Converse bounds for private communication over quantum channels

Mark M. Wilde (LSU)

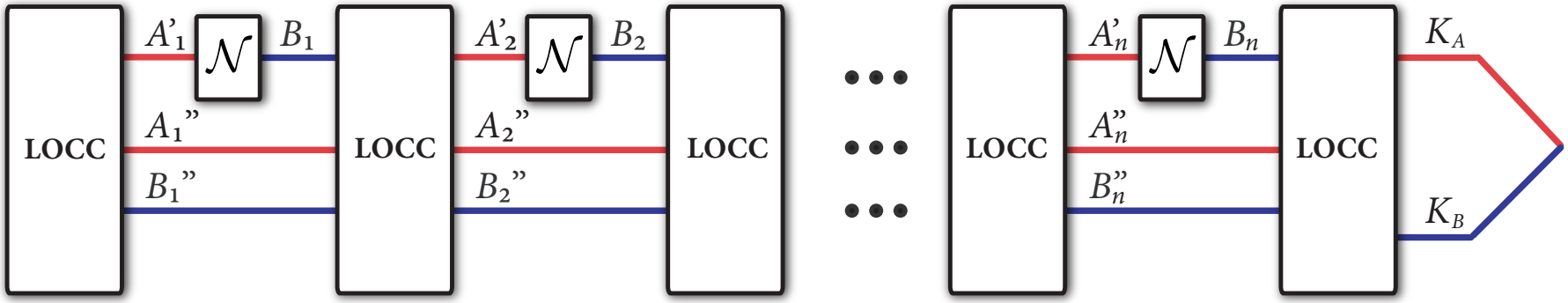
joint work with Mario Berta (Caltech) and
Marco Tomamichel ($|$Univ. Sydney$\rangle$ + $|$Univ. of Technology, Sydney$\rangle$)

arXiv:1602.08898
accepted for publication in IEEE Trans. Inf. Theory
DOI: 10.1109/TIT.2017.2648825

QIP 2017 Seattle - January 20, 2017

Setup I

- Given a quantum channel $\mathcal{N}$ and a quantum key distribution (**QKD**) protocol that uses it n times, how much **key** can be generated?



- **Non-asymptotic private capacity**: maximum rate of ε-close secret key achievable using the channel n times with two-way classical communication (LOCC) assistance

$$\hat{P}_{\mathcal{N}}^{\leftrightarrow}(n,\varepsilon) := \sup\left\{P : (n,P,\varepsilon) \text{ is achievable for } \mathcal{N} \text{ using LOCC}\right\}. \quad (1)$$

Setup II

- Practical question: how to characterize $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon)$ for all $n \geq 1$ and $\varepsilon \in (0,1)$? The answers give the **fundamental limitations of QKD**.

- Upper bounds on $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon)$ can be used as **benchmarks for quantum repeaters** [Lütkenhaus].

- Today, I will present

  the tightest known upper bound on $\hat{P}^{\leftrightarrow}_{\mathcal{N}}(n,\varepsilon)$

  for several channels of practical interest. Interesting special case: single-mode phase-insensitive bosonic Gaussian channels.

Overview

1. Main Results (Examples)
2. Proof Idea: Meta Converse
3. Conclusion

Main Result: Gaussian Channels I

- Converse bounds for single-mode phase-insensitive bosonic Gaussian channels, most importantly the **photon loss channel**

$$\mathcal{L}_\eta : \ \hat{b} = \sqrt{\eta}\hat{a} + \sqrt{1-\eta}\hat{e} \tag{2}$$

  where transmissivity $\eta \in [0, 1]$ and environment in vacuum state.

- Our approach gives a complete proof for the following weak converse bound, stated in [Pirandola *et al.* 2016]:

$$P^{\leftrightarrow}(\mathcal{L}_\eta) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \hat{P}^{\leftrightarrow}_{\mathcal{N}_\eta}(n, \varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) , \tag{3}$$

  which is actually tight in the asymptotic limit, i.e., $P^{\leftrightarrow}(\mathcal{N}_\eta) = \log\left(\frac{1}{1-\eta}\right)$.

- Drawback: an asymptotic statement, and thus says **little for practical protocols** (called a weak converse bound).

Main Result: Gaussian Channels II

- We show the **non-asymptotic converse bound**

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n}, \tag{4}$$

  where $C(\varepsilon) := \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$ (other choices possible).

- Can be used to **assess the performance of any practical quantum repeater** which uses a loss channel n times for desired security ε.

- Other variations of this bound are possible if η is not the same for each channel use, if η is chosen adversarially, etc.

- We give similar bounds for the quantum-limited amplifier channel (tight), thermalizing channels, amplifier channels, and additive noise channels.

Main Result: Dephasing Channels I

- Asymptotic result [Pirandola *et al.* 2016] for the **qubit dephasing channel**

$$\mathcal{Z}_\gamma : \rho \mapsto (1-\gamma)\,\rho + \gamma Z \rho Z$$

with $\gamma \in (0,1)$ is

$$P^{\leftrightarrow}(\mathcal{Z}_\gamma) := \lim_{\varepsilon \to 0} \lim_{n \to \infty} \hat{P}^{\leftrightarrow}_{\mathcal{Z}_\gamma}(n,\varepsilon) = 1 - h(\gamma)\,, \tag{5}$$

with the binary entropy $h(\gamma) := -\gamma \log \gamma - (1-\gamma)\log(1-\gamma)$.

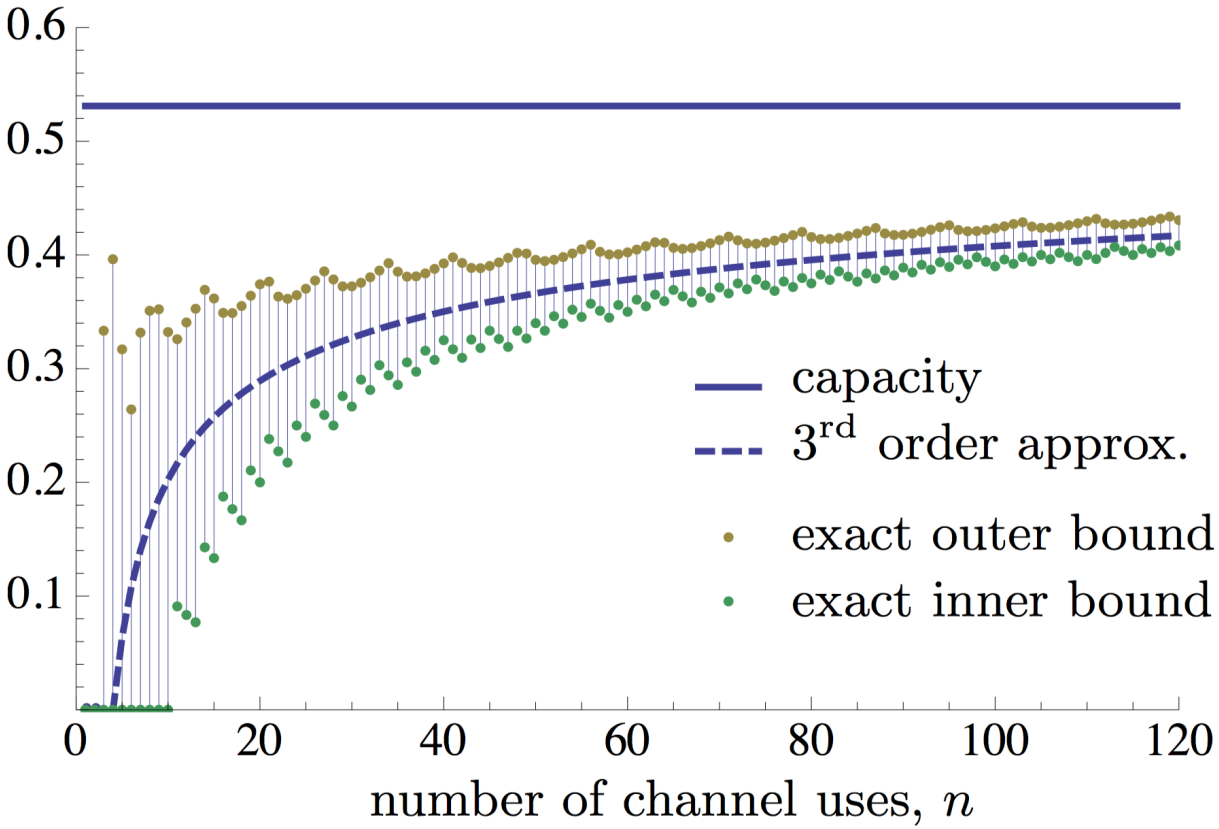
- By combining with [Tomamichel *et al.* 2016] we show the expansion

$$\hat{P}^{\leftrightarrow}_{\mathcal{Z}_\gamma}(n,\varepsilon) = 1 - h(\gamma) + \sqrt{\frac{v(\gamma)}{n}}\,\Phi^{-1}(\varepsilon) + \frac{\log n}{2n} + O\left(\frac{1}{n}\right)\,, \tag{6}$$

with Φ the cumulative standard Gaussian distribution and the binary entropy variance $v(\gamma) := \gamma(\log\gamma + h(\gamma))^2 + (1-\gamma)(\log(1-\gamma) + h(\gamma))^2$.

Main Result: Dephasing Channels II

- For the dephasing parameter $\gamma = 0.1$ we get (figure from [Tomamichel *et al.* 2016]):



(c) Comparison of strict bounds with third order approximation for $\varepsilon = 5\%$.

Proof Idea: Meta Converse I

- **Meta converse approach** from classical channel coding [Polyanskiy *et al.* 2010], uses connection to **hypothesis testing**. In the quantum regime, e.g., for classical communication [Tomamichel & Tan 2015] or quantum communication [Tomamichel *et al.* 2014 & 2016]. We extend this approach to **private communication**.
- Hypothesis testing relative entropy defined for a state ρ, positive semi-definite operator σ, and $\varepsilon \in [0,1]$ as

$$D_H^\varepsilon(\rho\|\sigma) := -\log \inf\left\{ \operatorname{Tr}[\Lambda\sigma] : 0 \leq \Lambda \leq I \wedge \operatorname{Tr}[\Lambda\rho] \geq 1-\varepsilon \right\}. \tag{7}$$

- The ε-relative entropy of entanglement is defined as

$$E_R^\varepsilon(A;B)_\rho := \inf_{\sigma_{AB}\in\mathcal{S}(A:B)} D_H^\varepsilon(\rho_{AB}\|\sigma_{AB}), \tag{8}$$

where $\mathcal{S}(A:B)$ is the set of separable states (cf. relative entropy of entanglement). **Channel's ε-relative entropy of entanglement** is then given as

$$E_R^\varepsilon(\mathcal{N}) := \sup_{|\psi\rangle_{AA'}\in\mathcal{H}_{AA'}} E_R^\varepsilon(A;B)_\rho, \tag{9}$$

where $\rho_{AB} := \mathcal{N}_{A'\to B}(\psi_{AA'})$.

Proof Idea: Meta Converse II

- Goal is the creation of $\log K$ **bits of key**, i.e., states γ_{ABE} with

$$(\mathcal{M}_A \otimes \mathcal{M}_B)(\gamma_{ABE}) = \frac{1}{K} \sum_i |i\rangle\langle i|_A \otimes |i\rangle\langle i|_B \otimes \sigma_E \tag{10}$$

  for some state σ_E and measurement channels $\mathcal{M}_A, \mathcal{M}_B$.

- In **one-to-one correspondence** with pure states $\gamma_{AA'BB'E}$ such that [Horodecki *et al.* 2005 & 2009]

$$\gamma_{ABA'B'} = U_{ABA'B'}(\Phi_{AB} \otimes \theta_{A'B'})U^\dagger_{ABA'B'}\,, \tag{11}$$

  where Φ_{AB} maximally entangled, $U_{ABA'B'} = \sum_{i,j} |i\rangle\langle i|_A \otimes |j\rangle\langle j|_B \otimes U^{ij}_{A'B'}$ with each $U^{ij}_{A'B'}$ a unitary, and $\theta_{A'B'}$ a state.

- Work in the latter, bipartite picture.

Proof Idea: Meta Converse III

- Let $\varepsilon \in [0,1]$ and let $\rho_{ABA'B'}$ be an ε-approximate γ-private state. The probability for $\rho_{ABA'B'}$ to pass the "γ-privacy test" satisfies

$$\operatorname{Tr}\{\Pi_{ABA'B'}\rho_{ABA'B'}\} \geq 1-\varepsilon, \tag{12}$$

  where $\Pi_{ABA'B'} \equiv U_{ABA'B'}(\Phi_{AB} \otimes I_{A'B'})U^{\dagger}_{ABA'B'}$ is a projective "γ-privacy test."
- For **separable states** $\sigma_{AA'BB'}$ (useless for private communication) and a state $\gamma_{AA'BB'}$ with $\log K$ bits of key we have [Horodecki *et al.* 2009]

$$\operatorname{Tr}\{\Pi_{ABA'B'}\sigma_{AA'BB'}\} \leq \frac{1}{K}, \tag{13}$$

- The monotonicity of the channel's ε-relative entropy of entanglement $E_R^{\varepsilon}(\mathcal{N})$ with respect to LOCC together with (13) implies the **meta converse**

$$\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N}) \quad \text{(LOCC pre- and post-processing assistance).} \tag{14}$$

  For n channel uses this gives $\hat{P}_{\mathcal{N}}(n,\varepsilon) \leq \frac{1}{n}E_R^{\varepsilon}\left(\mathcal{N}^{\otimes n}\right)$.
- Finite block-length version of **relative entropy of entanglement** upper bound [Horodecki *et al.* 2005 & 2009].
- One can then **evaluate** the meta converse for specific channels of interest.

Conclusion

- Our meta converse $\hat{P}_{\mathcal{N}}(1,\varepsilon) \leq E_R^{\varepsilon}(\mathcal{N})$ gives bounds for the private transmission capabilities of quantum channels. These give the **fundamental limitations of QKD** and thus can be used as **benchmarks for quantum repeaters**.
- Can our bound be improved for the **photon loss channel**

$$\hat{P}^{\leftrightarrow}_{\mathcal{L}_\eta}(n,\varepsilon) \leq \log\left(\frac{1}{1-\eta}\right) + \frac{C(\varepsilon)}{n} \quad \text{with} \quad C(\varepsilon) = \log 6 + 2\log\left(\frac{1+\varepsilon}{1-\varepsilon}\right) \tag{15}$$

  to $C'(\varepsilon) := \log\left(\frac{1}{1-\varepsilon}\right)$?
- Corresponding matching **achievability**? (Tight analysis of random coding in infinite dimensions needed.)
- Tight **finite-energy** bounds for single-mode phase-insensitive bosonic Gaussian channels?
- Understand more channels, for example such with $P^{\leftrightarrow} > 0$ but zero quantum capacity [Horodecki *et al.* 2008]?

Extra: Gaussian Formulas

- For **Gaussian channels** we need formulas for the relative entropy $D(\rho\|\sigma)$ and the relative entropy variance $V(\rho\|\sigma)$.
- From [Chen 2005, Pirandola *et al.* 2015] and [Wilde *et al.* 2016], respectively: writing zero-mean Gaussian states in exponential form as

$$\rho = Z_\rho^{-1/2} \exp\left\{-\frac{1}{2}\hat{x}^T G_\rho \hat{x}\right\} \quad \text{with} \tag{16}$$

$$Z_\rho := \det(V^\rho + i\Omega/2), \quad G_\rho := 2i\Omega \operatorname{arcoth}(2V^\rho i\Omega)\,, \tag{17}$$

and V^ρ the Wigner function covariance matrix for ρ, we have

$$D(\rho\|\sigma) = \frac{1}{2}\left(\log\left(\frac{Z_\sigma}{Z_\rho}\right) - \operatorname{Tr}\left[\Delta V^\rho\right]\right) \tag{18}$$

$$V(\rho\|\sigma) = \frac{1}{2}\operatorname{Tr}\{\Delta V^\rho \Delta V^\rho\} + \frac{1}{8}\operatorname{Tr}\{\Delta\Omega\Delta\Omega\}\,, \tag{19}$$

where $\Delta := G_\rho - G_\sigma$.