Geometric inequalities and contractivity of bosonic semigroups

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based on
arXiv:1606.08603 a work with
Stefan Huber, Robert König (TU Munich)

and arXiv:1607.04242 results of
Nilanjana Datta, Cambyse Rouzé (University of Cambridge) and Yan Pautrat (University Paris-Sud)

Quantum Information Processing 2017

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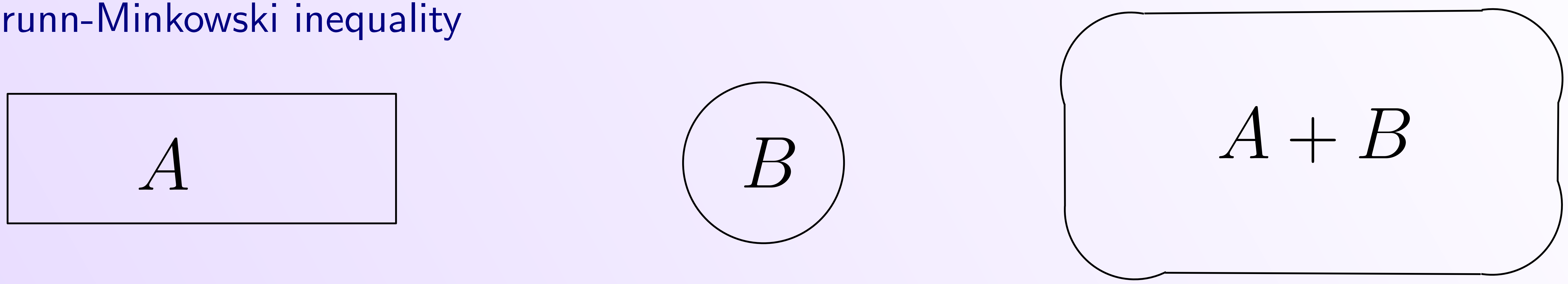
Outline of the talk

- Introduction: Geometric inequalities in information theory
- Functional inequalities for classical-quantum convolution
- Quantum entropy power inequality
- Quantum diffusion semigroup
    - Functional inequalites
    - Ultracontractivity
    - Convergence to a space of Gaussian states
- Quantum Ornstein-Uhlenbeck semigroup
- Quantum Log Sobolev Inequality
- Open questions

Geometric inequalities

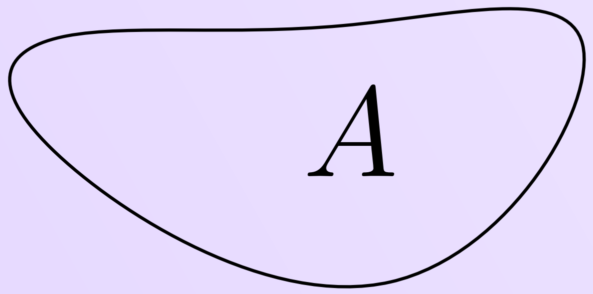
Brunn-Minkowski inequality



$A+B=\{a+b|a\in A, b\in B\}$

$$\mathrm{vol}(A)^{1/n} \quad + \quad \mathrm{vol}(B)^{1/n} \quad \leq \quad \mathrm{vol}(A+B)^{1/n}$$

Isoperimetric inequality



$$\mathrm{vol}(A)^{\frac{n-1}{n}} \leq \frac{1}{n\,\mathrm{vol}(B_1)^{\frac{1}{n}}}\mathrm{area}(\partial A)$$

B_1 → unit ball

Geometry vs Information theory

Set $A \subset \mathbb{R}$

volume $\mathrm{vol}(A)$

Random variable X on $\Omega = \{x_j\}_j \subset \mathbb{R}$
with prob. mass function $p_x = \Pr[X = x]$

entropy power $2^{H(X)}$

Shannon entropy $H(X) = -\sum p_x \log p_x$

Geometry vs Information theory

Set $A \subset \mathbb{R}$

volume $\mathrm{vol}(A)$

Consider a set $A^n = A \times \cdots \times A \in \mathbb{R}^n$

$\mathrm{vol}(A^n) = (\mathrm{vol}(A))^n$

Random variable X on $\Omega = \{x_j\}_j \subset \mathbb{R}$
with prob. mass function $p_x = \Pr[X = x]$

entropy power $2^{H(X)}$

Shannon entropy $H(X) = -\sum p_x \log p_x$

Consider n i.i.d. r.v. with $P_{X^n} = P_X \cdot \cdots \cdot P_X$

has the following property:

$\forall \epsilon > 0 \qquad \exists M_{n,\epsilon} \subset \Omega^n$ s.t.

$|M_{n,\epsilon}| \sim c_\epsilon \left(2^{H(X)}\right)^n$

and $\Pr[X^n \in M_{n,\epsilon}] \geq 1 - \epsilon$

Geometry vs Information theory

Set $A \in \mathbb{R}^n$

volume $\mathrm{vol}(A)$

Random variable X on $\mathbb{R}^n$
with prob. density function f_X

entropy power $e^{2H(X)/n}$

entropy $H(X) = -\int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$

Geometry vs Information theory

Set $A \in \mathbb{R}^n$

volume $\mathrm{vol}(A)$

Random variable X on $\mathbb{R}^n$
with prob. density function f_X

entropy power $e^{2H(X)/n}$ → variance of Gaussian r.v. Z with $H(Z) = H(X)$

entropy $H(X) = -\int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$

Geometry vs Information theory

Set $A \in \mathbb{R}^n$

volume $\mathrm{vol}(A)$

addition $A + B$

Random variable X on $\mathbb{R}^n$
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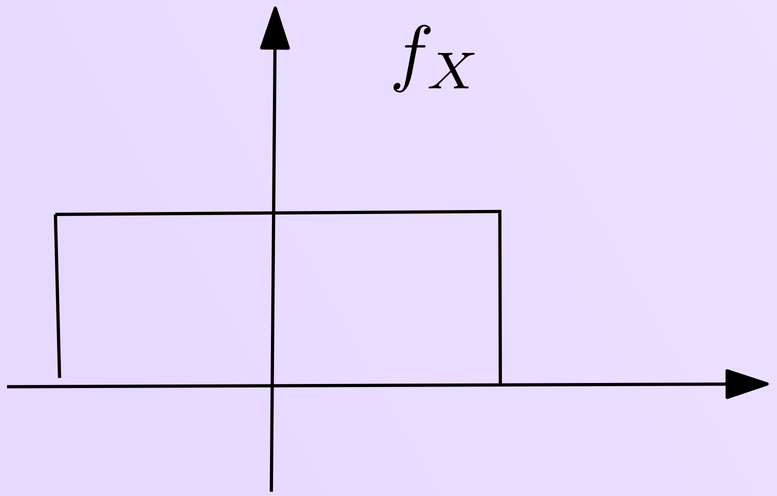
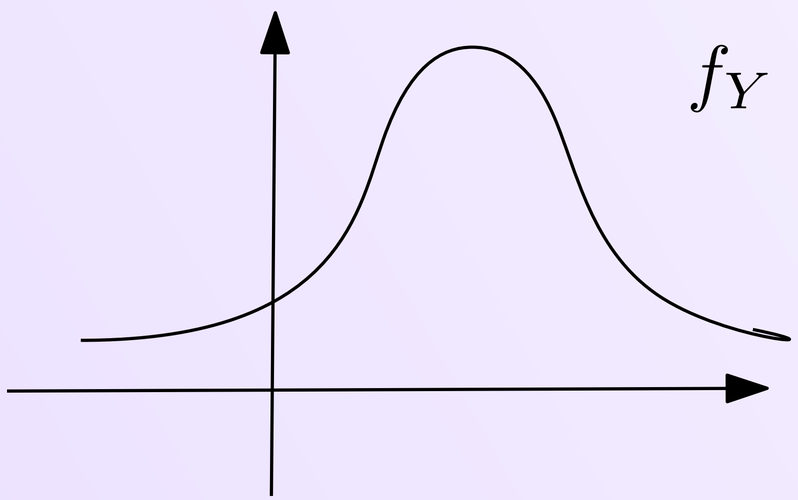
entropy power $e^{2H(X)/n}$ → variance of Gaussian r.v. Z with $H(Z) = H(X)$

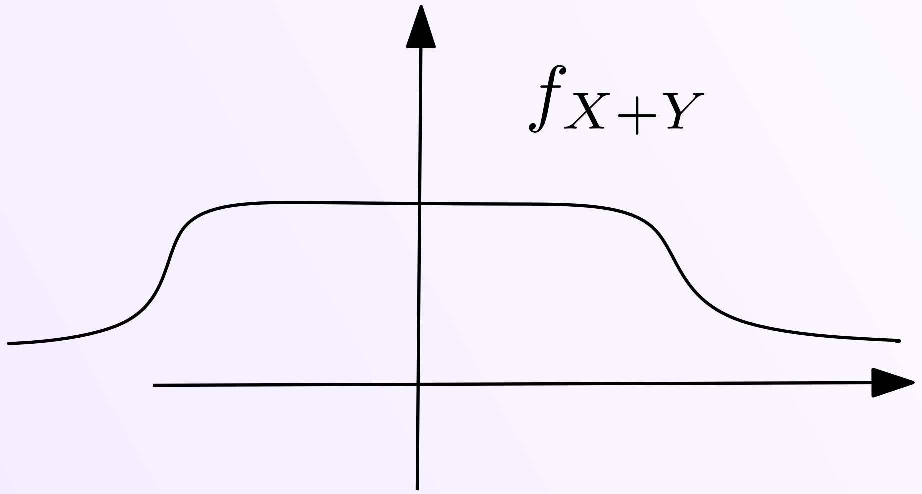
entropy $H(X) = -\int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$

convolution: $X + Y$ has a density function

$$f_{X+Y}(x) = \int f_X(x - z) f_Y(z) dz$$







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Brunn-Minkowski inequality

$$\mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n} \leq \mathrm{vol}(A+B)^{1/n}$$

Shannon's entropy power inequality ['48]

$$e^{2H(X)/n} + e^{2H(Y)/n} \leq e^{2H(X+Y)/n}$$

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$$\mathrm{vol}(A)^{\frac{n-1}{n}} \leq \frac{1}{n\,\mathrm{vol}(B_1)^{\frac{1}{n}}}\mathrm{area}(\partial A)$$

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Proof

n-th power of Brunn-Minkowski ineq.

$$\left(\mathrm{vol}(A)^{1/n} + \mathrm{vol}(\epsilon B_1)^{1/n}\right)^n \leq \mathrm{vol}(A+\epsilon B_1)$$

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$$\left(\mathrm{vol}(A)^{1/n} + \epsilon\, \mathrm{vol}(B_1)^{1/n}\right)^n - \mathrm{vol}(A) \leq \mathrm{vol}(A+\epsilon B_1) - \mathrm{vol}(A)$$

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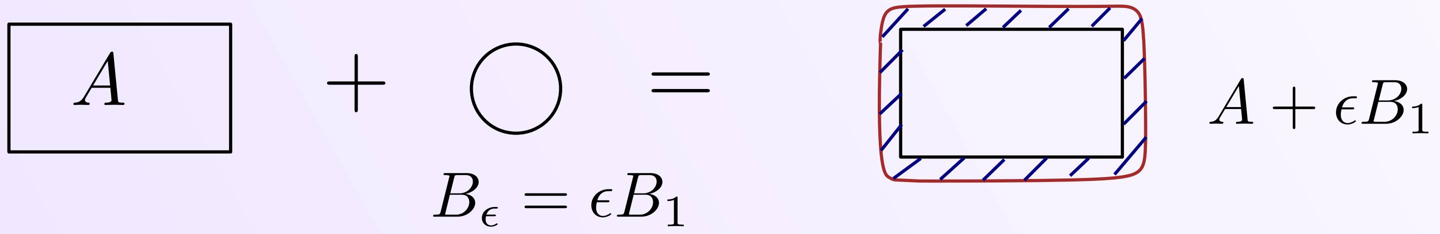
→ $\mathrm{area}(\partial A)$



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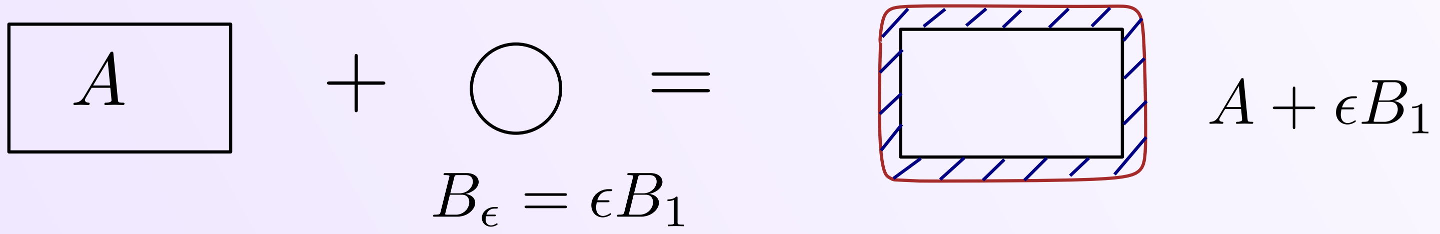
$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} \left(\mathrm{vol}(A)^{1/n} + \epsilon\,\mathrm{vol}(B_1)^{1/n}\right)^n$$

$$\mathrm{area}(\partial A)$$

$$n\,\mathrm{vol}(A)^{\frac{n-1}{n}}\,\mathrm{vol}(B_1)^{\frac{1}{n}}$$



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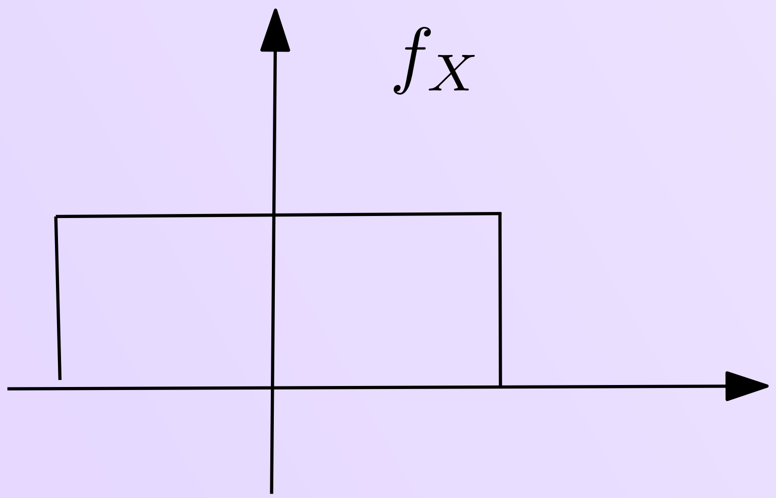
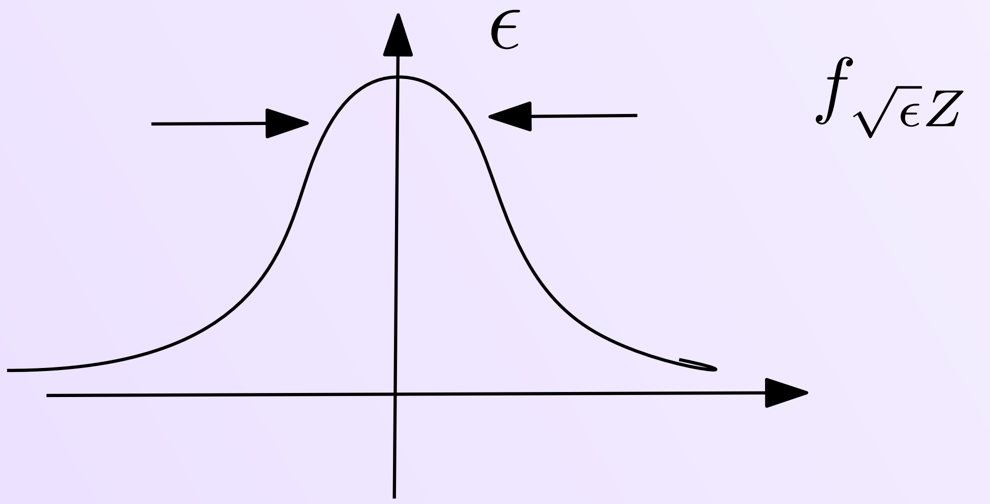
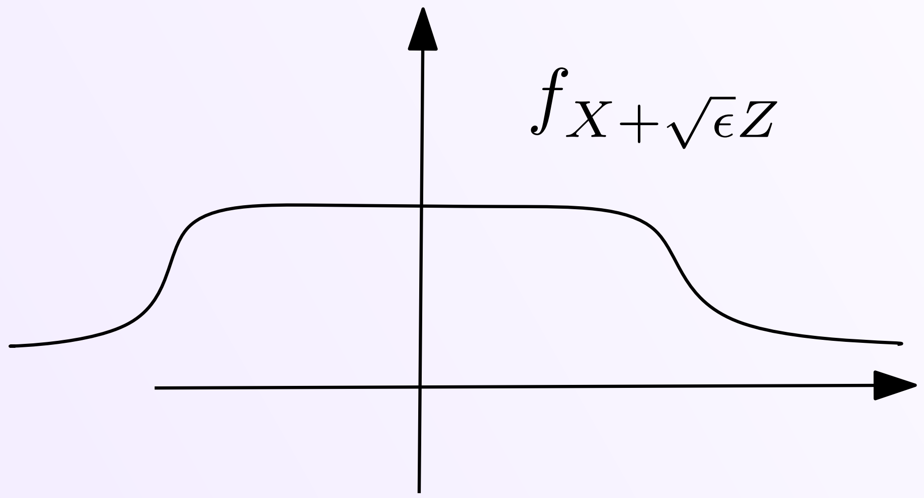
$$e^{2H(X)/n} + e^{2H(Y)/n} \leq e^{2H(X+Y)/n}$$

“surface area“ $\lim_{\epsilon\to 0} \frac{e^{2H(X+\sqrt{\epsilon}Z)/n} - e^{2H(X)/n}}{\epsilon}$







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$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} e^{2H(X+\sqrt{\epsilon}Z)/n} = \frac{2}{n} e^{2H(X)/n}\ \frac{d}{d\epsilon}\Big|_{\epsilon=0} H(X+\sqrt{\epsilon}Z)$$

Fisher information

$$J(X) = 2\ \frac{d}{d\epsilon}\Big|_{\epsilon=0} H(X+\sqrt{\epsilon}Z)$$

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$$e^{2H(X)/n} + e^{2H(Y)/n} \leq e^{2H(X+Y)/n}$$

Isoperimetric inequality for entropies

$$\frac{1}{n}J(X)e^{2H(X)/n} \geq 2\pi\, e$$

$$\lim_{\epsilon\to 0} \frac{e^{2H(X+\sqrt{\epsilon}Z)/n} - e^{2H(X)/n}}{\epsilon} \geq \lim_{\epsilon\to 0} \frac{e^{2H(X)/n} + e^{2H(\sqrt{\epsilon}Z)/n} - e^{2H(X)/n}}{\epsilon} = 2\pi\, e$$

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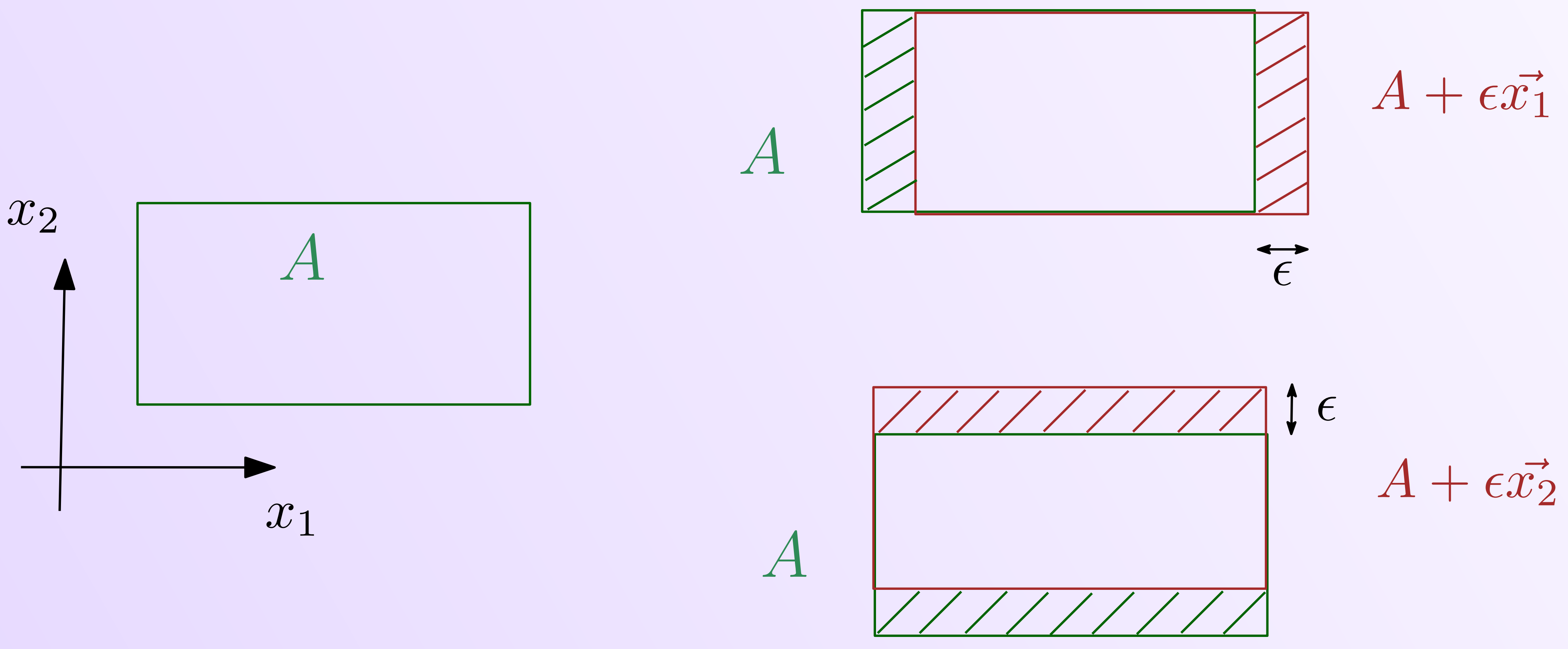
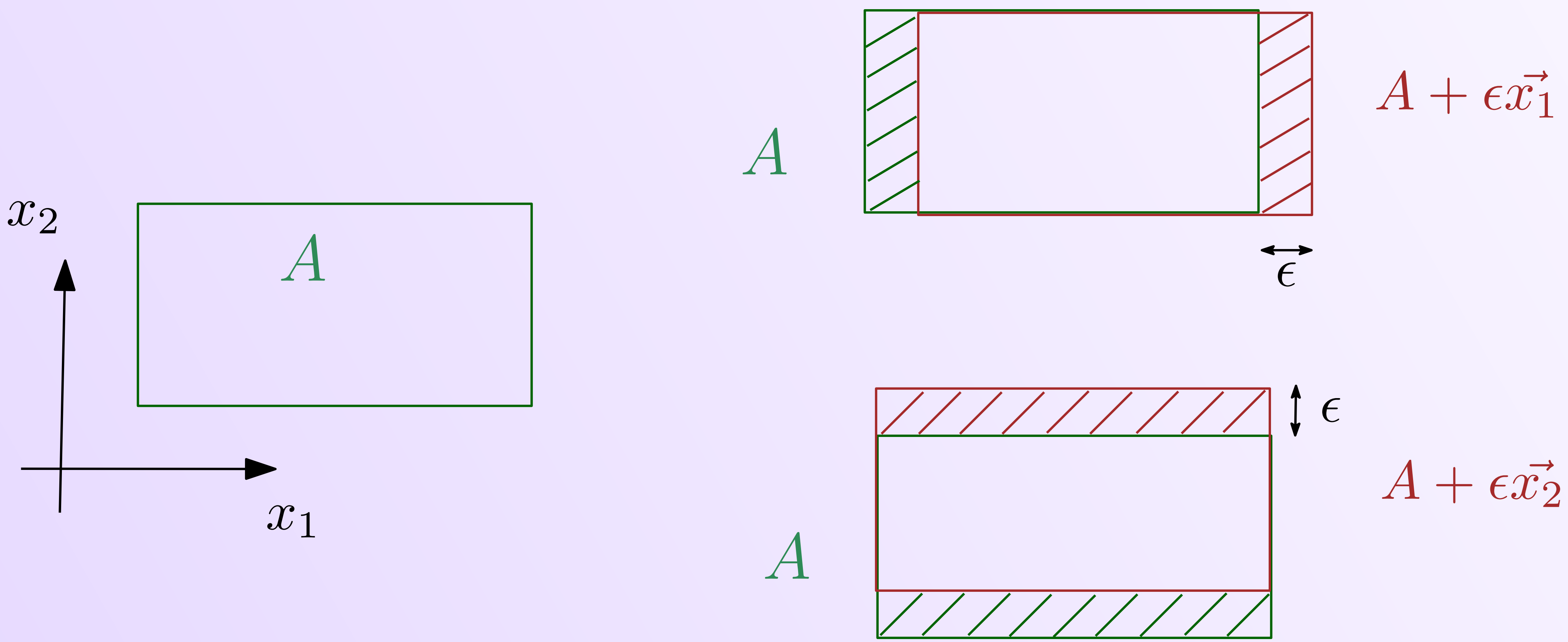
Geometric analogue of Fisher information



$D(A_1, A_2) =$ "difference" between A_1 and A_2

"Fisher Information" $= \frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \sum_{j=1}^{2} D(A, A + \epsilon \vec{x_j})$

Geometric analogue of Fisher information



distribution translated in $\vec{x_j}$ direction by ϵ

$D(A_1, A_2) =$ "difference" between A_1 and A_2

$$J(X) = \sum_{j=1}^{n} \frac{\partial^2}{\partial\epsilon^2}\Big|_{\epsilon=0} D(f||f^{(\epsilon\vec{x_j})})$$

de Bruijn's identity

"Fisher Information" $= \frac{d^2}{d\epsilon^2}\Big|_{\epsilon=0} \sum_{j=1}^{2} D(A, A+\epsilon\vec{x_j})$

$$= 2\frac{d}{d\epsilon}\Big|_{\epsilon=0} H(X+\sqrt{\epsilon}Z)$$

Geometry	Classical	Quantum
Set $A \in \mathbb{R}^n$	Random variable X on $\mathbb{R}^n$ with prob. density function f_X	ρ - n-mode state of a continuous variable q.s. $[Q_j, P_k] = i\delta_{j,k} I$ $[Q_j, Q_k] = [P_j, P_k] = 0 \qquad 1 \leq j, k, \leq n$
volume $\mathrm{vol}(A)$	entropy power $e^{2H(X)/n}$	$e^{S(\rho)/n}$
	$H(X) = -\int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$	$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$
addition $A + B$	convolution: $X + Y$	
	$f_{X+Y}(x) = \int f_X(x - z) f_Y(z) dz$	

Geometry

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$X + Y$: ρ_{X+Y} ??

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addition $A + B$	convolution: $X + Y$	$\rho_{X \boxplus_\lambda Y} = \mathrm{Tr}_2 \left(U_\lambda (\rho_X \otimes \rho_Y) U_\lambda^* \right)$
	$f_{X+Y}(x) = \int f_X(x-z) f_Y(z) dz$	beamsplitter with transmissivity $\lambda \in [0,1]$
		$Q_{X \boxplus_\lambda Y} = \sqrt{\lambda} Q_X + \sqrt{1-\lambda} Q_Y$ $P_{X \boxplus_\lambda Y} = \sqrt{\lambda} P_X + \sqrt{1-\lambda} P_Y$ $\qquad n = 1$ $U_\lambda^{(n)} = \left(U_\lambda^{(1)} \right)^{\otimes n}$

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beamsplitter with transmissivity $\lambda \in [0, 1]$

Quantum entropy power inequality

$$\lambda e^{S(X)/n} + (1-\lambda) e^{S(Y)/n} \leq e^{S(X \boxplus_\lambda Y)/n}$$

[König, Smith 14] $\lambda = 1/2$

[de Palma et al 15]

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classical-quantum

$$f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W(\sqrt{t}\xi)^* d\xi$$

[Werner '84] $t=1$ ⟶ form of Young's inequality

[Huber, König, V '16]

[Datta, Pautrat, Rouzé '16] $f = f_Z \sim \mathcal{N}(0,1)$

Geometry	Classical	Quantum
Set $A \in \mathbb{R}^n$	Random variable X on $\mathbb{R}^n$ with prob. density function f_X	ρ - n-mode state of a continuous variable q.s. $[Q_j, P_k] = i\delta_{j,k} I$ $[Q_j, Q_k] = [P_j, P_k] = 0 \quad 1 \le j, k, \le n$
volume $\mathrm{vol}(A)$	entropy power $e^{2H(X)/n}$	$e^{S(\rho)/n}$
	$H(X) = -\int_{\mathbb{R}^n} f_X(x) \log f_X(x) dx$	$S(\rho) = -\mathrm{Tr}(\rho \log \rho)$
addition $A + B$	convolution: $X + Y$	classical-quantum
	$f_{X+Y}(x) = \int f_X(x-z) f_Y(z) dz$	$f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W(\sqrt{t}\xi)^* d\xi$
Brunn-Minkowski inequality	Shannon entropy power inequality	Classical-quantum entropy power inequality
$\mathrm{vol}(A)^{1/n} + \mathrm{vol}(B)^{1/n} \le \mathrm{vol}(A+B)^{1/n}$	$e^{2H(X)/n} + e^{2H(Y)/n} \le e^{2H(X+Y)/n}$	$te^{H(f)/n} + e^{S(\rho)/n} \le e^{S(f \star_t \rho)/n} \qquad t \ge 0$
Isoperimetric inequality	Isoperimetric inequality for entropies	Isoperimetric inequality for entropies
$\mathrm{vol}(A)^{\frac{n-1}{n}} \le \frac{1}{n \mathrm{vol}(B_1)^{\frac{1}{n}}} \mathrm{area}(\partial A)$	$\frac{1}{n} J(X) e^{2H(X)/n} \ge 2\pi e$	$\frac{1}{n} J(\rho) e^{S(\rho)/n} \ge 4\pi e$

Classical vs Quantum information theory

Classical

X - $\mathbb{R}^n$-valued r.v. with a prob. density function f_X.

Relative entropy

$$D(f||g) = \int f(x)(\log f(x) - \log g(x))dx$$

Quantum

ρ - n-mode state of a continuous variable q.s.

$[Q_j, P_k] = -[P_k, Q_j] = i\delta_{j,k} I$

$[Q_j, Q_k] = [P_j, P_k] = 0 \qquad 1 \leq j, k, \leq n$

$$D(\rho||\sigma) = \mathrm{Tr}(\rho \log \rho - \rho \log \sigma)$$

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Fisher information matrix

For $\theta \in \mathbb{R}^n$, define $\quad f^{(\theta)}(x) = f(x-\theta)$

$$J(f^{(\theta)})\big|_{\theta=\theta_0} = \left(\frac{\partial^2}{\partial\theta_i \partial\theta_j}\Big|_{\theta=\theta_0} D\left(f^{(\theta_0)}||f^{(\theta)}\right)\right)_{i,j=1}^{2n}$$

Fisher information

$$J(f) = \mathrm{Tr}\left(J(f^{(\theta)})\big|_{\theta=\theta_0}\right)$$

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For $\theta \in \mathbb{R}^{2n}$, define $\rho^{(\theta)}$?

Fisher information

$$J(f) = \mathrm{Tr}\left(J(f^{(\theta)})\big|_{\theta=\theta_0} \right)$$

Quantum Fisher information

For $\theta \in \mathbb{R}^{2n}$ the Weyl displacement operators are defined as

$$W(\theta) = \exp\{i\sqrt{2\pi}\,(\theta_1 P_1 - \theta_2 Q_1 + \cdots + \theta_{2n-1} P_n - \theta_{2n} Q_n)\}$$

Translated state is

$$\rho^{(\theta)} = W(\theta)\rho W(\theta)^*$$

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Weyl operators translate position and momentum operators

$$W(\theta) Q_j W(\theta)^* = Q_j + \theta_{2j-1} I \qquad W(\theta) P_j W(\theta)^* = P_j + \theta_{2j} I$$

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Fisher information matrix is defined by

$$J(\rho^{(\theta)})\big|_{\theta=0} = \left(\frac{\partial^2}{\partial\theta_i \partial\theta_j}\Big|_{\theta=0} D\left(\rho || \rho^{(\theta)}\right)\right)_{i,j=1}^{2n}$$

(Divergence-based quantum) Fisher information is defined as

$$J(\rho) = \mathrm{Tr}\left(J(\rho^{(\theta)})\big|_{\theta=0} \right)$$

Properties of Quantum Fisher Information

[König, Smith '12]

[Petz '96]

[Lesniewski, Ruskai '99]

Non-negativity

For any state ρ

$$J(\rho) \geq 0$$

Reparametrization Formula

For any $\omega \in \mathbb{R}$ and $\theta_0 \in \mathbb{R}^{2n}$

$$J(\rho^{(\omega\theta)})\big|_{\theta=\theta_0} = \omega^2 J(\rho^{(\theta)})\big|_{\theta=\theta_0}$$

Data Processing for Fisher Information

For any CPTP map $\mathcal{T}$

$$J(\mathcal{T}(\rho)) \leq J(\rho)$$

Quantum-Classical Convolution

Define

$$(f,\rho) \to f \star_t \rho = \int_{\mathbb{R}^{2n}} f(\xi) W(\sqrt{t}\xi) \rho W(\sqrt{t}\xi)^* d\xi$$

[Werner ‘84] $t = 1$

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Rescaling

If f_X is a prob. density function of a r.v. X, then

$$f_X \star_t \rho = f_{\sqrt{t}X} \star_1 \rho$$

here the r.v. $\sqrt{t}X$ is given by $f_{\sqrt{t}X}(\zeta) = f_X(\zeta/\sqrt{t})/(\sqrt{t})^{2n}$

Addition

$$f_X \star_1 (f_Y \star_1 \rho) = f_{X+Y} \star_1 \rho$$

where $(f_X, f_Y) \to f_{X+Y}(z) = \int f_X(z-x) f_Y(x) dx$

Translation

Let $\omega_c, \omega_q > 0$, and $t \geq 0$. Then for all $\theta \in \mathbb{R}^{2n}$

$$(f \star_t \rho)^{(\omega\theta)} = f^{(\omega_c\theta)} \star_t \rho^{(\omega_q\theta)}$$

$$\omega = \sqrt{t}\omega_c + \omega_q$$

Quantum-Classical Convolution

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Data Processing Inequality

$$D\left(f \star_t \rho \| g \star_t \sigma\right) \leq D\left(f \| g\right) + D\left(\rho \| \sigma\right)$$

Inequalities for Convolution

Classical

$$(f_X, f_Y) \to f_{X+Y}(z) = \int f_X(z-x) f_Y(x) dx$$

Quantum

$$(f, \rho) \to f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W(\sqrt{t}\xi)^* d\xi$$

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$$(f_X, f_Y) \to f_{X+Y}(z) = \int f_X(z-x) f_Y(x) dx$$

- the Fisher information ineq.

  for $\lambda \in [0,1]$

$$J\left(\sqrt{\lambda}X + \sqrt{1-\lambda}Y\right) \leq \lambda J(X) + (1-\lambda) J(Y)$$

- Stam inequality

$$J(X+Y)^{-1} \geq J(X)^{-1} + J(Y)^{-1}$$

Quantum

$$(f, \rho) \to f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W(\sqrt{t}\xi)^* d\xi$$

[Huber, König, V '16]

for $\omega = \sqrt{t}\omega_c + \omega_q$

$$\omega^2 J(f \star_t \rho) \leq \omega_c^2 J(f) + \omega_q^2 J(\rho)$$

$$J(f \star_t \rho)^{-1} \geq t J(f)^{-1} + J(\rho)^{-1}$$

Gaussian States

Recall: the Weyl displacement operators have the property

$$W(\theta)R_jW(\theta)^* = R_j + \theta_j I$$

$$R = (Q_1, P_1, \ldots, Q_n, P_n)^T$$

$$\theta \in \mathbb{R}^{2n}$$

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$\theta \in \mathbb{R}^{2n}$

A state ρ is defined by its **characteristic function**

$$\chi_\rho(\theta) := \mathrm{Tr}(\rho W(\theta))$$

or equivalently by its **Wigner function**

$$f_\rho(z) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{iz\omega\theta} \chi_\rho(\theta) d\theta$$

where $\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Note that $\int_{\mathbb{R}^{2n}} f_\rho(z)dz = (2\pi)^n$

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A **Gaussian state** has a characteristic function of the form

$$\chi_\rho(\theta) = e^{i\mu\cdot\theta - \frac{1}{2}\theta\cdot M_\rho\theta}$$

where $\mu \in \mathbb{R}^{2n}$ is a mean vector: $\mu_j = \mathrm{Tr}(\rho R_j)$

M_ρ is a covariance matrix: $M_{ij} = \frac{1}{2}\mathrm{Tr}(\rho(R_i^c R_j^c + R_j^c R_i^c))$, with $R_j^c = R_j - \mu_j I$

Entropy Power Inequality

Classical-classical convolution $(f_X, f_Y) \to f_{X+Y}(z) = \int f_X(z-x) f_Y(x) dx$

[Shannon '48]

[Stam '59]

[Blachman '65]

$$e^{2H(X+Y)/n} \geq e^{2H(X)/n} + e^{2S(Y)/n}$$

Quantum-quantum convolution $(\rho_X, \rho_Y) \to \rho_{X \boxplus_\lambda Y} = \mathrm{Tr}_2 \left(U_\lambda (\rho_X \otimes \rho_Y) U_\lambda^* \right)$

$$e^{S(X \boxplus_\lambda Y)/n} \geq \lambda e^{S(X)/n} + (1-\lambda) e^{S(Y)/n}$$

[König, Smith 14]

[de Palma et al 15]

Classical-quantum convolution $(f, \rho) \to f \star_t \rho = \int f(\xi) W(\sqrt{t}\xi) \rho W(\sqrt{t}\xi)^* d\xi$

For some ρ, e.g. Gaussian states

$$e^{S(f \star_t \rho)/n} \geq t e^{H(f)/n} + e^{S(\rho)/n}$$

[Huber, König, V '16]

Isoperimetric Inequality

Classical

$$\frac{1}{n}J(X)e^{2H(X)/n} \geq 2\pi e$$

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Therefore

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Classical Heat Diffusion Semigroup

Quantum Diffusion Semigroup

For $Z = \mathcal{N}(0,1)$ - stand. Gaussian r.v. on $\mathbb{R}^{2n}$: $f_Z(\xi) = (2\pi)^{-n} e^{-\|\xi\|^2/2}$

$X_t = X + \sqrt{t}Z$ has a density function u_t

$u_t = f_{\sqrt{t}Z} \star u_0$ $\qquad$ $f_{\sqrt{t}Z} \sim \mathcal{N}(0,t)$

The function satisfies

$$\frac{\partial}{\partial t} u_t = \Delta u_t$$

for a Laplacian

$$\Delta = \frac{1}{2} \sum_{j=1}^{2n} \frac{\partial^2}{\partial x_j^2}$$

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$$= f_{\sqrt{t}Z} \star_1 \rho = \rho_t$$

The state satisfies

$$\frac{d}{dt} \rho_t = \mathcal{L}(\rho_t)$$

for a Liouvillean

$$\mathcal{L}(\rho) = -\pi \sum_{j=1}^{n} [Q_j, [Q_j, \rho]] + [P_j, [P_j, \rho]].$$

Quantum Dynamical Semigroup

The state of the system at time t is given by the solution of the Markovian master equation

$$\frac{d}{dt}\rho_t = \mathcal{L}(\rho_t)$$

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The dynamical maps Λ_t form a one-parameter completely-positive trace-preserving (CPTP) semigroup

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1) Λ_t is completely positive → for any $n > 1$, the maps $\Lambda_t \otimes id_{M_n}$ where M_n are $n \times n$ complex matrices, are positive

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The dynamical maps Λ_t form a one-parameter completely-positive trace-preserving (CPTP) semigroup

1) Λ_t is completely positive
2) $\mathrm{Tr}(\Lambda_t(\rho)) = \mathrm{Tr}(\rho)$
3) $\Lambda_t(I) = I$
4) $\Lambda_s\Lambda_t = \Lambda_{s+t}$
5) $\lim_{t\to 0} \|\Lambda_t(\rho) - \rho\| = 0$

Inequalities for Diffusion Semigroup

For Z - $\mathcal{N}(0,1)$ Gaussian r.v.

Classical

- Fisher information isoperimetric inequality

$$\frac{d}{dt}\bigg|_{t=0} \left\{ \left[\frac{1}{n} \left\{ J(X+\sqrt{t}Z) \right\} \right]^{-1} \right\} \geq 1$$

- Quantum Blachman-Stam inequality

For $\alpha, \beta > 0$ and $t > 0$,

$$(\alpha+\beta)^2 J(X+\sqrt{t}Z) \leq \alpha^2 J(X) + \frac{n\beta^2}{t}$$

Quantum

$$e^{t\mathcal{L}}(\rho) = f_Z \star_t \rho$$

$$\frac{d}{dt}\bigg|_{t=0} \left\{ \left[\frac{1}{2n} J(e^{t\mathcal{L}}(\rho)) \right]^{-1} \right\} \geq 1$$

[Huber, König, V '16]

[Datta, Pautrat, Rouzé '16]

$$(\alpha+\beta)^2 J(e^{t\mathcal{L}}(\rho)) \leq \alpha^2 J(\rho) + \frac{2n\beta^2}{t}$$

Inequalities for Diffusion Semigroup

Classical

For Z - $\mathcal{N}(0,1)$ Gaussian r.v.

Quantum

$$e^{t\mathcal{L}}(\rho) = f_Z \star_t \rho$$

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$$\frac{d}{dt}\Big|_{t=0} \left\{ \left[\frac{1}{n} \left\{ J(X+\sqrt{t}Z) \right\} \right]^{-1} \right\} \geq 1$$

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$$(\alpha+\beta)^2 J(e^{t\mathcal{L}}(\rho)) \leq \alpha^2 J(\rho) + \frac{2n\beta^2}{t}$$

For some ρ, e.g. Gaussian states:

- de Bruijn identity

$$J(X) = 2\frac{d}{dt}\Big|_{t=0} H\left(X+\sqrt{t}Z\right)$$

$$J(\rho) = 2\frac{d}{dt}\Big|_{t=0} S\left(e^{t\mathcal{L}}(\rho)\right)$$

[König, Smith '12]

- Concavity of the entropy power

$$\frac{d^2}{dt^2}\Big|_{t=0} \exp\{2H(X+\sqrt{t}Z)/n\} \leq 0$$

$$\frac{d^2}{dt^2}\Big|_{t=0} \exp\{S(e^{t\mathcal{L}}(\rho))/n\} \leq 0$$

[Huber, König, V '16]

[Datta, Pautrat, Rouzé '16]

Schwartz States

A Schwartz function $f : \mathbb{R}^n \to \mathbb{C}$ is an infinitely differentiable function for which:

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \nabla_x^\beta f(x)| < \infty \qquad \text{for all } \alpha, \beta \in \mathbb{N}^n$$

where $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$

$\nabla_x^\beta = \partial_{x_1}^{\beta_1} \ldots \partial_{x_n}^{\beta_n}$

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A Schwartz state ρ is a state whose characteristic function χ_ρ is a Schwartz function.

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e.g. Gaussian state

Nash Inequality

[Nash 58]

Classical

For $u = u(x,t)$, $x \in \mathbb{R}^n$

Dirichlet form of a diffusion semigroup

$$\mathcal{E}(u) = -\int_{\mathbb{R}^n} u\,\Delta u\,dx$$

[Datta, Pautrat, Rouzé '16]

Quantum

$$\mathcal{E}(\rho) = -\mathrm{Tr}(\rho\,\mathcal{L}(\rho))$$

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Nash inequality

there exists $c_n > 0$ s.t.

$$\left(\|u\|_2^2\right)^{1+\frac{2}{n}} \leq c_n \|u\|_1^{4/n} \mathcal{E}(u)$$

Quantum [Datta, Pautrat, Rouzé '16]

$$\mathcal{E}(\rho) = -\mathrm{Tr}(\rho\,\mathcal{L}(\rho))$$

For a Schwartz state of positive Wigner function

there exists $C_n > 0$ s.t.

$$\left(\|\rho\|_2^2\right)^{1+\frac{2}{2n}} \leq C_n \mathcal{E}(\rho)$$

where $\|\rho\|_2^2 = \mathrm{Tr}(\rho^2)$

Ultracontractivity

[Datta, Pautrat, Rouzé '16]

There exists $\kappa_n > 0$, s.t. for any $t > 0$, for any Schwartz state ρ with positive Wigner function for $\rho_t = e^{t\mathcal{L}}(\rho)$

$$\|\rho_t\|_\infty \leq \|\rho_t\|_2 \leq \kappa_n t^{-n/2}$$

with $\|\rho\|_p^p = \mathrm{Tr}(|\rho|^p)$

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with $\|\rho\|_p^p = \mathrm{Tr}(|\rho|^p)$

Purity	$\mathrm{Tr}(\rho_t^2) = \|\rho_t\|_2^2 \leq \kappa_n^2 t^{-n}$
Entropy	$S(\rho_t) \geq n \log(\frac{1}{2}\kappa_n^{-\frac{2}{n}} t)$
Largest Eigenvalue	$\|\rho_t\|_\infty \leq \kappa_n t^{-n/2}$

Convergence to a space of Gaussian states

[Datta, Pautrat, Rouzé '16]

The **Gaussification** of a state ρ with finite mean and covariance is the Gaussian state ρ^G with the same mean and covariance as ρ.

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Theorem

For all ρ s.t. quantum de Bruijn inequality holds for $\rho_t = e^{t\mathcal{L}(\rho)}$, for any $0 < \epsilon < 1$, $0 < \delta$, there exists α and $T_0 > 0$ s.t. for any $t \geq T_0$

$$D(\rho_t || \rho_t^G) \leq \alpha\, t^{(\epsilon-1)/2}$$

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By quantum Pinsker inequality, this implies that:

$$\|\rho_t - \rho_t^G\|_1 \leq \sqrt{2D(\rho_t\|\rho_t^G)} \leq \sqrt{2\alpha}\, t^{(\epsilon-1)/4}.$$

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Classically

For smooth u_0

$$\|u_t - f_{\sqrt{2t+1}Z}\|_{L_1} \leq \sqrt{2D(u_0\|f_Z)}\,(2t+1)^{-1/2}$$

here $\partial_t u_t = \Delta u_t$

Quantum Ornstein-Uhlenbeck semigroup

$n = 1$

Quantum Attenuator:

$$\mathcal{L}_-(\rho) = a\rho a^* - \frac{1}{2}\{a^*a, \rho\}$$

Quantum Amplifier:

$$\mathcal{L}_+(\rho) = a^*\rho a - \frac{1}{2}\{aa^*, \rho\}$$

Let $\rho = \omega_{\mathbf{n}}$ be a Gaussian thermal state with a mean photon number $\mathbf{n}$. Then $\rho_\pm(t) = \omega_{\mathbf{n}_\pm(t)}$

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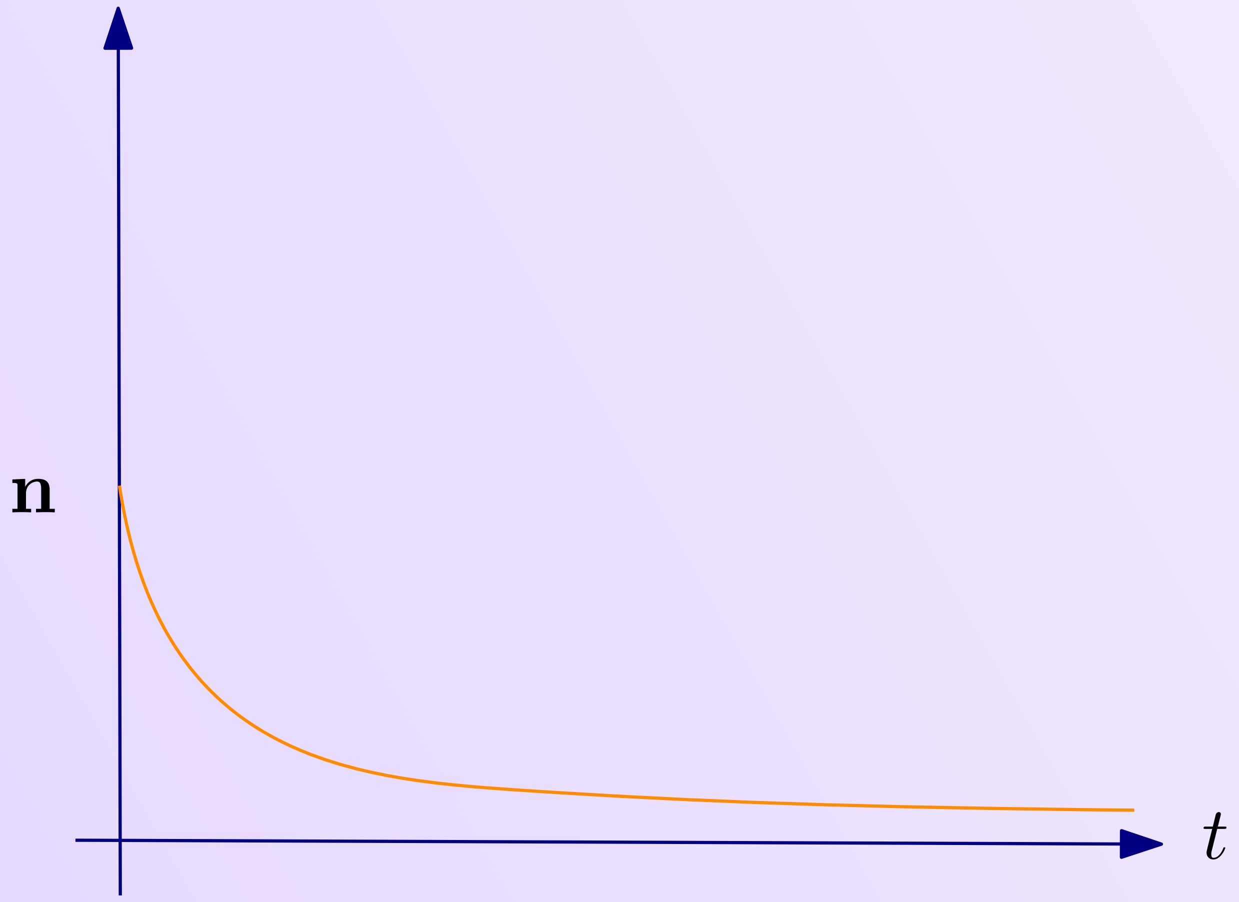
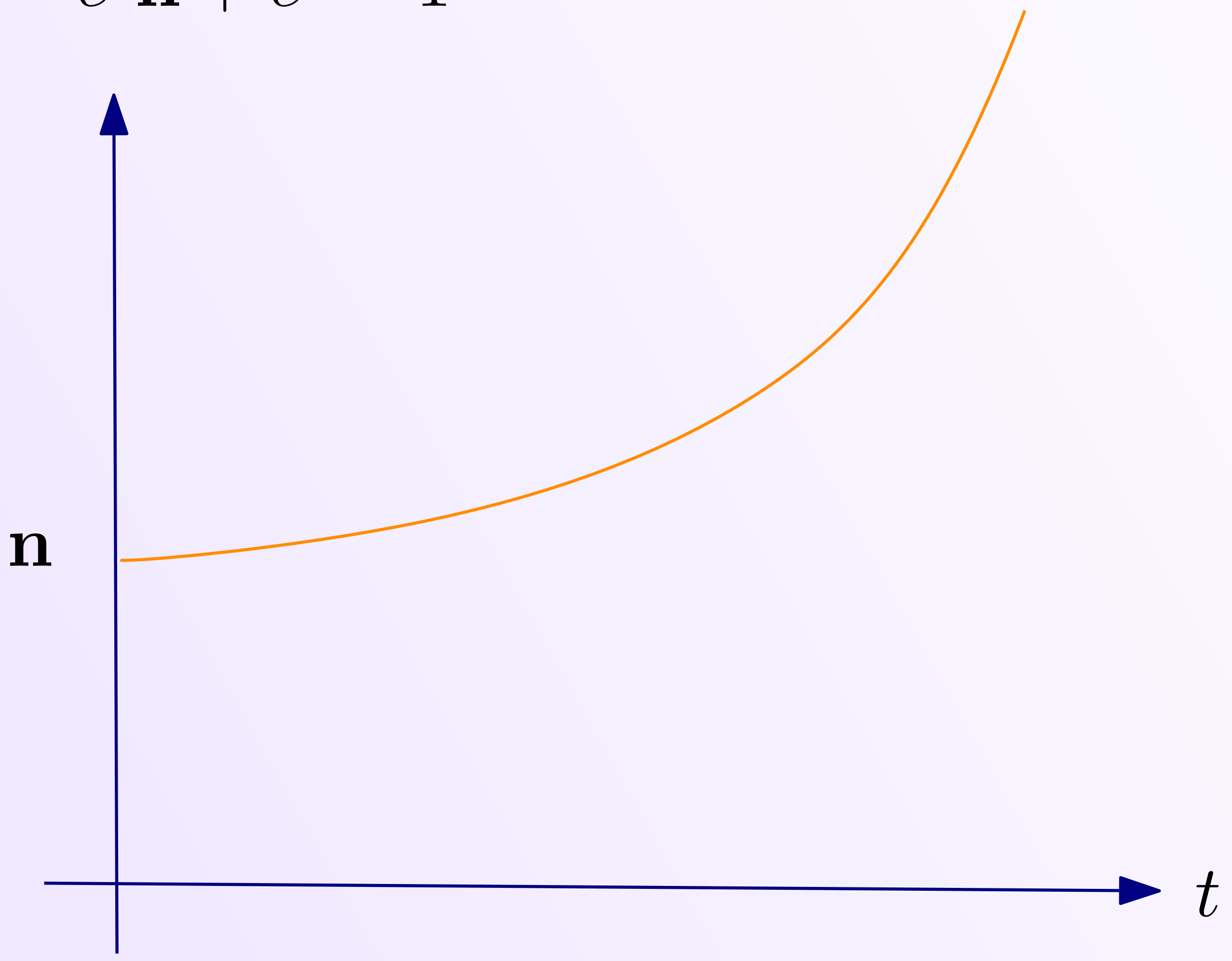
Under $\mathcal{L}_-$

$$\mathbf{n}_-(t) = e^{-t}\mathbf{n}$$



Under $\mathcal{L}_+$

$$\mathbf{n}_+(t) = e^t\mathbf{n} + e^t - 1$$



Quantum Ornstein-Uhlebeck semigroup

One-parameter group of CPTP maps $\{e^{\mathcal{L}_{\mu,\lambda}}\}_{t\geq 0}$, generated by

$$\mathcal{L}_{\mu,\lambda} = \mu^2\mathcal{L}_- + \lambda^2\mathcal{L}_+ \qquad \text{for } \mu > \lambda > 0$$

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Fixed point of $\mathcal{L}_{\mu,\lambda}$ is $\sigma_{\mu,\lambda} = (1-\nu)\sum_{n=0}^{\infty} \nu^n |n\rangle\langle n|$, with $\nu = \lambda^2/\mu^2$

Then $\omega_{\mathbf{n}(t)}$ is s.t.

$$\mathbf{n}(t) = e^{-(\mu^2-\lambda^2)t}\mathbf{n} + (1 - e^{-(\mu^2-\lambda^2)t})\mathbf{n}_\infty \qquad \text{with } \mathbf{n}_\infty = \mathrm{tr}(\sigma_{\mu,\lambda}\hat{n}) = \frac{\lambda^2}{\mu^2-\lambda^2}$$

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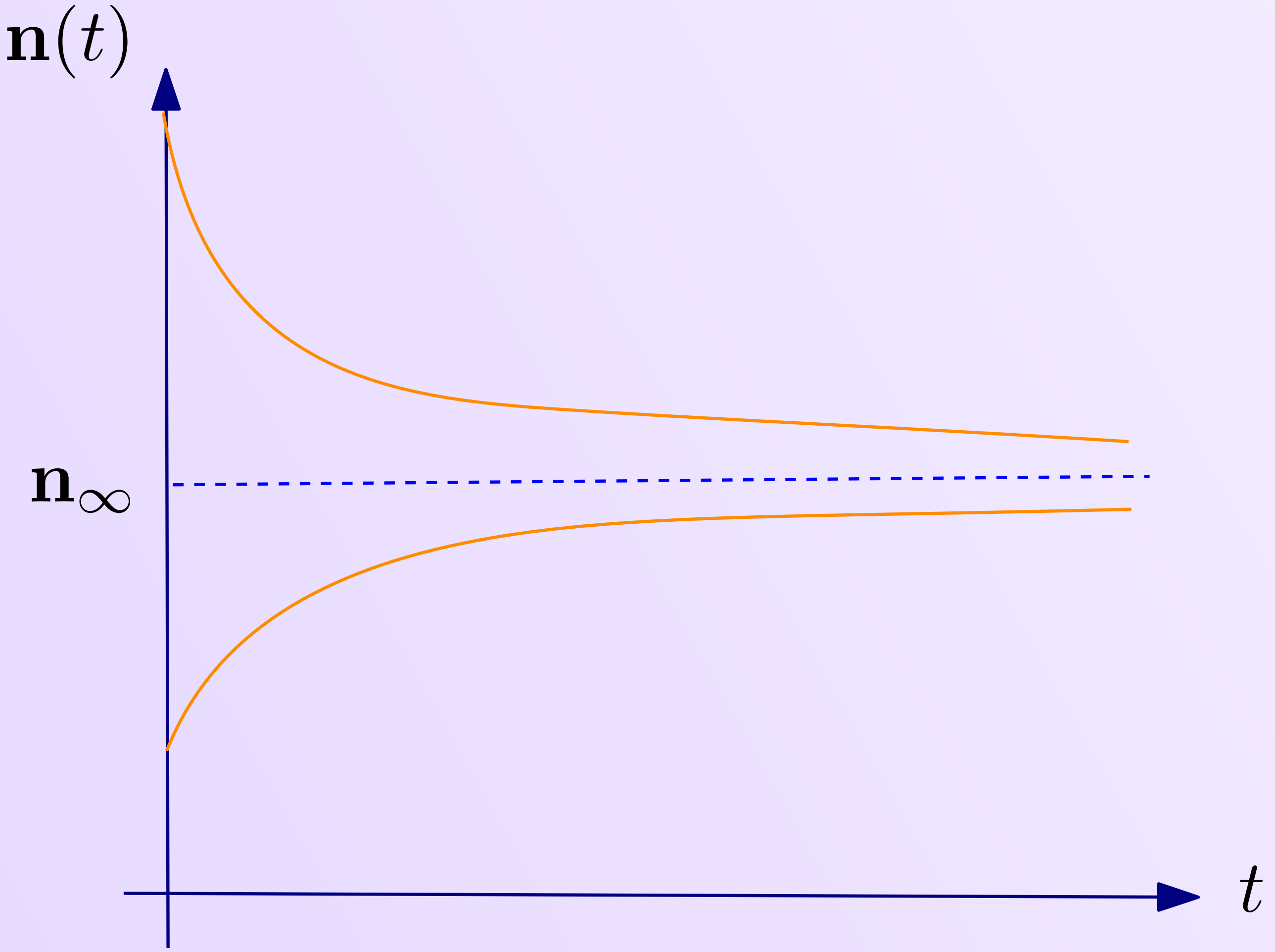

Fast convergence of qOU semigroup

Proposition

[Huber, König, V '16]

For a large class of states ρ we have

$$D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) \le e^{-(\mu^2-\lambda^2)t}D(\rho\|\sigma_{\mu,\lambda}) \qquad \text{for all } t \ge 0$$

In other words,

$$\frac{d}{dt}\bigg|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) \le -(\mu^2-\lambda^2)D(\rho\|\sigma_{\mu,\lambda})$$

Moreover, for any $\zeta > \mu^2 - \lambda^2$ there exists a state ρ s.t.

$$\frac{d}{dt}\bigg|_{t=0} D(e^{t\mathcal{L}_{\mu,\lambda}}(\rho)\|\sigma_{\mu,\lambda}) > -\zeta D(\rho||\sigma_{\mu,\lambda})$$

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Conjecture (June '16)

For any state we have

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Fast convergence of qOU semigroup

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Proved by Carlen, Maas in Sep '16

Quantum Log-Sobolev inequality

[Huber, König, V '16]

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Proposition For $A > 0$ and $\nu = \lambda^2/\mu^2 < 1$

$$D(\rho\|\sigma_{\mu,\lambda}) \leq AJ(\rho) - 2 - \log(4\pi A) + \mathbf{n}\log\frac{1}{\nu} - \log(1-\nu)$$

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Proof

From Isoperimetric inequality for entropies for $A > 0$

$$-S(\rho) \leq AJ(\rho) - (2 + \log(4\pi A))$$

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$$-S(\rho) \leq AJ(\rho) - (2 + \log(4\pi A))$$

$$-S(\rho) \leq \log\Big\{ \tfrac{1}{4\pi e} J(\rho) \Big\}$$

$$\log\Big(\tfrac{1}{4\pi e} J(\rho) \Big) = \log\Big(\tfrac{AJ(\rho)}{4\pi e A} \Big) = \log\Big(\tfrac{1}{4\pi e A} \Big) + \log\Big(AJ(\rho) \Big)$$

using $\log x \leq x - 1$

$$\leq AJ(\rho) - 2 - \log(4\pi A)$$

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Note that

$$D(\rho\|\sigma_{\mu,\lambda}) = -S(\rho) - (\log\nu)\mathbf{n} - \log(1-\nu) \qquad \nu = \lambda^2/\mu^2 < 1$$

Therefore

$$D(\rho\|\sigma_{\mu,\lambda}) \leq AJ(\rho) - 2 - \log(4\pi A) + \mathbf{n}\log\tfrac{1}{\nu} - \log(1-\nu)$$

$\square$

Classical Log-Sobolev inequality

[Carlen '91]

Gross' logarithmic Sobolev inequality is

$$\int |f|^2 \log |f|^2 e^{-\pi|x|^2} dx \leq \frac{1}{\pi} \int | \bigtriangledown f|^2 e^{-\pi|x|^2} dx \qquad \text{for } \int |f|^2 e^{-\pi|x|^2} dx = 1$$

Let $g(x) = f(x)e^{-\pi|x|^2/2}$. Then

$$\int |g|^2 \log |g|^2 dx \leq \frac{1}{\pi} \int | \bigtriangledown g|^2 dx - n \qquad \text{with } \int |g|^2 dx = 1$$

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$H(X|Y)$

$$\int h_X \log h_X dx - \int h_X \log h_Y dx$$

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$$\downarrow \qquad\qquad \downarrow$$

$$\frac{1}{4\pi} J(X) \qquad\qquad + \pi E|X|^2$$

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Quantum case:

$$D(\rho \| \sigma_{\mu,\lambda}) \leq A J(\rho) - 2 - \log(1-\nu) - \log(4\pi A) + \mathbf{n} \log \frac{1}{\nu} \qquad \nu = \lambda^2/\mu^2 < 1$$

$$A > 0$$

Open Problems

Multimode Ornstein-Uhlenbeck process

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Entropy Photon-number Inequality: replace entropy power $e^{S(\rho)/n}$ by $g^{-1}(S(\rho))$

[Guha, Erkmen, Shapiro '08]

photon number of a Gaussian state with the same entropy

$$g(x) = -(x+1)\log(x+1) - x\log x$$

Thank you!

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