \{\log Z_h^{(t)}(s_h)\} \\
& \leq \frac{H \log A}{\eta T} + \frac{1}{\eta T} \sum_{t=0}^{T-1} \mathbb{E}_{\pi^*} \{\log Z_h^{(t)}(s_h)\}
\end{aligned}$$

Therefore we only need to bound $\log Z_h^{(t)}(s_h)$, where the technique in Agarwal et al. [2019] does not apply and we use a different approach. Notice for $x \leq 1$, $\exp\{x\} \leq 1 + x + x^2$. So as long as $\eta \leq \frac{1}{H}$, $\eta[Q_h^{(t)}(s, a) - V_h^{(t)}(s)] \leq 1$ and we have

$$\begin{aligned}
\log Z_h^{(t)}(s) &= \log \left\{ \sum_{a \in \mathcal{A}} \pi_h^{(t)}(a|s) \exp\{\eta[Q_h^{(t)}(s, a) - V_h^{(t)}(s)]\} \right\} \\
&\leq \log \left\{ \sum_{a \in \mathcal{A}} \pi_h^{(t)}(a|s) \{1 + \eta[Q_h^{(t)}(s, a) - V_h^{(t)}(s)] + \eta^2[Q_h^{(t)}(s, a) - V_h^{(t)}(s)]^2\} \right\} \\
&= \log \left\{ 1 + \eta^2 \sum_{a \in \mathcal{A}} \pi_h^{(t)}(a|s) [Q_h^{(t)}(s, a) - V_h^{(t)}(s)]^2 \right\} \\
&\leq \eta^2 \sum_{a \in \mathcal{A}} \pi_h^{(t)}(a|s) [Q_h^{(t)}(s, a) - V_h^{(t)}(s)]^2 \\
&\leq \eta^2 H^2
\end{aligned}$$

Put everything together we have

$$\mathbb{E}_{s_1 \sim \mathbb{P}_1} \{V_1^*(s_1; r) - V_1^{(T-1)}(s_1; r)\} \leq \frac{H \log A}{\eta T} + \eta H^2$$

This finishes the proof. \square

D Proof of Lower Bound

In this section, we prove our lower bound, Theorem 4.1. First, we develop further notation in Section D.1 which will aid in distinguishing between multiple possible instances. Next, Section D.2 states Lemma D.2, the formal analogue of Lemma 4.2, which describes a lower bound for learning transitions at a single state. Then, Section D.3 embeds the construction to obtain an instance where the learner to learn transitions at n states, yielding the lower bound Theorem 4.1. Finally, Section D.4 details the proof of the 1-state lower bound, Lemma 4.2.

D.1 Preliminaries

Environments, Transition Classes, Reward Classes To formalize our embedding a one-state instance into a larger MDP, the following formalities are helpful: we define an environment $\mathcal{E} = (\mathcal{X}, A, H)$ as a triple specifying a finite state space \mathcal{X} , number of actions A , and horizon H . For a fixed environment, a transition class \mathcal{P} is a class of transition and initial state distributions, denoted by \mathbb{P} ; a reward class \mathcal{R} is a family of reward functions $r : (\mathcal{X}, A) \rightarrow [0, 1]$. Given a reward vector r and transition vector \mathbb{P} , we let $\text{mdp}(\mathbb{P}, r)$ denote the with-reward MDP induced by \mathbb{P} and r . We denote value of a policy π on $\text{mdp}(\mathbb{P}, r)$ by $V^\pi(\mathbb{P}, r)$.

Reward-Free MDP Algorithm A reward-free MDP algorithm Alg is algorithm which collects a random number K trajectories from a given reward-free MDP, and then, when given a sequence of reward vectors $r^{(1)}, r^{(2)}, \dots, r^{(N)}$, returns a sequence of policies $\pi^{(1)}, \pi^{(2)}, \dots, \pi^{(N)}$. We let $\mathbf{E}_{\mathbb{P}, \text{Alg}}[\cdot]$ denote the expectation under the joint law prescribed by the exploration phase of algorithm Alg and transition operator \mathbb{P} .

Correctness Given $\epsilon, p \in (0, 1)$, say that a reward-free MDP algorithm (ϵ, p, \cdot) -learns a a problem class $\mathcal{M} := (\mathcal{E}, \mathcal{R}, \mathcal{P})$ if, for any transition operator $\mathbb{P} \in \mathcal{P}$, for any finite sequence of reward vectors $r^{(1)}, \dots, r^{(N)} \in \mathcal{R}$, Alg returns a sequence policies $\pi^{(1)}, \dots, \pi^{(N)}$, such that, with probability $1 - p$, the following holds

$$V^{\pi^{(i)}}(\mathbb{P}, r^{(i)}) \geq \max_{\pi} V^{\pi}(\mathbb{P}, r^{(i)}) - \epsilon, \quad \forall i \in [N].$$

For the lower bound, we allow the policies π prescribed by Alg to be arbitrary randomized mappings from observed histories, that is, Alg selects a random seed ξ from some distribution; that is the policy at stage h is a map

$$\pi_h : (s_1, \dots, s_h, a_1, \dots, a_{h-1}, \xi) \rightarrow [A].$$

D.2 Learning A Single Instance

In this section, we define a triple $(\mathcal{E}, \mathcal{R}, \mathcal{P})$ on $\mathcal{O}(n)$ -states which forces the learner to spend $\Omega(nA/\epsilon^2)$ trajectories to learn the transition probabilities at a given state.

As described in Figure 3, the hard instances consist of reward-free MDPs that begin in a fixed initial state, and transition to one of $2n$ terminal states according to an unknown transition distribution. The transitions are all taken to be $\epsilon/2n$ -close to uniform in the ℓ_∞ norm, which helps with the embedding later on. For simplicity, the rewards are taken to depend only on states but not on actions. We formalize these instances in the following definition:

Definition D.1 (Hard Transitions and Rewards at Single State). For parameters $n, A \geq 1$ and A , we define the problem class $\mathcal{M}_{\text{single}}(\epsilon; n, A) : (\mathcal{E}_{\text{single}}(n), \mathcal{P}_{\text{single}}(\epsilon; n, A), \mathcal{R}_{\text{single}}(n, A))$ as the triple with the following constituents:

1. The environment $\mathcal{E}_{\text{single}}(n)$ is

$$\mathcal{E}_{\text{single}}(n, A) = (\mathcal{X}_{\text{single}}(n), A, 2), \quad \text{where } \mathcal{X}_{\text{single}}(n) := \{0, 1, \dots, 2n\}$$

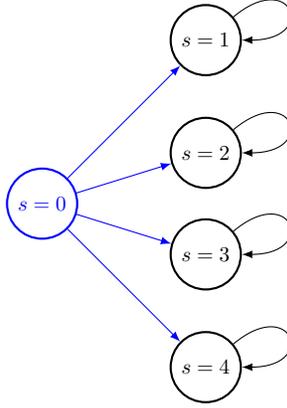


Figure 3: The agent begins in stage $s = 0$, and moves to states $s \in [2n]$, $n = 2$. Different actions correspond to different probability distributions over next states $s \in [2n]$. States $s \in [2n]$ are absorbing, and rewards are action-independent. Lemma 4.2 shows that this construction requires the learner to learn $\Omega(n)$ bits about the transition probabilities $p(\cdot | 0, a)$.

2. For a given $\epsilon \in (0, 1)$, we define the transition class $\mathcal{P}_{\text{single}}(\epsilon; n, A)$ as the set of transition operator on $\mathcal{E}_{\text{single}}(n, A)$, parameterized by vectors q , which begin at state $x_1 = 0$, and always transition to a state $x_2 \in \{1, \dots, 2n\}$ with near-uniform probability, and remain at that state for the remainder of the episode. Formally,

$$\mathcal{P}_{\text{single}}(\epsilon; n, A) := \left\{ \mathbb{P}[x_1 = 0] = 1, \left| \mathbb{P}[x' = s | x = 0, a] - \frac{1}{2n} \right| \leq \frac{1}{2n}\epsilon \right. \\ \left. \mathbb{P}[x' = s | x = s, a] = 1 \forall a \in [A], s \in [2n], \right\}.$$

3. We define the hard reward class $\mathcal{R}_{\text{single}}(n, A)$ as the set of rewards which assign 0 reward to state 0, and an action-independent reward to each state $s \in [2n]$. Formally, we define $\mathcal{R}_{\text{single}}(n, A) := \{r_\nu : r_\nu(0, \cdot) = 0, r_\nu(x, \cdot) = \nu[x], \nu \in [0, 1]^{2n}\}$.

Lemma D.2 (Formal Statement of Lemma 4.2). *Fix $\epsilon \leq 1$, $p \leq 1/2$, $A \geq 2$, and suppose that $n \geq c_0 \log_2 A$ for universal constants c_0 . Then, there exists a distribution \mathcal{D} over transition vectors $\mathbb{P} \in \mathcal{P}_{\text{single}}(\epsilon; n, A)$ such that any algorithm which $(\epsilon/12, p)$ -learns the class $\mathcal{M}_{\text{single}}(\epsilon; n, A)$ satisfies*

$$\mathbf{E}_{\mathbb{P} \sim \mathcal{D}} \mathbf{E}_{\mathbb{P}, \text{Alg}}[K] \gtrsim \frac{nA}{\epsilon^2}.$$

Due to its level of technical, the proof of Lemma D.2 is given in Section D.4.

D.3 Learning Transitions at n states: Proof of Theorem 4.1

Let $n \geq 2$ be a power of two, which we ultimately will choose to be $\Omega(S)$. This means that $\ell_0 := \log_2 n \in \mathbb{N}$ is integral, and define the layered state space:

$$\mathcal{X} := \left\{ (x, \ell) : x \in [2^\ell], \ell \in \{0, 1, \dots, \ell_0 + 1\} \right\}$$

The cardinality of the state space is bounded as $|\mathcal{X}| \leq 1 + 2 + \dots + n/2 + n + 2n \leq 4n$. Hence, we shall chose n to be the largest power of two such that $4n \leq S$. Note then that $n = \Omega(S)$ as long as $S \geq C$ for a universal constant C . We will establish our lower bound for the environment $\mathcal{E}_{\text{embed}} = (\mathcal{X}, A, H)$, that is, with state space \mathcal{X} ; the lower bound extends to an MDP with desired state space of size S by augmenting the MDP with isolated, univstable states.

Description of Transition Class Let us define the class $\mathcal{P}_{\text{embed}}$. First, we require that the states (x, ℓ) for $\ell \in [\ell_0]$ form a dyadic tree, whose transitions are all known to the learner. That is, for $\mathbb{P} \in \mathcal{P}_{\text{embed}}$,

$$\begin{aligned} \mathbb{P}[s_1 = (0, 1)] &= 1 \\ \mathbb{P}[s' = (x, \ell + 1) \mid s = (x, \ell), a = 1] &= 1, \quad \ell \in \{0, 1, \dots, \ell_0 - 1\} \\ \mathbb{P}[x' = (2^\ell + x, \ell - 1) \mid s = (x, \ell), a] &= 1, \quad \ell \in \{0, 1, \dots, \ell_0 - 1\}, a > 1. \end{aligned}$$

In words, \mathbb{P} starts at $(1, 1)$, moves leftward with action $a = 1$, and rightward with actions $a > 1$. At each state $s = (x, \ell_0)$, the learner faces transitions described by some $\mathbb{P}_{\text{single}}^{(x)} \in \mathcal{P}_{\text{single}}(\epsilon_0)$ for $\epsilon_0 = 1/8H$: specifically, we stipulate that states (x, ℓ_0) always transition to states $(x', \ell_0 + 1)$, which are absorbing:

$$\begin{aligned} \forall \mathbb{P} \in \mathcal{P}_{\text{embed}}, x \in [n], \text{ there exists a } \mathbb{P}_{\text{single}}^{(x)} \in \mathcal{P}_{\text{single}}(\epsilon_0) \text{ such that :} \\ \mathbb{P}[s' = (x', \ell_0 + 1) \mid s = (x, \ell_0), a] &= \mathbb{P}_{\text{single}}^{(x)}[s' = x' \mid s = 0, a], \quad \forall a \in [A], x' \in [2n]. \\ \mathbb{P}[s' = (x', \ell_0 + 1) \mid s = (x', \ell_0 + 1), a] &= 1, \quad \forall a \in [A] \end{aligned}$$

Thus, there is a bijection between instances $\mathbb{P} \in \mathcal{P}_{\text{embed}}$ and tuples $(\mathbb{P}_{\text{single}}^{(1)}, \dots, \mathbb{P}_{\text{single}}^{(n)}) \in \mathcal{P}_{\text{single}}^n$.

Description of Reward Class Define the reward class $\mathcal{R}_{\text{embed}} = \{r_{x,\nu}\}$ considering for action-independent rewards

$$r_{x,\nu}(s, a) = \begin{cases} 0 & s = (x', \ell), \ell < \ell_0, \\ 0 & s = (x', \ell_0) \text{ and } x' \neq x \\ 1 & s = (x, \ell_0) \\ r_\nu[x'] & s = (x', \ell_0 + 1). \end{cases}$$

In other words, the learner receives reward 1 at state (x, ℓ_0) , rewards r_ν at terminal states $(x', \ell_0 + 1)$, and 0 elsewhere. We now establish that any policy which is ϵ -optimal under reward $r_{x,\nu}$ must visit (y, ℓ_{\max}) with sufficiently high probability:

Lemma D.3. *Suppose that a (possibly randomized, non-Markovian) policy π satisfies, for $\epsilon \leq 1/4$ and $\epsilon_0 \leq 1/8H$,*

$$V^\pi(\mathbb{P}, r_{x,\nu}) \geq \max_{\pi'} V^{\pi'}(\mathbb{P}, r_{x,\nu}) - \epsilon, \quad \forall i \in [N].$$

Then, $\mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_{\max})] \geq \frac{1}{2}$.

Proof. Due to the structure of the transitions and rewards, the value of any policy π is

$$V^\pi(\mathbb{P}, r_{x,\nu}) = \mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_0)] + (H - \ell_0 - 1) \sum_{x'=1}^{2n} \nu(x') \mathbb{P}^\pi[s_{\ell_0+2} = (x, \ell_0)]$$

Since the transitions from (x', ℓ_0) to $(x'', \ell_0 + 1)$ is $\epsilon_0/2n$ -away from uniform in ℓ_∞ , we can also see that $\mathbb{P}^\pi[s_{\ell_0+2} = (x, \ell_0)] \in (\frac{1}{2n} - \epsilon, \frac{1}{2n} + \epsilon)$. Thus, letting $\bar{\nu} := \frac{1}{2n} \sum_{x'=1}^{2n} \nu[x']$, we have

$$\left| (H - \ell_0 - 1) \sum_{x'=1}^{2n} \nu(x') \mathbb{P}^\pi[s_{\ell_0+2} = (x, \ell_0)] - (H - \ell_0 - 1) \bar{\nu} \right| \leq (H - \ell_0 - 1) \epsilon_0 \leq \frac{1}{8}.$$

This entails that

$$|V^\pi(\mathbb{P}, r_{x,\nu}) - (H - \ell_0 - 1) \bar{\nu} - \mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_0)]| \leq \frac{1}{8}.$$

Consequently, by considering a policy π' which always visits state $s_{\ell_0+1} = (x, \ell_0)$ (this can be achieved due to the deterministic behavior of the actions),

$$\max_{\pi'} V^{\pi'}(\mathbb{P}, r_{x,\nu}) - V^\pi(\mathbb{P}, r_{x,\nu}) \geq 1 - \mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_0)] - 2 \cdot \frac{1}{8} = \frac{3}{4} - \mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_0)].$$

In order for the above to be at most $1/4$, we must have that $\mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_0)] \geq 1/2$. \square

Concluding the Proof of Theorem 4.1 To prove Theorem 4.1, we use the following lemma:

Lemma D.4 (Embedding Correspondence). *Suppose that $H \geq (2\ell_0 + 2)$. Then there exists a correspondence Ψ , which does not depend on $\mathbb{P} \in \mathcal{P}_{\text{embd}}$ or $r_{y,\nu} \in \mathcal{R}_{\text{embed}}$ (but possibly on ϵ, n, A, H) which operates as follows: Given a policy π for $\mathcal{E}_{\text{embed}}$, $\Psi[\pi] = (\pi^{(1)}, \dots, \pi^{(n)})$ returns an n -tuple of policies for $\mathcal{E}_{\text{single}}(n, A)$ with the following property: For any $\mathbb{P} \equiv (\mathbb{P}_{\text{single}}^{(1)}, \dots, \mathbb{P}_{\text{single}}^{(n)}) \in \mathcal{P}_{\text{embd}}$ and $r_{x,\nu} \in \mathcal{R}_{\text{embed}}$,*

$$\text{If } V^\pi(\mathbb{P}, r_{x,\nu}) \geq \max_{\pi'} V^{\pi'}(\mathbb{P}, r_{x,\nu}) - \epsilon, \quad \forall x \in [n], \quad V^{\pi^{(x)}}(\mathbb{P}_{\text{single}}^{(x)}, r_\nu) \geq \max_{\pi'} V^{\pi'}(\mathbb{P}_{\text{single}}^{(x)}, r_\nu).$$

Proof of Lemma D.4. We directly construct the map Ψ . Observe that policies $\pi^{(x)}$ on the single state environment can be discred by a distribution over which actions $a \in [A]$ they select at the initial state x . Thus identifying policies as elements of $\Delta(A)$, we set

$$\pi^{(x)}[a] := \begin{cases} \mathbb{P}^\pi[a_{\ell_0+1} = a \mid s_{\ell_0+1} = (x, \ell_0)] & \mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_0)] > 0 \\ \text{arbitrary} & \text{otherwise} \end{cases}$$

as the marginal distribution of actions selected when $s_{\ell_0+1} = (x, \ell_0 + 1)$. Observe that the above conditional probabilities *do not* depend on $\mathbb{P} \in \mathcal{P}_{\text{embd}}$ since the dynamics up to $h = \ell_0 + 1$ are identical for all instances. By considering a policy which coincides with π until $s_{\ell_0+1} = (x, \ell_0)$ and switches to playing optimally, we can lower bound the suboptimality of π by

$$\begin{aligned} \max_{\pi'} V^{\pi'}(\mathbb{P}, r_{x,\nu}) - V^\pi(\mathbb{P}, r_{x,\nu}) &\geq \\ &\mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_0)] \cdot (H - \ell_0 - 1) \left(\max_{\pi'} V^\pi(\mathbb{P}_{\text{single}}^{(x)}, r_\nu) - V^{\pi^{(x)}}(\mathbb{P}_{\text{single}}^{(x)}, r_\nu) \right) \end{aligned}$$

In particular, if π is $\epsilon \leq 1/4$ -suboptimal, then Lemma D.3 ensures $\mathbb{P}^\pi[s_{\ell_0+1} = (x, \ell_0)] \geq 1/2$. Since $H \geq 2(\ell_0 + 1)$ by assumption, we have

$$\epsilon \geq \max_{\pi'} V^{\mathcal{M}, \pi'} - V^{\mathcal{M}, \pi} \geq \frac{H}{4} \left(\max_{\pi'} V^\pi(\mathbb{P}_{\text{single}}^{(x)}, r_\nu) - V^{\pi^{(x)}}(\mathbb{P}_{\text{single}}^{(x)}, r_\nu) \right),$$

Therefore, $\max_{\pi'} V^\pi(\mathbb{P}_{\text{single}}^{(x)}, r_\nu) - V^{\pi^{(x)}}(\mathbb{P}_{\text{single}}^{(x)}, r_\nu) \leq \frac{4\epsilon}{H}$, as needed. \square

We now conclude with the proof of our main theorem:

Proof of Theorem 4.1. Let Alg be (ϵ, p) -correct on the class $(\mathcal{E}_{\text{embed}}, \mathcal{P}_{\text{embed}}, \mathcal{R}_{\text{embed}})$. Then, for any $x \in [2n]$, we simulate obtain a $(4\epsilon/H, p)$ -correct algorithm for $\mathcal{M}_{\text{single}}(4\epsilon/H; n, A)$ as follows:

1. **Exploration:** Let \mathcal{D} be the distribution over $\mathbb{P}_{\text{single}} \in \mathcal{P}_{\text{single}}$ from Lemma D.2. Draw a tuple $\mathbb{P}^{\neq x} = (\mathbb{P}_{\text{single}}^{(x')})_{x' \neq x}$ of $n - 1$ distributions i.i.d from \mathcal{D} , and let $\text{Alg}_{\text{single}}^{(x, \mathbb{P}^{\neq x})}$ denote the algorithm induced by embedding the instance in $\mathcal{M}_{\text{single}}(4\epsilon/H; n, A)$ at stage x of the embedding construction, running Alg on this embedded instance
2. **Planning:** When queried given a reward vector $r_\nu \in \mathcal{R}_{\text{single}}$, use Alg to compute a policy π for reward vector $r_{x, \nu} \in \mathcal{R}_{\text{embed}}$, and return the policy $\pi^{(x)}$ dictated by the corresponding ψ .

Since Alg is (ϵ, p) -correct and $\epsilon \leq 1/4$, the correspondence Ψ ensures that for any draw of $\mathbb{P}^{\neq x}$, $\text{Alg}_{\text{single}}^{(x, \mathbb{P}^{\neq x})}$ is $(4\epsilon/H, p)$ -correct. Let $K^{(x, \mathbb{P}^{\neq x})}$ denote the random number of episodes collected by $\text{Alg}_{\text{single}}^{(x, \mathbb{P}^{\neq x})}$ in the exploration phase, Thus, if $\epsilon \leq \min\{\frac{1}{4}, \frac{H}{48}\}$, and $n \geq c_0 \log_2 A$ for the appropriate c_0 specified in Lemma D.2, the Lemma D.2 entails

$$\mathbf{E}_{\mathbb{P}_{\text{single}} \sim \mathcal{D}} \mathbf{E}_{\mathbb{P}_{\text{single}}, \text{Alg}_{\text{single}}^{(x, \mathbb{P}^{\neq x})}} [K^{(x, \mathbb{P}^{\neq x})}] \gtrsim \frac{nAH^2}{\epsilon^2}.$$

By taking an expectation over $\mathbb{P}^{\neq x}$, we have

$$\mathbf{E}_{\mathbb{P}^{\neq x} \sim \mathcal{D}^{n-1}, \mathbb{P}_{\text{single}} \sim \mathcal{D}} \mathbf{E}_{\mathbb{P}_{\text{single}}, \text{Alg}_{\text{single}}^{(x, \mathbb{P}^{\neq x})}} [K^{(x, \mathbb{P}^{\neq x})}] \gtrsim \frac{nAH^2}{\epsilon^2}.$$

Note then that, if $N_K(x)$ denotes the number of times that the original Alg visits state (x, ℓ_0) , then, by Fubini's theorem and the construction of $\text{Alg}_{\text{single}}^{(x, \mathbb{P}^{\neq x})}$, the expectation of $N_K(x)$ under probabilities drawn uniform from \mathcal{D}^n is equal to the expectation of $K^{(x, \mathbb{P}^{\neq x})}$ where $\mathbb{P}^{\neq x}$ is drawn uniformly from \mathcal{D}^{n-1} , and then the transition $\mathbb{P}_{\text{single}}$ is selected. Formally,

$$\mathbf{E}_{\mathbb{P}^{\neq x} \sim \mathcal{D}^{n-1}, \mathbb{P}_{\text{single}} \sim \mathcal{D}} \mathbf{E}_{\mathbb{P}_{\text{single}}, \text{Alg}_{\text{single}}^{(x, \mathbb{P}^{\neq x})}} [K^{(x, \mathbb{P}^{\neq x})}] = \mathbf{E}_{\mathbb{P} \equiv (\mathbb{P}_{\text{single}}^{(1)}, \dots, \mathbb{P}_{\text{single}}^{(n)}) \sim \mathcal{D}^n} \mathbf{E}_{\mathbb{P}, \text{Alg}} [K_x]$$

This implies that

$$\mathbf{E}_{\mathbb{P} \equiv (\mathbb{P}_{\text{single}}^{(1)}, \dots, \mathbb{P}_{\text{single}}^{(n)}) \sim \mathcal{D}^n} \mathbf{E}_{\mathbb{P}, \text{Alg}} [K_x] \gtrsim \frac{nAH^2}{\epsilon^2}.$$

Since the number of episodes K encountered by Alg is equal to $\sum_{x=1}^n K_x$ (the agent visits exactly one state of the form (x, ℓ_0) per episode), we have

$$\mathbf{E}_{\mathbb{P} \equiv (\mathbb{P}_{\text{single}}^{(1)}, \dots, \mathbb{P}_{\text{single}}^{(n)}) \sim \mathcal{D}^n} \mathbf{E}_{\mathbb{P}, \text{Alg}} [K] \gtrsim \sum_{x=1}^n \frac{nAH^2}{\epsilon^2} = \frac{n^2 AH^2}{\epsilon^2}.$$

Since $S/8 \leq n \leq S$, for the above conditions to hold, it suffices that, for a sufficiently large constant C , $S \geq C \log_2 A$, $\epsilon \leq \min\{\frac{1}{4}, \frac{H}{48}\}$, and $H \geq C \log_2 S$. Moreover, $\frac{n^2 AH^2}{\epsilon^2} = \Omega(\frac{S^2 AH^2}{\epsilon^2})$, as needed. \square

D.4 Proof of Lemma D.2

A packing of reward-free MDPs The first step is to construct a family of transition probabilities $\mathbb{P}_J \in \mathcal{P}(\epsilon; n, A)$ which witness the lower bound. Let $\mathbf{1}$ denote the all ones vector on $[2n]$. To construct the packing, we define the set of binary vectors

$$\mathcal{K} := \left\{ v \in \{-1, 1\}^{2n} : \mathbf{1}^\top v = 0 \right\}.$$

For a cardinality parameter M to be chosen shortly, we consider a packing of vectors

$$\mathcal{V}_{A,M} := \{v_{a,j} \in \mathcal{K} : a \in [A], j \in [M]\}$$

Throughout, we shall consider packings $\mathcal{V}_{A,M}$ which are *uncorrelated* in the following sense:

Definition D.5 (Uncorrelated). For $\gamma \in (0, 1)$, we say that $\mathcal{V}_{A,M}$ is γ -uncorrelated if, for any pair $(a, j), (a', j')$ with *either* $a \neq a'$ or $j \neq j'$, it holds that $|\langle v_{a,j}, v_{a',j'} \rangle| < 2n\gamma$.

The following lemma shows that there exist γ -uncorrelated packings of size $e^{\Omega(n\gamma^2)}$:

Lemma D.6. Fix $\gamma \in (0, 1)$, and suppose that $2 \log(M) \leq n\gamma^2 - \log(4n) - 2 \log(A)$. Then, there exists a γ -uncorrelated packing $\mathcal{V}_{A,M}$.

Proof Sketch. We use the probabilistic method. Specifically, we draw $v_{a,j} \stackrel{\text{unif}}{\sim} \mathcal{K}$, and can bound $\langle v_{a,j}, v_{a',j'} \rangle$ with high-probability Chernoff bounds. Taking a union bound shows that uncorrelated packings arise from this construction with non-zero probability. A full proof is given in Section D.4.1. \square

Given a γ -uncorrelated packing $\mathcal{V}_{A,M}$, define transition vectors

$$q_{a,j} := q_0 + \frac{\epsilon}{2n} v_{a,j}, \text{ where } q_0 = \frac{1}{2n} \mathbf{1}.$$

Since $\epsilon \leq 1$ and $\mathbf{1}^\top v_{a,j} = 0$, $q_{j,a} \in \Delta(2n)$. Wet indices J denote tuples $J = (J_1, \dots, J_A) \in [M]^A$, let $q_J(\cdot, a) = q_{a, J_a}$, and define \mathbb{P}_J as the instance \mathbb{P}_{q_J} , where \mathbb{P}_q is as in Definition []. Formally,

$$\mathbb{P}_J : \quad \mathbb{P}^{\mathbb{P}_J}[s_1 = 0] = 1, \mathbb{P}^{\mathbb{P}_J}[s_2 = 0] = 0, \forall s \in [2n], \mathbb{P}^{\mathbb{P}_J}[s_2 = s \mid s_1 = 0, a] = q_J(s, a) = q_{a, J_a}(s)$$

Lower Bound for Estimating the Packing Instance: Let us suppose we have an exploration algorithm Alg_{est} which, for any \mathbb{P}_J , collects (a possibly random number) K trajectories, and returns estimates $\widehat{J}_1, \dots, \widehat{J}_A$ of J_1, \dots, J_A . Our first step is to establish a lower bound on K assuming that Alg_{est} satisfies a uniform correctness guarantee:

Lemma D.7. For any Alg_{est} satisfying the guarantee

$$\forall J \in [A]^M, \mathbb{P}_{\mathbb{P}_J, \text{Alg}_{\text{est}}} \left[\widehat{J}_a = J_a \forall a \in [A] \right] \geq 1 - p. \quad (9)$$

Then, we must have

$$\mathbf{E}_{J \stackrel{\text{unif}}{\sim} [A]^M} \mathbf{E}_{\mathbb{P}_J, \text{Alg}_{\text{est}}} [K] \geq A \cdot \frac{(1-p) \log M - \log 2}{\epsilon^2}$$

The above bound essentially follows from an application of Fano's inequality, and is proven in Section D.4.2. In particular, if we take say $p = 1/2$, and require $M = e^{\Omega(S)}$, then we have $\mathbf{E}_{J \stackrel{\text{unif}}{\sim} [A]^M} \mathbf{E}_{\mathbb{P}_J, \text{Alg}_{\text{est}}} [K] \gtrsim \frac{SA}{\epsilon^2}$, as desired.

Estimation Reduces to Exploration Of course, the above bound applies only to an estimation algorithm Alg_{est} , but our intent is to establish lower bounds for exploration algorithms. In the following lemma, we state that if the packing is sufficiently uncorrelated, then we can convert an $(\epsilon/24, p)$ -correct exploration algorithm into an Algorithm Alg_{est} satisfying Eq. (9).

Lemma D.8. *Suppose Alg is $(\epsilon/24, p)$ -correct on the class $\mathcal{M}_{\text{single}}(\epsilon, n, A)$, and that the packing $\mathcal{V}_{M,A}$ is $\gamma = 1/10$ -uncorrelated. Then, there is an algorithm Alg_{est} which collects K trajectories according to Alg, and satisfies Eq. 9.*

Proof Sketch. Consider reward vectors r_ν induced by $\nu_{a,j,a_2,j_2} \propto 2q_{a,j} - q_{a_2,j_2}$. These reward vectors can be used to “pick out” q_{a,J_a} as follows. For a given a , we show that on the good exploration event, Alg returns policies with $\mathbb{P}[\hat{\pi}'_1(0) = a] > 1/2$ for all $\nu = \nu_{a,J_a,a_2,j_2}$ ranging across a_2, j_2 . However, for $j \neq J_a$, we show that on this good event there exists some a_2, j_2 for which Alg returns policies with $\mathbb{P}[\hat{\pi}'_1(0) = a] < 1/2$. Hence, we can estimate q_{a,J_a} by finding the (say, the first) index j for which $\mathbb{P}[\hat{\pi}'_1(0) = a] > 1/2$ for all $\nu = \nu_{a,j,a_2,j_2}$, ranging across a_2, j_2 . A full proof is given in Section D.4.3. \square

As a consequence, we find that if $\gamma \leq 1/10$ and Alg is $(\epsilon/24, p)$ -correct,

$$\mathbf{E}_{J \stackrel{\text{unif}}{\sim} [A]^M} \mathbf{E}_{\mathbb{P}_{J,\text{Alg}}} [K] \geq A \cdot \frac{(1-p) \log M - \log 2}{\epsilon^2}$$

In particular, if $\log M \geq 4 \log 2$ and $p \leq 1/2$, then,

$$\mathbf{E}_{J \stackrel{\text{unif}}{\sim} [A]^M} \mathbf{E}_{\mathbb{P}_{J,\text{Alg}}} [K] \geq A \cdot \frac{\log M}{4\epsilon^2} \quad (10)$$

Concluding the proof Take $\gamma = 1/10$. For constants c_0, c_1 sufficiently large, we can ensure that if $n \geq c_0 \log_2 A$, then $M = e^{-n/c_1}$ satisfies $2 \log(M) \leq n\gamma^2 - \log(4n) - 2 \log(A)$ and $\log M \geq 4 \log 2$. Thus, we can construct a γ -uncorrelated packing of cardinality $\log M \geq n/c_1$,

$$\mathbf{E}_{J \stackrel{\text{unif}}{\sim} [A]^M} \mathbf{E}_{\mathbb{P}_{J,\text{Alg}}} [K] \geq A \cdot \frac{n}{4c_1\epsilon^2},$$

as needed. \square

D.4.1 Proof of Lemma D.6

We begin with the following concentration inequality:

Lemma D.9. *For any fixed (a, j) and (a', j') , we have*

$$\mathbb{P}[|\langle v_{a,j}, v_{a',j'} \rangle| \geq 2n\gamma] \leq e^{\log(4n) - n\gamma^2}.$$

Proof. By permuting coordinates, we may assume that

$$v_{a',j'}[s] = \begin{cases} 1 & s \in [n] \\ -1 & s \in \{n+1, \dots, 2n\} \end{cases}.$$

Then,

$$\langle v_{a,j}, v_{a',j'} \rangle = 2|\{s \in [n] : v_{a,j}[s] = 1\}| - 2(n - |\{s \in [n] : v_{a,j}[s] = 1\}|)$$

$$= 2n - 4|\{s \in [n] : v_{a,j}[s] = 1\}| := 2n - 4Z,$$

where we set $Z = |\{s \in [n] : v_{a,j}[s] = 1\}|$. Hence, if $|\langle v_{a,j}, v_{a',j'} \rangle| \geq 2\gamma n$, we need

$$\left| \frac{Z}{n} - \frac{1}{2} \right| \geq \frac{\gamma}{2}.$$

Now, we have that for $i \in [n]$,

$$\mathbb{P}[Z = i] < \frac{\binom{n}{i} \cdot \binom{n}{n-i}}{\sum_{i=0}^n \binom{n}{i} \cdot \binom{n}{n-i}} = \frac{\binom{n}{i}^2}{\sum_{i=0}^n \binom{n}{i}^2} < n \frac{\binom{n}{i}^2}{\left(\sum_{i=0}^n \binom{n}{i}\right)^2} = n \mathbb{P}_{W \sim \text{Binom}(n, 1/2)}[W = i]^2.$$

Hence,

$$\begin{aligned} \mathbb{P}\left[\left|\frac{Z}{n} - \frac{1}{2}\right| \geq \frac{\gamma}{2}\right] &\leq n \sum_{i: \left|\frac{i}{n} - \frac{1}{2}\right| \geq \frac{\gamma}{2}} \mathbb{P}_{W \sim \text{Binom}(n, 1/2)}[W = i]^2 \\ &\leq n \left(\sum_{i: \left|\frac{i}{n} - \frac{1}{2}\right| \geq \frac{\gamma}{2}} \mathbb{P}_{W \sim \text{Binom}(n, 1/2)}[W = i] \right)^2 \\ &= n \left(\mathbb{P}_{W \sim \text{Binom}(n, 1/2)}\left[\left|\frac{W}{n} - \frac{1}{2}\right| \geq \frac{\gamma}{2}\right] \right)^2 \leq n(2e^{-2(\gamma/2)^2 n})^2 = e^{\log(4n) - n\gamma^2} \end{aligned}$$

□

We now finish the proof of our intended lemma:

Proof of Lemma D.6. By a union bound over at most $A^2 M^2 - 1$ pairs $(a, j), (a', j')$, there exists a γ -uncorrelated packing for any M satisfying

$$A^2 M^2 e^{\log(4n) - n\gamma^2} \leq 1$$

Taking logarithms, we require $2 \log(M) \leq n\gamma^2 - \log(4n) - 2 \log(A)$.

□

D.4.2 Proof of Lemma D.7

To begin, let us state a variant of Fano's inequality, which replaces mutual-information with an arbitrary comparison measure:

Lemma D.10 (Fano's Inequality). *Consider M probability measures $\mathbb{P}_1, \dots, \mathbb{P}_M$ on a space Ω . Then for any estimator \hat{j} on Ω and any comparison law \mathbb{P}_0 on Ω ,*

$$\frac{1}{M} \sum_{j=1}^M \mathbb{P}_j[\hat{j} \neq j] \geq 1 - \frac{\log 2 + \frac{1}{M} \sum_{j=1}^M \text{KL}(\mathbb{P}_j, \mathbb{P}_0)}{\log M}$$

Proof. This follows from the standard statement of Fano’s inequality, where we use that

$$\inf_{\mathbb{P}_0} \frac{1}{M} \sum_{j=1}^M \text{KL}(\mathbb{P}_j, \mathbb{P}_0) = \frac{1}{M} \sum_{j=1}^M \text{KL} \left(\mathbb{P}_j, \frac{1}{M} \sum_{j'=1}^M \mathbb{P}_{j'} \right)$$

For reference, see e.g. Equation (11) in Chen et al. [2016]. \square

We will apply Fano’s inequality of each $a \in [A]$. To begin, for a fixed $J \in [M]^A$ and $a \in [A]$, let us define the laws “ \mathbb{P}_j ”. We let $\mathbb{P}_{J,a,j}$ denote the reward-free MDP with starting at $x = 0$ deterministically, and with transitions

$$\mathbb{P}^{\mathbb{P}_{J,a,j}}[s \mid x_1 = 0, a_1 = a'] = \begin{cases} q_{a,j}[s] & a' = a \\ q_{a',J_{a'}}[s] & a' \neq a. \end{cases}$$

For fixed J, a , we let $\mathbb{P}_{j;J,a}$ denote the joint law induced by Alg_{est} and $\mathbb{P}_{J,a,j}$. For the comparison measure, let $\mathbb{P}_{J,a,0}$ denote the analogous MDP to $\mathbb{P}_{J,a,j}$, but where $\mathbb{P}^{\mathbb{P}_{J,a,0}}[s \mid x_1 = 0, a_1 = a] = q_0$ for the fixed action a . We let $\mathbb{P}_{0;J,a}$ denote the law induced by Alg_{est} and $\mathbb{P}_{J,a,j}$. Then, Fano’s inequality implies that

$$\forall J, a, \quad (1-p) \log M - \log 2 \leq \frac{1}{M} \sum_{j=1}^M \text{KL}(\mathbb{P}_{J,a,j}, \mathbb{P}_{0;J,a}). \quad (11)$$

Now, observe that the laws $\mathbb{P}_{J,a,j}$ and $\mathbb{P}_{0;J,a}$ only differ due to transitions selecting action $a_1 = a$. Under the first law, these have distribution $\text{Multinomial}(q_{a,j})$, and under the second, $\text{Multinomial}(q_0)$. Let $N_K(a = a_1)$ denote the expected number of times algorithm Alg_{est} selects action $a_1 = a$ at time step 1. From a Wald’s identity argument (see e.g. Kaufmann et al. [2016]), we have

$$\begin{aligned} \text{KL}(\mathbb{P}_{J,a,j}, \mathbb{P}_{0;J,a}) &= \mathbf{E}_{\mathbb{P}_{J,a,j}, \text{Alg}_{\text{est}}} [N_K(a_1 = a)] \text{KL}(\text{Multinomial}(q_{a,j}), \text{Multinomial}(q_{a,0})) \\ &= \mathbf{E}_{\mathbb{P}_{J,a,j}, \text{Alg}_{\text{est}}} [N_K(a_1 = a)] \sum_{s=1}^{2n} \frac{1 + \epsilon v_{j,a}[s]}{2n} \log(1 + \epsilon v_{j,a}[s]) \\ &\stackrel{(i)}{\leq} \mathbf{E}_{\mathbb{P}_{J,a,j}, \text{Alg}_{\text{est}}} [N_K(a_1 = a)] \sum_{s=1}^{2n} \frac{\epsilon v_{j,a} + \epsilon^2 v_{j,a}[s]^2}{2n} \\ &\stackrel{(ii)}{\leq} \epsilon^2 \cdot \mathbf{E}_{\mathbb{P}_{J,a,j}, \text{Alg}_{\text{est}}} [N_K(a_1 = a)] \end{aligned}$$

where (i) uses $1 + \epsilon v_{j,a}[s] \geq 0$ and the identity $\log(1+x) \leq x$, and (ii) uses the fact that $v_{j,a}[s]^2 = 1$ and $\sum_{s=1}^{2n} v_{j,a}[s] = 0$ for $v_{j,a} \in \mathcal{K}$. Thus, by Eq 11,

$$\forall J, a, \quad \frac{(1-p) \log M - \log 2}{\epsilon^2} \leq \frac{1}{M} \sum_{j=1}^M \mathbf{E}_{\mathbb{P}_{J,a,j}, \text{Alg}_{\text{est}}} [N_K(a_1 = a)].$$

By taking an expectation over index tuples J drawn uniformly from $[A]^M$, we have

$$\forall a, \quad \frac{(1-p) \log M - \log 2}{\epsilon^2} \leq \frac{1}{M} \sum_{j=1}^M \mathbf{E}_{J \sim [A]^M} \mathbf{E}_{\mathbb{P}_{J,a,j}, \text{Alg}_{\text{est}}} [N_K(a_1 = a)]$$

$$= \mathbf{E}_{J \sim \text{unif}_{[A]}^M} \mathbf{E}_{\mathbb{P}_J, \text{Alg}_{\text{est}}} [N_K(a_1 = a)],$$

where the last line follows that $\mathbb{P}_{J,a,j} = \mathbb{P}_{J'}$ for some J' and that, by symmetry, each index J' has equal weight when averaged over both $J \in [A]^M$ and $j \in [M]$. Summing over $a \in [A]$, we have

$$A \cdot \frac{(1-p) \log M - \log 2}{\epsilon^2} \leq \mathbf{E}_{J \sim \text{unif}_{[A]}^M} \mathbf{E}_{\mathbb{P}_J, \text{Alg}_{\text{est}}} \left[\sum_{a=1}^A N_K(a_1 = a) \right] = \mathbf{E}_{J \sim \text{unif}_{[A]}^M} \mathbf{E}_{\mathbb{P}_J, \text{Alg}_{\text{est}}} [K].$$

D.4.3 Proof of Lemma D.8

Let us now show that $(\epsilon/12, p)$ -learning implies the existence of an algorithm Alg_{est} satisfying Eq. 9, provided the packing is sufficiently uncorrelated. Introduce the vectors

$$\nu_{a_1, a_2, j_1, j_2} := \frac{1}{3} v_{a_1, j_1} + \frac{1}{6} v_{a_2, j_2} + \frac{1}{2} \mathbf{1},$$

which can be checked to lie $[0, 1]^{2n}$. We shall establish the following lemma, which says that for sufficiently uncorrelated packings, the vectors $\nu_{(\dots)}$ witness separations between q_{a_1, j_1} and q_{a_2, j_2} for different actions a_1, a_2 :

Lemma D.11. *Fix $a_1 \in [A]$ and $j_1 \in [M]$, and suppose the packing is $\gamma = 1/10$ -uncorrelated: Then, for any $a_2 \neq a_1$ and $j_2 \in [M]$, the following holds*

$$\begin{aligned} \min_{a'_2, j'_2} \langle q_{a_1, j_1} - q_{a_2, j_2}, \nu_{a_1, a'_2, j_1, j'_2} \rangle &> \frac{\epsilon}{12} \\ \forall j'_1 \neq j_1, \min_{a'_2, j'_2} \langle q_{a_1, j_1} - q_{a_2, j_2}, \nu_{a_1, a'_2, j'_1, j'_2} \rangle &< -\frac{\epsilon}{12} \end{aligned}$$

Proof of Lemma D.11.

$$\begin{aligned} \langle q_{a_1, j_1} - q_{a_2, j_2}, \nu_{a'_1, a'_2, j'_1, j'_2} \rangle &= \frac{\epsilon}{2n} \langle v_{a_1, j_1} - v_{a_2, j_2}, \nu_{a'_1, a'_2, j'_1, j'_2} \rangle \\ &= \frac{\epsilon}{12n} \langle v_{a_1, j_1} - v_{a_2, j_2}, 2v_{a'_1, j'_1} - v_{a'_2, j'_2} \rangle, \end{aligned}$$

where we use the fact that $v_{a,j}^\top \mathbf{1} = 1$ for all a, j . If $a'_1 = a_1$ and $j'_1 = j_1$, and the packing is $\gamma \leq 1/6$ -uncorrelated

$$\begin{aligned} \langle q_{a_1, j_1} - q_{a_2, j_2}, \nu_{a_1, a'_2, j_1, j'_2} \rangle &= \frac{\epsilon}{12n} \langle v_{a_1, j_1} - v_{a_2, j_2}, 2v_{a_1, j_1} - v_{a'_2, j'_2} \rangle \\ &= \frac{\epsilon}{12n} \left(2\langle v_{a_1, j_1}, v_{a_1, j_1} \rangle - 2\langle v_{a_2, j_2}, v_{a_1, j_1} \rangle + \langle v_{a_1, j_1}, v_{a'_2, j'_2} \rangle - \langle v_{a_2, j_2}, v_{a'_2, j'_2} \rangle \right) \\ &> \frac{\epsilon}{12n} (4n - 4\gamma n - 2n - 2n\gamma) \\ &\geq \frac{\epsilon}{12n} (2n - 6n\gamma) = \frac{\epsilon}{12}. \end{aligned}$$

On the other hand, if $j_1 \neq j'_1$, but $(a_2, j_2) = (a'_2, j'_2)$ then a similar computation reveals that for $\gamma \leq 1/10$,

$$\langle q_{a_1, j_1} - q_{a_2, j_2}, \nu_{a_1, a_2, j'_1, j_2} \rangle < \frac{\epsilon}{12n} (10\gamma n - 2n) < \frac{-\epsilon}{12}.$$

□

We can now conclude the proof of our reduction:

Proof of Lemma D.8. Suppose that Alg is run on \mathbb{P}_J for $J \in [M]^A$. Further, recall the rewards r_ν which assign reward of $r_\nu(s, a) = \mathbf{I}(s \in [2n])\nu(s)$. By $(\epsilon/24, p)$ -correctness of Alg, then with probability $1 - p$, Alg computes policies $\hat{\pi}_\nu$ which satisfies the following bound simultaneously for all $\nu \in \{\nu_{a_1, a_2, j_1, j_2}\}$:

$$\max_{\pi} V^\pi(\mathbb{P}_J, r_\nu) - V^{\hat{\pi}_\nu}(\mathbb{P}_J, r_\nu) \leq \epsilon/24. \quad (12)$$

For a possibly randomized policy, we use the shorthand $\pi[a]$ to denote the probability of selecting a at the initial state 0; that is $\mathbb{P}^\pi[a_1 = a]$. Now, Consider the following procedure: for each $a \in [A]$, estimate J_a by returning the first $j \in [M]$ for which

$$\forall a'_2, j'_2, \quad \hat{\pi}_{\nu_{a, a'_2, j, j'_2}}[a] > 1/2. \quad (13)$$

We conclude our proof by showing that, on the good event Eq. (12), the condition in Eq. (13) holds if and only if $j = J_a$. To this end, define the short hand

$$q_\pi := \sum_{a'} \pi[a'] q_{a', J_{a'}}$$

Then, we have that

$$\max_{\pi} V^\pi(\mathbb{P}_J, r_\nu) - V^{\hat{\pi}_\nu}(\mathbb{P}_J, r_\nu) = \max_{\pi} \langle q_\pi - q_{\hat{\pi}_\nu}, \nu \rangle,$$

so that on the good event of Eq. 12, we have

$$\max_{\pi} \langle q_\pi - q_{\hat{\pi}_\nu}, \nu \rangle \leq \frac{\epsilon}{24}.$$

True Positive for $j = J_a$: First let's show that Equation 13 holds for $j = J_a$. Indeed, if it does not, then there exists some a'_2, j'_2 for which $\mathbb{P}[\hat{\pi}_{\nu_{a, j, a'_2, j'_2}}[a] \leq 1/2]$, and (setting $\nu = \nu_{a, j, a'_2, j'_2}$ for shorthand in $\hat{\pi}^\nu$)

$$\begin{aligned} \epsilon/24 &\geq \max_{\pi} \langle q_\pi - q_{\hat{\pi}_\nu}, \nu \rangle, \\ &\geq \langle q_{a, J_a} - q_{\hat{\pi}_\nu}, \nu_{a, j, a'_2, j'_2} \rangle && \text{(choose } \pi[a] = 1) \\ &= \sum_{a' \neq a} \hat{\pi}_\nu[a'] \langle q_{a, J_a} - q_{a', J_{a'}} \rangle \\ &\geq \underbrace{(1 - \hat{\pi}_\nu[a])}_{\geq 1/2} \cdot \underbrace{\min_{a' \neq a} \langle q_{a, J_a} - q_{a', J_{a'}} \rangle}_{> \epsilon/12 \text{ by Lemma D.11}} > \frac{\epsilon}{24}, \end{aligned}$$

yielding a contradiction.

True Negative for $j \neq J_a$: On the other hand, for $j \neq J_a$ suppose that for all all $a'_2 \neq a$ and all $j'_2 \in [M]$, $\mathbb{P}[\hat{\pi}_1^{\nu_{a, j, a'_2, j'_2}}(0) = a] > 1/2$. Then, considering $a'_2 = a_2$ and $j'_2 = J_{a_2}$, we have (setting $\nu = \nu_{a, j, a_2, J_{a_2}}$ for shorthand in $\hat{\pi}^\nu$)

$$\epsilon/24 \geq \max_{a'} \langle q_{a', J_{a'}} - q_{\hat{\pi}_\nu}, \nu_{a, j, a_2, J_{a_2}} \rangle$$

$$\begin{aligned}
&\geq \langle q_{a_2, J_{a_2}} - q_{\widehat{\pi}^\nu}, \nu_{a, j, a_2, J_2} \rangle \\
&\geq \underbrace{\widehat{\pi}^\nu[a_2]}_{\geq \widehat{\pi}^\nu[a] > 1/2} \cdot \underbrace{\min_{a' \neq a_2} \langle q_{a_2, J_{a_2}} - q_{a', J_{a'}}, \nu_{a, j, a'_2, J'_2} \rangle}_{> \epsilon/12 \text{ by Lemma D.11}} > \frac{\epsilon}{24},
\end{aligned}$$

again drawing a contradiction. □