Capturing Complementarity in Set Functions by Going Beyond Submodularity/Subadditivity

Wei Chen[1]
Microsoft Research, Beijing, China
weic@microsoft.com

Shang-Hua Teng[2]
USC, Los Angeles, CA, USA
shanghua@usc.edu

Hanrui Zhang[3]
Duke University, Durham, NC, USA
hrzhang@cs.duke.edu

Abstract

We introduce two new "degree of complementarity" measures: *supermodular width* and *superadditive width*. Both are formulated based on natural witnesses of complementarity. We show that both measures are robust by proving that they, respectively, characterize the gap of monotone set functions from being submodular and subadditive. Thus, they define two new hierarchies over monotone set functions, which we will refer to as Supermodular Width (SMW) hierarchy and Superadditive Width (SAW) hierarchy, with foundations — i.e. level 0 of the hierarchies — resting exactly on submodular and subadditive functions, respectively.

We present a comprehensive comparative analysis of the SMW hierarchy and the Supermodular Degree (SD) hierarchy, defined by Feige and Izsak. We prove that the SMW hierarchy is strictly more expressive than the SD hierarchy: Every monotone set function of supermodular degree d has supermodular width at most d, and there exists a supermodular-width-1 function over a ground set of m elements whose supermodular degree is $m - 1$. We show that previous results regarding approximation guarantees for welfare and constrained maximization as well as regarding the Price of Anarchy (PoA) of simple auctions can be extended without any loss from the supermodular degree to the supermodular width. We also establish almost matching information-theoretical lower bounds for these two well-studied fundamental maximization problems over set functions. The combination of these approximation and hardness results illustrate that the SMW hierarchy provides not only a natural notion of complementarity, but also an accurate characterization of "near submodularity" needed for maximization approximation. While SD and SMW hierarchies support nontrivial bounds on the PoA of simple auctions, we show that our SAW hierarchy seems to capture more intrinsic properties needed to realize the efficiency of simple auctions. So far, the SAW hierarchy provides the best dependency for the PoA of Single-bid Auction, and is nearly as competitive as the Maximum over Positive Hypergraphs (MPH) hierarchy for Simultaneous Item First Price Auction (SIA). We also provide almost tight lower bounds for the PoA of both auctions with respect to the SAW hierarchy.

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1 Introduction

For a ground set $X = [m] = \{1, 2, \ldots, m\}$, a *set function* $f : 2^X \to \mathbb{R}$ assigns each subset $S \subseteq X$ a real value.[4] Function f is *monotone* if $f(T) \geq f(S), \forall S \subseteq T \subseteq X$, and *normalized* if $f(\emptyset) = 0$. In this paper, we will focus on normalized monotone set functions, which by definition are non-negative.

Like graphs to network analysis, set functions provide the mathematical language for many applications, ranging from combinatorial auctions (economics) to coalition formation (cooperative game theory; political science) [25, 26] to influence maximization (viral marketing) [24, 17]. Because of its exponential dimensionality, set functions — which are as rich as weighted hypergraphs — are far more expressive mathematically and challenging algorithmically than graphs [28]. However, when monotone set functions are *submodular* [22, 29], or — more generally — *complement-free* [8], algorithms with remarkable performance guarantees have been developed for various optimization, social influence, economic, and learning tasks [2, 17, 20, 3, 23].

In this paper, we study two new *degree-of-complementarity* measures of monotone set functions, and demonstrate their usefulness for several optimization and economic tasks. We prove that our complementarity measures — which are based on natural *witnesses* of complementarity — introduce hierarchies (over monotone set functions) that smoothly move beyond submodularity and subadditivity.

1.1 Witnesses to Complementarity: Supermodular Sets and Superadditive Sets

For any sets $S, T \subseteq X$, let $f(S|T) := f(S \cup T) - f(T)$ be the *margin* of S given T. Recall that f is *subadditive* if $f(S \cup T) \leq f(S) + f(T)$, $\forall S, T \subseteq X$, and *submodular* if for all S, T and $v \in X \setminus T$, $f(v|S \cup T) \leq f(v|S)$. It is well known that every submodular set function is also subadditive. If there are sets $S, T \subseteq V$ such that $f(S \cup T) > f(S) + f(T)$, then one may say that (S, T) is a witness to *complementarity* in f. Motivated by a line of recent work [1, 14, 10, 9, 11, 6], we consider the following fundamental question about set functions:

> *Are there other natural, and preferably more general, forms of witnesses to complementarity that have algorithmic consequences?*

The supermodular degree of Feige and Izsak [10] is among the first measures of complementarity that are connected with algorithmic solutions to monotone-set-function maximization and combinatorial auctions. The supermodular degree is defined based on a notion of positive dependency between elements: $u \in X$ positively depends on $v \in X \setminus \{u\}$ (denoted by $u \to^+ v$), if there exists $S \subseteq X$ such that $f(u|S) > f(u|S \setminus \{v\})$.

[4] Throughout the paper we use m to denote the number of elements in the ground set.

▶ **Definition 1.1** (Supermodular Degree)**.** The supermodular degree of a set function $f : 2^X \to \mathbb{R}^+$, $\mathrm{SD}(f)$, is defined to be $\mathrm{SD}(f) = \max_{u \in X} |\mathrm{Dep}_f^+(u)|$, where $\mathrm{Dep}_f^+(u) = \{v | u \to^+ v\}$.

Although supermodular degree has been shown useful in a number of settings, it is not clear whether it provides the tightest possible characterization of supermodularity. For example, consider a customer who wants any two or more items out of m items, but not zero or one item. That is, the customer has a valuation function, where any subset of $[m]$ of size at least 2 provides utility 1, and any subset of size at most 1 provides utility 0. For this function, according to Feige and Izsak's definition, any two items depend positively on each other. In particular, any item depends positively on all other items, so the supermodular degree of this valuation function is $m - 1$ — the largest degree possible. This seems to contradict the intuition that there is only very limited complementarity.

Below, we will provide two perspectives, with the first highlighting *supermodularity* and the second highlighting *superadditivity*. We will then study how these two complementarity measures can be used to capture the performance of basic computational solutions in optimization and auction settings where the utilities are modeled by monotone set functions. In particular, our measure of supermodularity refines supermodular degree, and avoids the kind of overestimation discussed above. Our first definition focuses on modularity:

▶ **Definition 1.2** (Supermodular Set)**.** Given a normalized monotone set function f over a ground set X, a set $T \subseteq X$ is *supermodular* w.r.t. f if:

$$\exists\, S \subseteq X \text{ and } v \in X \setminus T, \text{ such that: } f(v|S \cup T) > \max_{T' \subsetneq T} f(v|S \cup T'). \tag{1}$$

Note that if f is submodular, then $f(v|S \cup T) \leq f(v|S \cup T'), \forall T' \subsetneq T$, implying f has no supermodular set. Thus, if a set function f has a supermodular set, then it is not submodular. We say that such a set T (in Definition 1.2) *complements* item v *given* S, where S provides the setting that demonstrates the complementarity between v and T. In the "customer example" given after Definition 1.1, we can check that any singleton is a supermodular set, but any set with size at least two is not a supermodular set, because any single item in the set already provides all the complementarity for any other single item. A supermodular set behaves similarly to the typical example of complements, namely *complementary bundles*,[5] in the sense that the set as a whole provides more complement to a single item than any of its strict subsets. However, supermodular sets have richer structures while preserving the strong complementarity of such bundles, making them potentially more challenging to deal with mathematically/algorithmically than complementary bundles of a similar size.

Our next definition focuses on additivity:

▶ **Definition 1.3** (Superadditive Set)**.** Given a normalized monotone set function f over a ground set X, a set $T \subseteq X$ is superadditive w.r.t. f if:

$$\exists\, S \subseteq X \setminus T \text{ such that: } f(S|T) > \max_{T' \subsetneq T} f(S|T'). \tag{2}$$

In Definition 1.3, we say such a set T *complements* set S. Note that if f is subadditive, then for $T' = \emptyset$, $f(S|T) = f(S \cup T) - f(T) \leq (f(S) + f(T)) - f(T) = f(S) = f(S) - f(T') = f(S|T')$, implying f does not have a superadditive set. In other words, if f has any superadditive set, then it is not subadditive.

Supermodular/superadditive sets correspond to witnesses that exhibit different kinds of complementarity. Supermodular sets are sensitive to the presence of an environment, and

[5] S is a *complementary bundle* if $f(S) > 0$ and $\max_{S' \subsetneq S} f(S') = 0$.

superadditive sets model complements to sets instead of items. The cardinality of the largest supermodular sets or superadditive sets provides a measure of the "level of complementarity", similar to the supermodular degree ([10]), the size of the largest bundle, and the hyperedge size ([9]) (also see Definition 1.16) in previous work.

▶ **Definition 1.4** (Supermodular Width). The *supermodular width* of a set function f is:

$$\mathrm{SMW}(f) := \max\{|T| \mid T \text{ is a supermodular set w.r.t. } f\}. \tag{3}$$

▶ **Definition 1.5** (Superadditive Width). The *superadditive width* of a set function f is:

$$\mathrm{SAW}(f) := \max\{|T| \mid T \text{ is a superadditive set w.r.t. } f\}. \tag{4}$$

Each measure classifies monotone set functions into a hierarchy of m levels:

▶ **Definition 1.6** (Supermodular Width Hierarchy (SMW-d)). For any integer $d \in \{0, \ldots, m-1\}$, a set function $f : 2^{[m]} \to \mathbb{R}$ belongs to the first d-levels of the *supermodular width hierarchy*, denoted by $f \in \mathrm{SMW}\text{-}d$, if and only if $\mathrm{SMW}(f) \leq d$.

▶ **Definition 1.7** (Superadditive Width Hierarchy (SAW-d)). For any integer $d \in \{0, \ldots, m-1\}$, a set function $f : 2^{[m]} \to \mathbb{R}$ belongs to the first d levels of the *superadditive width hierarchy*, denoted by $f \in \mathrm{SAW}\text{-}d$, if and only if $\mathrm{SAW}(f) \leq d$.

We will show that functions at level 0 of the above two hierarchies, respectively, are precisely the families of submodular and subadditive functions. In both hierarchies, SMW-$(m-1)$ and SAW-$(m-1)$ contains all monotone set functions over m elements. Coming back again to the customer example, we see that the utility of the customer has supermodular width of 1. Comparing to its supermodular degree of $m-1$, our hierarchy characterizes this utility function at a much lower level, which matches our intuition that the complementarity of this customer's utility function should be limited. We will further show below that this difference also has significant algorithmic implications.

1.2 Our Results and Related Work

We now summarize the technical results of this paper. Structurally, we provide strong evidence that our definitions of supermodular/superadditive sets are natural and robust. We show that they — respectively — capture a set-theoretical gap of monotone set functions to submodularity and subadditivity. Algorithmically, we prove that our characterization based on supermodular width is *strictly stronger* than that of Feige-Izsak's based on supermodular degree, by establishing the following:

1. For every set function $f : 2^{[m]} \to \mathbb{R}$, $\mathrm{SD}(f) \leq \mathrm{SMW}(f)$, and there exists a function whose supermodular degree is much larger than its supermodular width.
2. The SMW hierarchy offers the same level of algorithmic guarantees in the maximization and auction settings as the SD hierarchy.

We will also compare both hierarchies with the MPH hierarchy of [9].

1.2.1 Robustness: Capturing the Set-Theoretical Gap to Submodularity/Subadditivity

We interpret the level of complementarity in our formulation of supermodular and superadditive sets from a dual perspective: We prove that they characterize the gaps from a monotone set function to submodularity and subadditivity, respectively. Our characterization uses the following definition.

▶ **Definition 1.8** (d-Scopic Submodularity). For integer $d \geq 0$, a normalized monotone set function f is d-scopic submodular if and only if:

$$f(v|T) \leq \max_{T':T' \subseteq T, |T'| \leq d} f(v|S \cup T'), \quad \forall S, T \subseteq X, v \in X \text{ satisfying } S \subseteq T,\, v \notin T \tag{5}$$

In Condition (5), $\{S \cup T' | T' \subseteq T, |T'| \leq d\}$ defines a set-theoretical *neighborhood* around S. The d-scopic submodularity means that even if the submodular condition $f(v|T) \leq f(v|S)$ may not hold for some $S \subseteq T$, it holds for some set in S's d-neighborhood inside T. Thus, the parameter d provides a set-theoretical scope for examining submodularity. Similarly:

▶ **Definition 1.9** (d-scopic Subadditivity). For integer $d \geq 0$, a set function f is d-scopic subadditive if and only if:

$$f(S|T) \leq \max_{T':T' \subseteq T,\, |T'| \leq d} f(S|T'), \quad \forall S, T \subseteq X \text{ satisfying } S \cap T = \emptyset. \tag{6}$$

In Section 2, we prove the following two theorems.

▶ **Theorem 1.10** (Set-Theoretical Characterization of the SMW Hierarchy). *For any integer $d \geq 0$ and set function $f : 2^X \to \mathbb{R}$, f is d-scopic submodular if and only if* $\mathrm{SMW}(f) \leq d$.

▶ **Theorem 1.11** (Set-Theoretical Characterization of the SAW Hierarchy). *For any integer $d \geq 0$ and set function $f : 2^X \to \mathbb{R}$, f is d-scopic subadditive if and only if* $\mathrm{SAW}(f) \leq d$.

With matching supermodularity/submodularity and superadditivity/subadditivity characterization, Theorems 1.10 and 1.11 illustrate that our definitions of supermodular/superadditive sets are both natural and robust. While monotone submodular functions are all subadditive, some d-scopic submodular functions are not d-scopic subadditive. In fact, these two hierarchies are not comparable (Propositions 2.4 and 2.5): They model different aspects of complementarity that can be utilized in diverse algorithmic and economic settings.

1.2.2 Expressiveness: Strengthening Supermodular Degree

Supermodular sets extend positive dependency (as used in supermodular degree), which — in essence — can be viewed as a graphical approximation of supermodular sets. Thus, our characterization based on supermodular width strengthens Feige-Izsak's the characterization based on supermodular degree [10].

▶ **Theorem 1.12.** *Every monotone set function f with supermodular degree d has supermodular width at most d (i.e., it is d-scopic submodular). Moreover, there exists a monotone set function $f : 2^{[m]} \to \mathbb{R}^+$ with* $\mathrm{SMW}(f) = 1$ *and* $\mathrm{SD}(f) = m - 1$.

In other words, the SMW hierarchy strictly *dominates* the SD hierarchy. [6]

1.2.3 Usefulness: Algorithmic and Economic Applications

We then show, algorithmically, the SMW hierarchy — while being more expressive than the SD hierarchy — provides a complexity classification as effective as the latter (Theorems 3.2, 3.5 and 4.14). We will illustrate the usefulness of our hierarchies in algorithm and auction design with two archetypal classes of problems, *set function maximization* and

[6] Formally, when comparing two set-function hierarchies, say with name $\{\mathrm{Y}_d\}_{d \in [0,m-1]}$ and $\{\mathrm{Z}_d\}_{d \in [0,m-1]}$, we say Y *dominates* Z, if for all $d \in [0, m-1]$ and f, $f \in \mathrm{Z}_d$ implies $f \in \mathrm{Y}_d$.

combinatorial auctions, which traditionally involve measures of complementarity. Motivated by previous work [10, 14, 11, 9], we will characterize the *approximation guarantee* of polynomial-time set-function maximization algorithms and *efficiency* of simple auction protocols in terms of the complementarity level in our hierarchies. In these settings, we will compare our hierarchies with two most commonly cited complementarity hierarchies: the supermodular degree (SD) hierarchy and the Maximum over Positive Hypergraphs (MPH) hierarchy.

- *Set-Function Maximization*: We will consider both constrained and welfare maximization. The former aims to find a set of a given cardinality with maximum function value. The latter aims to allocate a set of items to n agents, [7] with potentially different valuations, such that the total value of all agents is maximized. As a set function has exponential dimensions in m, in both maximization problems, we assume that the input set functions are given by their value oracles.
- *Combinatorial Auctions and Simple Auction Protocols*: We will consider two well-studied simple combinatorial auction protocols: Single-bid Auction and Simultaneous Item First Price Auction (SIA). In both settings, there are multiple agents, each of which has a (potentially different) valuation function over subsets of items. The former auction protocol proceeds by asking each bidder to bid a single price, and letting bidders, in descending order of their bids, buy any available set of items paying their bid for each item. The latter simply runs first-price auctions simultaneously for all items.

1.2.4 Approximation Guarantees According to Supermodular Widths

We will prove that the elegant approximability results for constrained maximization by [14] and for welfare maximization by [10] can be extended from supermodular degree to supermodular width. We obtain the same dependency (see Theorems 3.2 and 3.5) — that is, $1-e^{-1/(d+1)}$ and $\frac{1}{d+2}$ respectively — on the supermodular width d as what the supermodular degree previously provides for these problems.

Because our SMW hierarchy is strictly more expressive, our upper bounds for SMW-d cover strictly more monotone set functions than previous results for SD-d. We will also complement our algorithmic results with nearly matching information theoretical lower bounds (see Theorems 3.3 and 3.6), for these two well-studied fundamental maximization problems. Our approximation and hardness results illustrate that the SMW hierarchy not only captures a natural notion of complementarity, but also provides an accurate characterization of the "nearly submodular property" needed by approximate maximization problems.

1.2.5 Efficiency of Simple Auctions According to Superadditive/Supermodular Width

Next, we will analyze the efficiency of two well-known simple auction protocols in terms of superadditive width. To state our results and compare them with previous work, we first recall a notation from [9]:

▶ **Definition 1.13** (Closure under Maximization)**.** For any family of set functions $\mathcal{F}$ over X, the closure of $\mathcal{F}$ under maximization, denoted by $\max(\mathcal{F})$, is the following family of set

[7] Throughout the paper we use n to denote the number of agents (whenever applicable) unless otherwise specified.

functions: $f \in \max(\mathcal{F})$ if and only if:

$$\exists k \in \mathbb{N},\ f_1, \ldots, f_k \in \mathcal{F},\ \text{s.t. } f(S) = \max_{i \in [k]} f_i(S), \forall S \subseteq X. \tag{7}$$

We will prove the following nearly tight upper and lower bounds:

▶ **Theorem 1.14.** *Single-bid Auction and SIA are approximately efficient for valuation functions in* $\max(\mathrm{SAW}(d))$*, with Price of Anarchy (PoA)* $O(d \log m)$*. In addition, for any* $d > 0$*, there is an instance with* SAW-d *valuations, where the Price of Stability (PoS) of Single-bid Auction is at least* $d + 1 - \varepsilon$ *for any* $\varepsilon > 0$*, and the PoA of SIA is at least* d*.*

Although supermodular width strictly strengthens supermodular degree, superadditive width is not comparable with supermodular degree. Nevertheless, our PoA bound of $O(d \log m)$ is a factor of d tighter than the $O(d^2 \log m)$ supermodular-degree based bound of [11] for Single-bid Auction. This improvement of dependency on d, together with the nearly matching lower bound, suggests that the SAW hierarchy might be more capable in capturing the smooth transition of efficiency of simple auctions. Furthermore, as a byproduct of our efficiency results for the SAW hierarchy, we also obtain similar results, but with a worse dependency on d, for the SMW hierarchy.

▶ **Theorem 1.15.** *Single-bid auction and SIA are approximately efficient for valuations in* $\max(\text{SMW-}d \cap \text{SUPADD})$ *— with PoA* $O(d^2 \log m)$ *— where* SUPADD *denotes the class of monotone superadditive set functions.*

For Single-bid Auction, this result strengthens the central efficiency result of [11] by replacing the supermodular degree with the more inclusive supermodular width. For the PoA analysis of SIA, the the Maximum over Positive Hypergraphs (MPH) hierarchy of [9] remains the gold standard, by providing asymptotically matching upper and lower bounds. MPH is defined based on the following hypergraph characterization of set functions: Every normalized monotone set function over ground set X can be uniquely expressed by another set function h such that $f(S) = \sum_{T \subseteq S} h(T), \forall S \subseteq X$, where $h(T)$ for each T is called the weight of hyperedge T.

▶ **Definition 1.16** (Maximum over Positive Hypergraphs [9])**.** Let PH-d be the class of set functions whose hypergraph representation h satisfies: (1) $h(S) \geq 0$ for all S, and (2) $h(S) > 0$ only if $|S| \leq d$. The d-th level of the MPH hierarchy is: MPH-$d = \max(\text{PH-}d)$.

MPH provides the best characterization to the efficiency of SIA as well as ties with SD and SMW regarding the approximation ratio of welfare maximization (although it requires access to the much stronger demand oracles). However, it remains open whether it can be used to analyze constrained set function maximization and Single-bid Auction. See Table 1 for a comparison. We will prove the following theorem which states that, in general, the SAW hierarchy is not comparable with MPH.

▶ **Theorem 1.17.** *There is a subadditive function that lives in an upper (i.e.* $\geq m/2$*)* MPH *level. On the other direction, there is a function on level* 2 *of* MPH*, whose superadditive and supermodular widths are both* $m - 1$*.*

It remains open whether MPH-$(d + 1)$ — which subsumes SD-d as a subset — contains SMW-d. In particular, the proof that SD-$d \subseteq$ MPH-$(d + 1)$ in [9] does not appear easily applicable to SMW-d.

	SD-d	MPH-$(d+1)$	SMW-d	SAW-d
constrained maximization	$1-e^{1/(d+1)}$ [14]	?	$1-e^{1/(d+1)}$ (Thm 3.2)	?
welfare maximization	$1/(d+2)$ [10]	$1/(d+2)$ [9]	$1/(d+2)$ (Thm 3.5)	?
PoA of Single-bid Auction	$O(d^2 \log m)$ [11]	?	$O(d^2 \log m)$ (Thm 4.13)	$O(d \log m)$ (Thm 4.8)
PoA of SIA	$O(d)$ [9]	$O(d)$ [9]	$O(d^2 \log m)$ (Thm 4.14)	$O(d \log m)$ (Thm 4.9)

Table 1 Comparison of hierarchies of complementarity. Note that the $O(d)$ bound for PoA of SIA with SD-d valuations follows from the fact that SD-$d \subseteq$ MPH-$(d+1)$, which is not clearly comparable with the PoA bound of SIA with SMW-d valuations. See corresponding references and theorems for more accurate statements.

1.2.6 Other Related Work

Set Function Maximization: There is a rich body of research focusing on set function maximization with complement-free functions, e.g. [22, 29, 8]. Various information/complexity theoretical lower bounds have been established for both problems, e.g. [21, 7, 20, 18].

Efficiency of Simple Auctions: Single-bid Auction with subadditive valuations has a PoA of $O(\log m)$ [5]. SIA with subadditive valuations has a constant PoA [12]. Posted price auctions with XOS valuations give a constant factor approximate welfare guarantee [13].

Other Measures of Complementarity: Some other useful measures include Positive Hypergraph (PH) [1] and Positive Lower Envelop (PLE) [9]. Eden *et al.* recently introduce an extensive measure which ranges from 1 to 2^m to capture the smooth transition of revenue approximation guarantee [6].

2 Expressiveness of the New Hierarchies

2.1 Characterization of Supermodular/Superadditive Widths

We first prove Theorems 1.10 and 1.11, which characterize supermodular/superadditive widths with d-scopic submodular/subadditive functions.

Proof of Theorem 1.10. We now show $\mathrm{SMW}(f) \leq d$ iff f is d-scopic submodular. First, suppose $\mathrm{SMW}(f) \leq d$. Consider any triple (T, S, v) such that $S \subseteq T \subseteq X$ and $v \notin T$. To show f is d-scopic submodular, we prove by induction on the size of T, that

$$f(v|T) \leq \max_{T':T' \subseteq T, |T'| \leq d} f(v|S \cup T'). \tag{8}$$

As the base case, when $|T| \leq d$, the inequality of (8) trivially holds because if $T' = T \setminus S$, then $|T'| \leq d$ and $f(v|S \cup T') = f(v|T)$. Inductively, assume that the statement is true for all $\{V \subseteq X : |V| \leq k\}$ for some $k \geq d$. Now consider any set T with $|T| = k+1 > d$. Because T is not supermodular, there is $T'' \subsetneq T$, such that $f(v|T) \leq f(v|T'')$. Applying the inductive hypothesis on (T'', S, v), we have:

$$f(v|T'') \leq \max_{T':T' \subseteq T'', |T'| \leq d} f(v|S \cup T') \leq \max_{T':T' \subseteq T, |T'| \leq d} f(v|S \cup T').$$

Thus, $f(v|T) \leq f(v|T'') \leq \max_{T':T' \subseteq T, |T'| \leq d} f(v|S \cup T')$, and we have demonstrated that f is d-scopic submodular. For the other direction, we assume f is d-scopic submodular. There is no supermodular set of size larger than d, because for any S, T, $v \notin T$ where $|T| > d$, there is some $T' \subseteq T$ where $|T'| \leq d$, such that $f(v|S \cup T) \leq f(v|S \cup T')$, i.e. T is not supermodular. Therefore $\mathrm{SMW}(f) \leq d$. ◀

▶ **Corollary 2.1.** *f is submodular iff $\mathrm{SMW}(f) = 0$ (i.e., f has no supermodular set).*

Proof of Theorem 1.11. We prove $\mathrm{SAW}(f) \leq d$ iff f is d-scopic subadditive. Suppose $\mathrm{SAW}(f) \leq d$. Consider S and T where $S \cap T = \emptyset$. We show d-scopic subadditivity by induction on the size of T. When $|T| \leq d$, the statement trivially holds. Suppose d-scopic subadditivity holds for $|T| \leq k$ where $k \geq d$. For $|T| = k+1 > d$, since T is not superadditive, there is $T'' \subsetneq T$, such that $f(S|T) \leq f(S|T'')$. Applying inductive hypothesis on S, T'' gives $f(S|T) \leq f(S|T'') \leq \max_{T': T' \subseteq T, |T'| \leq d} f(S|T')$, i.e. f is d-scopic subadditive.

Now assume d-scopic subadditivity. There is no superadditive set with size larger than d, because for any S and T where $|T| > d$ and $S \cap T = \emptyset$, there is some $T' \subseteq T$ where $|T'| \leq d$, such that $f(S|T) \leq f(S|T')$, i.e. T is not superadditive. ◀

▶ **Corollary 2.2.** *f is subadditive iff $\mathrm{SAW}(f) = 0$ (i.e., f has no superadditive set).*

2.2 Supermodular Width vs Supermodular Degree

The following two propositions establish Theorem 1.12, showing supermodular width strictly dominates supermodular degree.

▶ Proposition 2.1. For any set function f, $\mathrm{SD}(f) \leq \mathrm{SMW}(f)$.

Proof. Fix f. Let T be a supermodular set of size $\mathrm{SMW}(f)$, and S, v be such that $f(v|T \cup S) > f(v|T' \cup S), \forall T' \subsetneq T$. Clearly for any $t \in T$, $f(v|\{t\} \cup (T \setminus \{t\}) \cup S) > f(v|(T \setminus \{t\}) \cup S)$. In other words, $v \to^+ t$ for all $t \in T$, so $\mathrm{SD}(f) \geq deg^+(v) \geq |T| = \mathrm{SMW}(f)$. ◀

▶ Proposition 2.2. There is a monotone set function f with $\mathrm{SMW}(f) = 1$ and $\mathrm{SD}(f) = m - 1$.

Proof. Consider a symmetric[8] f over a ground set $X = [m]$, where $f(S) = 0$ if $|S| \leq 1$, and $f(S) = 1$ otherwise. Observe that for any $u \neq v$, $1 = f(u|\{v\}) > f(u|\emptyset) = f(u) = 0$, so $u \to^+ v$, and $\mathrm{SD}(f) = |\mathrm{Dep}_f^+(u)| = m - 1$. On the other hand, consider any T where $|T| \geq 2$. For any v, S, we have $|S \cup T| \geq 2$, so $0 = f(v|S \cup T) \leq f(v|S)$. Thus, T is not supermodular. Since there is no supermodular set with size larger than 1 and f is not submodular, $\mathrm{SMW}(f) = 1$. ◀

While the SAW hierarchy does not subsume the MPH hierarchy (see Proposition 2.6), we show that there is a monotone set function in the lowest layer of the SAW hierarchy (i.e. a subadditive function) and a notably high layer of the MPH hierarchy.

▶ Proposition 2.3. There is a monotone set function f with $\mathrm{SAW}(f) = 0$ and $\mathrm{MPH}(f) = m/2$.

Proof. The proposition is a direct corollary of Proposition L.2 in [9]. ◀

[8] f is symmetric if $f(S)$ depends only on $|S|$.

2.3 Further Comparisons between Hierarchies

▶ Proposition 2.4. There is a monotone set function f with $\mathrm{SMW}(f) = 1$ and $\mathrm{SAW}(f) = m/2$.

Proof. Let $h_T(S) = \mathbb{I}[T \subseteq S]$. Consider $f : 2^X \to \mathbb{R}^+$ where $X = [2t]$ and $f(S) = \sum_{i \in [t]} h_{\{i,i+t\}}(S)$. Because the only complement set to any item $i \in [t]$ is $i + t$, $\mathrm{SMW}(f) = 1$. Note also $T = \{t + 1, \ldots, 2t\}$ is a complement set to $S = [t]$, so $\mathrm{SAW}(f) = t = m/2$. ◀

▶ Proposition 2.5. There is a monotone set function f with $\mathrm{SAW}(f) = 0$ and $\mathrm{SMW}(f) = m-1$.

Proof. Consider a symmetric $f : 2^X \to \mathbb{R}^+$, where $f(\emptyset) = 0$, $f(X) = 2$ and $f(S) = 1$ otherwise. f is subadditive so $\mathrm{SAW}(f) = 0$. On the other hand, $X \setminus \{u\}$ for any u is a complement set to u, so $\mathrm{SMW}(f) = m - 1$. ◀

▶ Proposition 2.6. There exists a monotone set function f with $\mathrm{MPH}(f) = 2$ and $\mathrm{SMW}(f) = \mathrm{SAW}(f) = m - 1$.

Proof. Let $h_T(S) = \mathbb{I}[T \subseteq S]$. Consider function $f : 2^X \to \mathbb{R}^+$ where $f(S) = \sum_{u \neq v} h_{\{u,v\}}(S)$. Note that f is in MPH-2 since its hypergraph representation consists of only hyperedges of size 2. Now consider any u and $T = X \setminus \{u\}$. For any $T' \subsetneq T$, $f(u|T) = |T| > |T'| = f(u|T')$. Thus, T is both supermodular and superadditive, and $\mathrm{SMW}(f) = \mathrm{SAW}(f) = m - 1$. ◀

3 Expanding Approximation Guarantees for Classic Maximization

In this section, we focus on the connection between supermodular width and two classical optimization problems: the constrained and welfare set-function maximization. For submodular functions, greedy algorithms provide tight approximation guarantees for both problems [22, 29]. Here, simple modifications to these greedy algorithms can effectively utilize the mathematical structure underlying the gap to submodularity in any set function f. These extensions achieve approximation ratios parametrized by the supermodular width with the same dependency as the supermodular degree provides [14, 10]. We complement our approximation results by nearly tight information-theoretical lower bounds.

3.1 Constrained Maximization

We first focus on cardinality constrained maximization, a problem at the center of resource allocation and network influence [24, 17, 22, 29]. Formally:

▶ **Definition 3.1** (Cardinality Constrained Maximization). Given a monotone set function $f : 2^X \to \mathbb{R}^+ \cup \{0\}$ and integer $k > 0$, compute a set $S \subseteq X$ with $|S| \leq k$ that maximizes $f(S)$.

We will analyze an algorithm which performs *batched greedy selection*, — see Algorithm 1 below — where the batch size is a function of the supermodular width of f. In particular, for an input set function, the batched greedy algorithm chooses a set of size not exceeding $\mathrm{SMW}(f) + 1$ which maximizes marginal gain, till all k elements are chosen.

Below, we show that this simple greedy algorithm provides strong approximation guarantees in terms of the supermodular width of the input function.

▶ **Theorem 3.2** (Extending [14]). *For any monotone set function f over $[m]$, Algorithm 1 achieves $\left(1 - e^{-1/(\mathrm{SMW}(f)+1)}\right)$-approximation for constrained maximization problem after making $O\left(m^{\mathrm{SMW}(f)+1}\right)$ value queries.*

ALGORITHM 1: Batched Greedy Selection for Constrained Maximization (f, k)

let $S_0 \leftarrow \emptyset$; $i = 0$;
while $|S_i| < k$ **do**
  Let $i = i + 1$; $T_i \leftarrow \mathrm{argmax}_{T' \subseteq [m], |T'| \leq s} f(T'|S_i)$ where $s = \min\{\mathrm{SMW}(f) + 1, k - |S_{i-1}|\}$;
  let $S_i \leftarrow S_{i-1} \cup T_i$; ;
end
return $S^{\mathrm{BatchedGreedy}} := S_i$;

Proof. The proof uses similar ideas to those in [14], which are originally from [22]. Let $d = \mathrm{SMW}(f)$ and (w.l.o.g.) let $S^* = [k] = \{1, \ldots, k\}$ be an optimal solution.

$$
\begin{aligned}
f(S^*) - f(S_i) &\leq f(S^* \cup S_i) - f(S_i) && (9)\\
&\leq f(S^*|S_i) = f([k]|S_i) = \sum_{j \in [k]} f(j|[j-1] \cup S_i) \leq k \max_j f(j|[j-1] \cup S_i)\\
&\leq k \max_j \max_{U_j : U_j \subseteq [j-1],\, |U_j| \leq d} f(j|U_j \cup S_i) && (10)\\
&\leq k \max_j \max_{U_j : U_j \subseteq [j-1],\, |U_j| \leq d} f(\{j\} \cup U_j|S_i) && (11)\\
&\leq k f(S_{i+1}|S_i) && (12)\\
&= k(f(S_{i+1}) - f(S_i))
\end{aligned}
$$

where (9) is by the monotonicity of f, (10) is by the equivalent d-scopic submodularity of f, (11) is again by the monotonicity of f, and (12) is by the greedy property: $f(S_{i+1}|S_i) = \max_{S:|S| \leq d+1} f(S|S_i)$.

Now we have

$$
\begin{aligned}
f(S^*) - f(S_i) &\leq \frac{k-1}{k}(f(S^*) - f(S_{i-1})) \leq \left(\frac{k-1}{k}\right)^i (f(S^*) - f(S_0))\\
&= \left(\frac{k-1}{k}\right)^i f(S^*) \leq e^{-i/k} f(S^*).
\end{aligned}
$$

Because f is monotone, we have $|T_i| = d + 1$, for all intermediate steps, i.e., $i < \lceil \frac{k}{\mathrm{SMW}(f)+1} \rceil$. Thus, Algorithm 1 takes exactly $t := \lceil \frac{k}{\mathrm{SMW}(f)+1} \rceil$ steps to terminate. The function value of its output $f(S^{\mathrm{BatchedGreedy}}) := f(S_t) \geq \left(1 - e^{-1/(\mathrm{SMW}(f)+1)}\right) f(S^*)$. ◀

While in general, Theorem 3.2 establishes a tighter approximation guarantee for the SMW hierarchy than that for the SD hierarchy, we note that in case of submodular degree, if the positive dependency graph is given, the running times are often of the form $\mathrm{poly}(n) \cdot 2^{O(\mathrm{SD}(f))}$, which can be significantly better than $n^{O(\mathrm{SMW}(f))}$ even if the submodular width $\mathrm{SMW}(f)$ is much smaller than the submodular degree $\mathrm{SD}(f)$.

We now provide a nearly-matching information-theoretical lower bound, suggesting that our approximation guarantee is essentially optimal. In the theorem below, the exponent $k^{0.99}$ can be replaced by any function of k in $o(k)$.

▶ **Theorem 3.3.** *For any $d \in \mathbb{N}$, $\varepsilon > 0$, and a large enough integer k, there exists a set function $f : 2^{[m]} \to \mathbb{R}^+$, with $\mathrm{SMW}(f) = d$, such that any (possibly randomized) algorithm that produces a $(1/(d+1) + \varepsilon)$-approximation (with a constant probability if randomized) for the k-constrained maximization problem makes at least $\Omega\left((m/2k)^{k^{0.99}}\right)$ value queries.*

Proof. The proof is based on similar high-level ideas to those in [20], but the detailed construction and key properties used are different. Consider a ground set X of m elements, which contains a subset R of r "special" elements. We will specify r below. We now construct a "hard-to-distinguish" function f_R such that for any $S \subseteq X$, $f_R(S) = g_R(|S|, \mathbb{I}[R \subseteq S])$ for a function $g_R : \mathbb{N} \times \{0,1\} \to \mathbb{R}$. In other words, f_R depends on the cardinality of S and whether or not S completely contains R. For discussion below, let $D = d+1$, and let c_1 and c_2 be two integers to be determined later. We set $|R| = r = c_1 \cdot D + 1$. We define f_R as follows:

$$f_R(S) = \begin{cases} \lfloor |S|/D \rfloor, & |S| \le c_1 D \\ \lfloor (|S| - c_1 D)/D \rfloor + c_1, & c_1 D < |S| \le (c_1+c_2)D,\ R \not\subseteq S \\ |S| - c_1(D-1), & c_1 D < |S| \le (c_1+c_2)D,\ R \subseteq S \\ |S| - (c_1+c_2)(D-1), & (c_1+c_2)D < |S| \le (c_1+c_2)D + c_2(D-1), R \not\subseteq S \\ c_1 + c_2 D, & (c_1+c_2)D < |S| \le (c_1+c_2)D + c_2(D-1), R \subseteq S \\ c_1 + c_2 D, & (c_1+c_2)D + c_2(D-1) < |S| \le m \end{cases}.$$

We will use the following three properties of f_R:

- Whenever $|S| \mod D = D-1$, for any $v \notin S$, $f_R(v|S) = 1$. Consequently, $\mathrm{SMW}(f_R) \le d, \forall R \subseteq X$ with $|R| = r$. In fact, this property ensures that $f_R(v|S \cup T') \ge f_R(v|S \cup T)$, for any $v \in X$, $S, T \subseteq X$ with $|T| \ge D = d+1$, and any proper subset T' of T with $|S \cup T'| \mod D = D-1$. Note that such a subset T' always exists.
- $\max\{f_R(S) \mid |S| = (c_1+c_2)D\} = c_1 + c_2 D$. The maximum is achieved whenever $R \subseteq S$.
- For any $S \subseteq X$ satisfying $|S| = c_1 + c_2 D$ and $R \not\subseteq S$, $f_R(S) = c_1 + c_2$.

First, consider $k = (c_1+c_2)D$. We have, for any S with $|S| = k$:

$$f_R(S) = \begin{cases} c_1 + c_2 D & \text{if } R \subseteq S \\ c_1 + c_2 & \text{otherwise.} \end{cases}$$

Suppose $c_1 = o(c_2)$. To obtain an approximation ratio better than $(c_1+c_2)/(c_1+c_2 D) \to 1/D$ for k-constrained maximization of f_R, any algorithm must find a set with size k that contains all special elements in R.

For our lower bound, we will analyze the following slightly relaxed variation of the problem: Let $K = (c_1+c_2)D + c_2(D-1) - 1 > k$. Find a set of size K which contains R as a subset. Note that K is the largest number where $f_R(S)$ — for $|S| = K$ — depends on whether or not S contains R. In this case, note that the algorithm has no incentive to make queries of $f_R(S)$ for $|S| < K$ or $|S| > K$, because the former reveals no more information than querying any of its supersets of size K, and the latter simply does not give any information.

We first focus on the query complexity of any deterministic optimization algorithm. Assume the algorithm makes T queries regarding $S_1, \dots, S_T$, where $|S_i| = K, \forall i \in [T]$, which are deterministically chosen when the algorithm is fixed. We now establish a condition on T such that there is a subset R such that $R \not\subset S_i, \forall i \in [T]$. Consider the distribution where the r elements are selected uniformly at random. Let C_i be the event that S_i contains R. Then,

$$\begin{aligned} \Pr[C_1 \cup \dots \cup C_T] &\le \sum_i \Pr[C_i] < \sum_i \left(\frac{|S_i|}{m}\right)^r \\ &= T\left(\frac{(c_1+c_2)D + (D-1)c_2 - 1}{m}\right)^{c_1 D + 1} \le T\left(\frac{2c_2 D}{m}\right)^{c_1 D}. \end{aligned}$$

So, if $T \le [m/(2c_2 D)]^{c_1 D}$ then $\Pr[C_1 \cup \dots \cup C_T] < 1$. In other words, for any selections $S_1, \dots, S_T \subseteq X$ with $|S_i| = K$, there is a subset R, such that $R \not\subset S_i, \forall i \in [T]$, implying

ALGORITHM 2: Batched Greedy for Welfare Maximization $(f_1, \ldots, f_n)$

for $j \in [n]$ **do**
  let $X_{j,0} \leftarrow \emptyset$;
end
Let $d = \max_j \{\mathrm{SMW}(f_j)\}$; let $i = 0$;
while $\cup_j X_{j,i} \neq X$ **do**
  Let $i = i + 1$; let
  $(T_i, j_i^*) = \mathrm{argmax}_{(T',j):\, |T'| \le s, j \in [n]} f_j\left(T' | X_{j,i-1}\right)$ where $s = \min\left\{d+1, n - \sum_j |X_{j,i-1}|\right\}$;
  Let $X_{j_i^*,i} \leftarrow X_{j_i^*,i-1} \cup T_i$;
  for $j \in [n] \setminus \{j_i^*\}$ **do**
    let $X_{j,i} \leftarrow X_{j,i-1}$;
  end
  return $X_j^{\mathrm{BatchedGreedy}} := X_{j,i}$ for every agent j;
end

the deterministic algorithm with querying set $S_1, \ldots, S_T$ will not find a good approximation to f_R. Let $c_2 = \frac{1}{2} c_1^{1.01}$, so $k^{0.99} = ((c_1 + c_2)D)^{0.99} \le (c_1^{1.01} D)^{0.99} \le c_1 D$. We have $(m/2c_2D)^{c_1 D} \ge (m/2k)^{k^{0.99}}$. Thus, we conclude that any $(1/(d+1) + \varepsilon)$-approximation deterministic algorithm must make at least $(m/2k)^{k^{0.99}}$ value queries.

Now consider a randomized optimization algorithm. Conditioned on the random bits of the algorithm, the above argument still works. Taking expectation of the probability of success, we see that the overall probability of success is at most $T(2k/m)^{k^{0.99}}$. Thus, a constant probability of success requires $T = \Omega\left((m/2k)^{k^{0.99}}\right)$. ◀

3.2 Welfare Maximization

We now turn our attention to welfare maximization. Formally:

▶ **Definition 3.4** (Welfare Maximization)**.** Given n monotone set functions $f_1, \ldots, f_n$ over $2^{[m]}$, compute n disjoint sets $X_1, \ldots, X_n$ that maximizes $\sum_{i \in [n]} f_i(X_i)$.

Because $f_1, \ldots, f_n$ are monotone, the optimal solution to welfare maximization is a partition of $X = [m]$. Thus, welfare maximization can also be viewed as a generalized clustering or multiway partitioning problem.

We will analyze the following greedy algorithm — see Algorithm 2 below — which repeatedly assigns groups of elements to agents. At each step, the algorithm picks a set of size not exceeding $\max_i \mathrm{SMW}(f_i) + 1$ — as opposed to one — that provides the largest possible marginal gain to some agent and assigns the set to that agent.

We now prove the following approximation guarantee in terms of supermodular width.

▶ **Theorem 3.5** (Extending [10])**.** *For any collection of monotone set functions $f_1, \ldots, f_n$ over $X = [m]$, Algorithm 2 achieves $\frac{1}{2+\max_i\{\mathrm{SMW}(f_i)\}}$-approximation for welfare maximization, after making $O\left(nm^{\max_i\{\mathrm{SMW}(f_i)\}+1}\right)$ value queries.*

The proof uses similar ideas to those in [10], which are originally from [16]. Due to space limit, we relegate the proof to the full version of the paper [4].

To show that our algorithm is nearly optimal, we prove the following information-theoretical lower bound: Similar to Theorem 3.3, the exponent $(m/n)^{0.99}$ in the theorem below, can be replaced by any function of m/n in $o(m/n)$.

▶ **Theorem 3.6.** *For any $d \in \mathbb{N}$, $\varepsilon > 0$, there is a family of function $f_1, \ldots, f_n : 2^{[m]} \to \mathbb{R}^+$ with $\mathrm{SMW}(f_i) = d, \forall i \in [n]$, such that any (possibly randomized) algorithm that produces a $(1/(d+1)+\varepsilon)$-approximation (with constant probability if randomized) for the n-agent welfare maximization problem makes at least $\Omega\left((n/2D)^{(m/n)^{0.99}}\right)$ value queries.*

The proof follows from a similar argument as the proof for Theorem 3.3. Due to space limit, we relegate the proof to the full version of the paper.

4 Efficiency of Simple Auctions

In this section, we study the connection between the SAW hierarchy and efficiency of auctions. We will draw extensively on previous work in this area, particularly on the characterization based on the *CH hierarchy* — see definition below — which is arguably the most simple class of set functions with complementarity.

▶ **Definition 4.1** (d-Constraint Homogeneous Functions [11])**.** A set function f over ground set X is d-constraint homogeneous (CH-d) if there exists a value $\hat{f}$, and disjoint sets $Q_1, \ldots, Q_h \subseteq X$ with $|Q_i| \leq d, \forall i \in [h]$, such that (1) $f(Q_i) = \hat{f} \cdot |Q_i|, \forall i \in [h]$, and (2) the value of every set $S \subseteq [m]$ is simply the sum of values of contained Q_i's, i.e., $f(S) = \sum_{Q_i \subseteq S} f(Q_i) = \hat{f} \cdot \sum_{Q_i \subseteq S} |Q_i|$.

We will show that previous characterization of auction efficiency [11] can be approximately extended from the CH hierarchy to the SAW hierarchy.

4.1 Backgrounds: Related Definitions and Results

We first restate a useful definition and a lemma for analyzing the efficiency of auction mechanisms.

▶ **Definition 4.2** ([27])**.** An auction mechanism $\mathcal{M}$ is (λ, μ)-smooth for a class of valuations $\mathcal{F} = \times_i \mathcal{F}_i$ if for any valuation profile $f \in \mathcal{F}$, there exists a (possibly randomized) action profile $a_i^*(f)$ such that for every action profile a:

$$\sum_i \mathbb{E}_{a_i' \sim a_i^*(f)}[u_i(a_i', a_{-i}; f_i)] \geq \lambda \cdot \mathrm{OPT}(f) - \mu \sum_i P_i(a),$$

where $u_i(a_i'; f_i)$ is the utility of i given action profile (a_i', a_{-i}), $\mathrm{OPT}(f)$ is the optimum social welfare given valuation profile f, and $P_i(a)$ is the payment of i given action profile a.

▶ **Lemma 4.3** ([27])**.** *If a mechanism is (λ, μ)-smooth then the price of anarchy w.r.t. coarse correlated equilibria is at most $\max\{1, \mu\}/\lambda$.*

For Single-bid Auction and Simultaneous Item First Price Auction (SIA), we will derive our results from the following results for CH-d and MPH-d.

▶ **Theorem 4.4** (Smoothness of Single-bid Auction with CH-d Valuations [11])**.** *Single-bid Auction is a $((1-e^{-d})/d, 1)$-smooth mechanism when agents have CH-d valuations. Consequently, Single-bid Auction has a PoA of $(1-e^{-d})/d$ with CH-d valuations w.r.t. coarse correlated equilibria.*

▶ **Theorem 4.5** (Smoothness of SIA with MPH-d Valuations [9])**.** *For SIA, when bidders have MPH-d valuations, both the correlated price of anarchy and the Bayes-Nash price of anarchy are at most $2d$. The bound follows from a smoothness argument.*

A key concept to extend these results to other valuation classes is the following notion of pointwise approximation defined in [5].

▶ **Definition 4.6** (Pointwise Approximation [5]). A class of set functions $\mathcal{F}$ over ground set X is pointwise β-approximated by another class $\mathcal{F}'$ of set functions over X if $\forall f \in \mathcal{F}, S \subseteq X$, $\exists f'_S \in \mathcal{F}'$ such that (1) $\beta f'_S(S) \geq f(S)$ and (2) $\forall T \subseteq X$, $f'_S(T) \leq f(T)$.

For example:

▶ Proposition 4.1 ([11]). The class $\max(\mathcal{F})$ is pointwise 1-approximated by the class $\mathcal{F}$.

We say a function $f' : 2^X \to \mathbb{R}$ pointwise β-approximates $f : 2^X \to \mathbb{R}$ (at X), if (1) $\beta f'(X) \geq f(X)$, and (2) $\forall T \subseteq X$, $f'(T) \leq f(T)$.

The following lemma of [5] provides a way to translate PoA bounds between classes via pointwise approximation.

▶ **Lemma 4.7** (Extension Lemma [5]). *If a mechanism for a combinatorial auction setting is (λ, μ)-smooth for the class of set functions $\mathcal{F}'$, and $\mathcal{F}$ is pointwise β-approximated by $\mathcal{F}'$, then it is $\left(\frac{\lambda}{\beta}, \mu\right)$-smooth for the class $\mathcal{F}$. And as a result, if a mechanism for a combinatorial auction setting has a PoA of α given by a smoothness argument for the class $\mathcal{F}'$, and $\mathcal{F}$ is pointwise β-approximated by $\mathcal{F}'$, then it has a PoA of $\alpha\beta$ for the class $\mathcal{F}$.*

4.2 Efficiency of Simple Auctions Parametrized by SAW

Applying Lemma 4.7, we are able to translate Theorems 4.4 and 4.5 to the SAW hierarchy.

▶ **Theorem 4.8** (Efficiency of Single-bid Auction with SAW-d Valuations). *When agents have valuations $f_1, \ldots, f_n \in \max(\text{SAW-}d)$, Single-bid Auction has a price of anarchy of at most $\frac{2d}{1-e^{-2d}} \cdot H_{\frac{m}{2d}}$ w.r.t. coarse correlated equilibria.*

▶ **Theorem 4.9** (Efficiency of SIA with SAW-d Valuations). *When agents have valuations $f_1, \ldots, f_n \in \max(\text{SAW-}d)$, SIA has a price of anarchy of at most $8d \cdot H_{\frac{m}{2d}}$ w.r.t. coarse correlated equilibria.*

Formally, Theorems 4.8 and 4.9 follow from Theorems 4.4 and 4.5 respectively, with the help of Lemma 4.7, Proposition 4.1, and the technical lemma (Lemma 4.10) that we will establish below, showing that for any $d \in \mathbb{N}$, functions in SAW-d can be approximated by CH-$2d$ functions. In particular, Lemma 4.10 establishes the approximation of SAW hierarchy by CH hierarchy with a loss of factor $O(\log m)$.

▶ **Lemma 4.10** (Pointwise Approximation of SAW Hierarchy by CH-Hierarchy). *For any $d \in \mathbb{N}$, SAW-d is pointwise $2H_{\frac{m}{2d}}$-approximated by CH-$2d$, where $H_i = \sum_{k\in[i]} \frac{1}{k}$ is the i-th harmonic number.*

Proof. Our proof is inspired by the constructions of [5] and [11].

For any $f \in \text{SAW-}d$ over $X = [m]$, we first apply the following greedy construction to obtain a partition $\mathcal{Q} = \{Q_i\}_{i\in[q]}$ of $[m]$ into sets of size not exceeding $2d$: At step i, we select a new set $Q_i \subseteq [m] \setminus (Q_1 \cup \cdots \cup Q_{i-1})$, with maximum $f(Q_i)$, among all sets of size at most $2d$.

We first prove by contradiction that there exists a function g in CH-$2d$ which $2H_{\frac{m}{2d}}$-approximates f at $[m]$. That is, (1) $2H_{\frac{m}{2d}} g([m]) \geq f([m])$ and (2) $\forall T \subseteq [m]$, $g(T) \leq f(T)$.

Suppose this statement is not true. Let

$$h_{\mathcal{Q}}(T) = \frac{f([m])}{\beta \cdot |\cup_i Q_i|} \sum_{Q_i \subseteq T} |Q_i|.$$

Note that $h_{\mathcal{Q}} \in$ CH-$2d$ because $|Q_i| \le 2d, \forall Q_i \in \mathcal{Q}$. We now construct a series of functions based on $h_{\mathcal{Q}}$, and prove that for any $\beta > 0$, if there is no g among these functions that is a β-approximation of f at $[m]$ — that is, there is no g such that (1) $\beta g([m]) \ge f([m])$ and (2) $\forall T \subseteq [m]$, $g(T) \le f(T)$, (below we will refer to this condition as Assumption (*)) — then $\beta < 2H_{\frac{m}{2d}}$.

First consider $h_{\mathcal{Q}}$. Note that $\beta h_{\mathcal{Q}}([m]) = \beta \frac{f([m])}{\beta} \ge f([m])$, because $\mathcal{Q}$ is a partition of $[m]$. Assumption (*) then implies there is a T_1 such that $h_{\mathcal{Q}}(T_1) > f(T_1)$. W.l.o.g. assume T_1 is a union of sets from $\mathcal{Q}$ (such T_1 exists because f is monotone).

Let $S_1 = [m]$. We now iteratively define $S_i = S_{i-1} \setminus T_{i-1}$, and construct its associated T_i. The construction maintains the following invariant: Both S_i and T_i are unions of sets from $\mathcal{Q}$. The former follows directly from the iterative property that S_{i-1} and T_{i-1} are both unions of sets from $\mathcal{Q}$. Our construction below will ensure the latter.

Let $\mathcal{Q}_{S_i} = \{Q \in \mathcal{Q} \mid Q \subseteq S_i\}$. Let

$$h_{\mathcal{Q}_{S_i}} = \frac{f([m])}{\beta \cdot |\cup_{j:Q_j \in \mathcal{Q}_{S_i}} Q_j|} \sum_{j:Q_j \in \mathcal{Q}_{S_i}} |Q_j|.$$

Again, $h_{\mathcal{Q}_{S_i}} \in$ CH-$2d$, and $h_{\mathcal{Q}_{S_i}}([m]) = \frac{f([m])}{\beta}$. Assumption (*) then implies there is a T_i such that $h_{\mathcal{Q}_{S_i}}(T_i) > f(T_i)$. Again, w.l.o.g. assume T_i is a union of sets from $\mathcal{Q}$ (such T_i exists because f is monotone). This iterative process terminates, producing a partition $\{T_i\}_{i \in [t]}$ of $[m]$, which satisfies:

$$\sum_i f(T_i) < \sum_i h_{\mathcal{Q}_{S_i}}(T_i) = \frac{f([m])}{\beta} \sum_i \frac{|T_i|}{|S_i|} \le \frac{f([m])}{\beta} \sum_{i \in [t]} \frac{1}{i} \le \frac{f([m])}{\beta} H_{\frac{m}{2d}}.$$

We now show that $\sum_i f(T_i) \ge \frac{1}{2} f([m])$. Recall that each member in partition $\{T_i\}_i$ is a unions of sets from $\mathcal{Q}$. We renumber $\{T_i\}_i$, in a way that for any $i < j$, there is some $T_i \supseteq Q_k \in \mathcal{Q}$, such that for any $T_j \supseteq Q_l \in \mathcal{Q}$, $k < l$. That is, the smallest index k where $Q_k \in T_i$ is smaller than the smallest index l where $Q_l \in T_j$, as long as $i < j$.

Since $(T_1, \dots, T_t)$ is a partition of $[m]$, we have:

$$\begin{aligned}
f([m]) &= \sum_i f(T_i | T_{i+1} \cup \dots \cup T_t) \\
&\le \sum_i \max\{f(T_i|U_i) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, \, |U_i| \le d\} && (13) \\
&\le \sum_i \max\{f(T_i \cup U_i) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, \, |U_i| \le d\} && (14) \\
&= \sum_i \max\{(f(U_i|T_i) + f(T_i)) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, \, |U_i| \le d\} \\
&\le \sum_i \max\{(f(U_i|V_i) + f(T_i)) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, \, |U_i| \le d, \, V_i \subseteq T_i, \, |V_i| \le d\} && (15) \\
&\le \sum_i \max\{(f(U_i \cup V_i) + f(T_i)) \mid U_i \subseteq T_{i+1} \cup \dots \cup T_t, \, |U_i| \le d, \, V_i \subseteq T_i, \, |V_i| \le d\} && (16) \\
&\le \sum_i (f(Q_{k_i}) + f(T_i)), \text{ where } k_i = min\{k \mid T_i \supseteq Q_k \in \mathcal{Q}\} && (17) \\
&\le \sum_i 2f(T_i), && (18)
\end{aligned}$$

where (13) and (15) follow from d-scopic subadditivity of f, (14), (16) and (18) follow from monotonicity of f, and (17) holds because, according to the construction of $\{Q_l\}_l$, Q_{k_i} maximizes f among all sets of size $2d$ contained in $Q_{k_i} \cup \cdots \cup Q_q \supseteq T_i \cup \cdots \cup T_t$, and in particular $U_i \cup V_i \subseteq T_i \cup \cdots \cup T_t$.

Consequently, it follows from $\sum_i f(T_i) \geq \frac{1}{2} f([m])$ that:

$$\frac{H_{\frac{m}{2d}} f([m])}{\beta} > \sum_i f(T_i) \geq \frac{1}{2} f([m]) \Rightarrow \beta < 2H_{\frac{m}{2d}}.$$

Thus, Assumption (*) with $\beta \geq 2H_{\frac{m}{2d}}$ leads to a contradiction. Therefore, we have established that there exists a CH-$2d$ function g such that (1) $g([m]) \geq 2H_{\frac{m}{2d}} f([m])$ and (2) $\forall T \subseteq [m]$, $g(T) \leq f(T)$.

As in [11], the above proof can be simply extended to prove for any $S \subseteq X$, there exists a CH-$2d$ function g such that (1) $g(S) \geq 2H_{\frac{m}{2d}} f([m])$ and (2) $\forall T \subseteq [m]$, $g(T) \leq f(T)$. Essentially, we restrict the function f to 2^S, apply the argument above, and then span the obtained function back to 2^X.

Therefore, SAW-d is pointwise $2H_{\frac{m}{2d}}$-approximated by CH-$2d$. ◀

We further analyze previously known hard instances to both auctions, and show that they provide almost matching lower bounds to the above two efficiency upper bounds.

▶ **Theorem 4.11.** *There is an instance with* SAW-d *valuations for any d, where the PoS of Single-bid Auction is at least $d + 1 - \varepsilon/d$ for any $\varepsilon > 0$.*

▶ **Theorem 4.12.** *There is an instance with* SAW-d *valuations for any d, where the PoA of SIA is at least $d + 1/(d + 1)$.*

We defer the proofs to the full version of the paper.

4.3 Efficiency of Simple Auctions Parametrized by SMW

As a byproduct of our efficiency results for the SAW hierarchy, we prove similar, but slightly weaker, results for the SMW hierarchy. We note that these bounds extend a central result in [11], which states that when agents have valuations in max(SD-$d \cap$ SUPADD), Single-bid Auction has a PoA of $O(d^2 \log m)$.

▶ **Theorem 4.13** (Extending [11])**.** *When agents have valuations $f_1, \ldots, f_n \in$* max(SMW-$d \cap$ SUPADD)*, Single-bid Auction has a price of anarchy of at most $\frac{(d+1)^2}{1-e^{-(d+1)}} \cdot H_{\frac{m}{d+1}}$ w.r.t. coarse correlated equilibria.*

▶ **Theorem 4.14.** *When agents have valuations $f_1, \ldots, f_n \in$* max(SMW-$d \cap$ SUPADD)*, SIA has a price of anarchy of at most $2(d + 1)^2 \cdot H_{\frac{m}{d+1}}$ w.r.t. coarse correlated equilibria.*

Due to space limit, we relegate the proofs to the full version of the paper.

5 Remarks

5.1 Further Comparative Analysis

As observed by Eden *et al.* [6], the right measure of complementarity often varies from application to application. This seems to be true even with the supermodular vs superadditive widths. We note that while the SD and SMW hierarchies give nontrivial bounds on the PoA

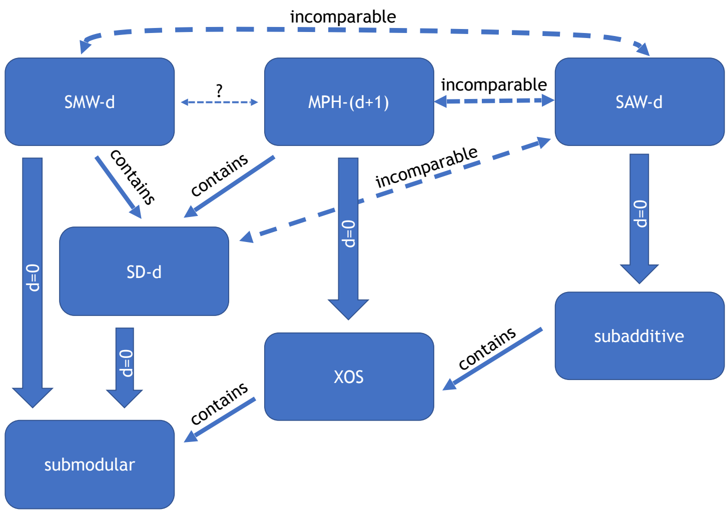

Figure 1 Relationship between hierarchies.

of simple auctions, SAW hierarchy seems to capture the intrinsic property needed by efficiency guarantees for simple auctions. It provides tighter characterization of PoA with a gap of $\log m$ (instead of $d \log m$) between upper and lower bounds. On the other hand, while SMW hierarchy captures the intrinsic property needed by the constrained/welfare maximization, it remains open whether a small superadditive width provides any approximation guarantee for the two optimization problems.

The MPH hierarchy takes a different approach from ours — it relies on a syntactic definition which provides elegant and intuitive structures. In contrast, both SMW and SAW hierarchies — like the SD hierarchy before it — are built on concrete natural concepts of witnesses and semantic intuition of complementarity. In the current definition, the MPH hierarchy is not an extension to submodularity or subadditivity. Rather — as shown in [9] — MPH can be considered as an extension to the fractionally subadditive (or XOS) class proposed in [19]. We therefore consider SMW, MPH and SAW parallel measures of complementarity, just like submodularity, fractional subadditivity and subadditivity in the complement-free case. One key difference is that the three hierarchies seem to diverge at higher levels of complementarity, as opposed to the fact that submodular functions are all fractionally subadditive, and fractionally subadditive functions are all subadditive. This phenomenon provides further evidence that the three hierarchies are likely to capture different aspects of complementarity. See Figure 1 for a comparison.

We also note that all upper bounds supported by our hierarchies are accompanied by almost matching lower bounds, which we consider as a justification of our definitions — they manage to categorize set functions roughly according to their "hardness" in different settings (i.e. optimization for SMW and efficiency for SAW). In contrast, while the less inclusive supermodular degree hierarchy supports a number of upper bounds, to our knowledge, none of those results are proven tight.

5.2 Final Remarks and Open Problems

Our SMW and SAW hierarchies may be applied to other problem settings. For example, for the online secretary problem based on supermodular degree [15], we believe that with a slight modification of the algorithms and the analysis, we could replace supermodular degree with supermodular width as well for this problem; also, SMW-d functions are efficiently PAC-learnable under product distributions [30]. It may be possible to look into other venues where SMW and SAW hierarchies are applicable.

There are also a few technical questions to be answered:

- Does MPH-$(d+1)$ — which subsumes SD-d — include all SMW-d functions?
- Can we improve the SAW-based efficiency characterization of Single-bid Auction and SIA to $O(d)$?
- Can the MPH hierarchy be used to characterize constrained set function maximization?

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