On preparing ground states of gapped Hamiltonians: An efficient Quantum Lovász Local Lemma

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Joint work with:
Or Sattath
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E.g.: Kitaev's Toric Code

Frustration-freeness and quantum satisfiability (QSAT)

Projector description

Π_i: orthogonal projector to the subspace of excited states of H_i.
The frustration-free states of $H = \sum_{i=1}^{m} H_i$ and $H' = \sum_{i=1}^{m} \Pi_i$ are the same.

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The decision problem k-QSAT

Input: orthogonal projectors $(\Pi_i)_{i\in[m]}$, s.t. each Π_i acts on k qubits
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This is a generalisation of classical satisfiability (SAT)

$$\text{SAT} \quad \Rightarrow \quad \text{QSAT}$$

$$\underbrace{(x_1 \vee x_2 \vee x_3)}_{C_1} \wedge \underbrace{(\overline{x_1} \vee x_3 \vee \overline{x_4})}_{C_2} \quad \Rightarrow \quad \begin{array}{l} \Pi_1 := |000\rangle\langle 000|_{123} \\ \Pi_2 := |101\rangle\langle 101|_{134} \end{array}$$

Hardness of deciding frustration-freeness

The complexity of SAT and QSAT

- 2-SAT and 2-QSAT are easy to decide (they are in P (Bravyi ’06))
- 3-SAT and 3-QSAT are very hard to decide (NP-complete and QMA_1-complete (Kitaev; Gosset & Nagaj ’13), respectively)

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- The Lovász Local Lemma (LLL) provides a sufficient condition for the satisfiability of k-SAT
- The Quantum LLL is a generalisation by Ambainis et al. for k-QSAT

The Lovász Local Lemma (LLL)

Application to k-SAT

– $\{C_i : i \in [m]\}$ are clauses of a k-SAT formula

– Each having at most d neighbours

If $p \cdot d \cdot e \leq 1$ ($p = 2^{-k}$, $e = 2.71\ldots$), then the formula is satisfiable.

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Generalisation to k-QSAT

– $\{\Pi_i : i \in [m]\}$ are k-local rank-r orthogonal projectors
– Each having at most d neighbours
If $p \cdot d \cdot e \leq 1$ ($p = r \cdot 2^{-k}$, $e = 2.71\ldots$), then $\sum_{i=1}^{m} \Pi_i$ is frustration-free.

QLLL in pictures

x_1 x_2 x_3 x_4

x_5 x_6 x_7 x_8

x_9 x_{10} x_{11} x_{12}

x_{13} x_{14} x_{15} x_{16}

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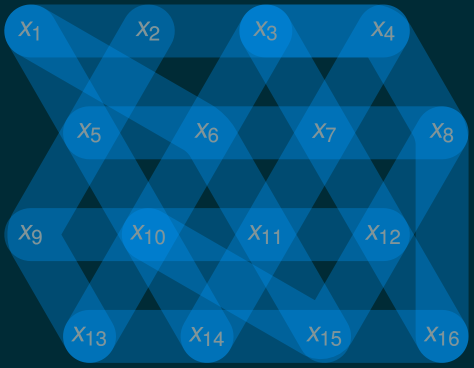

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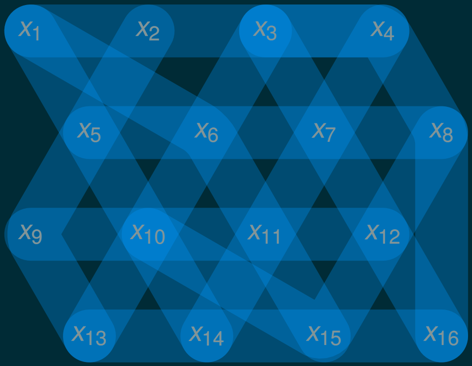

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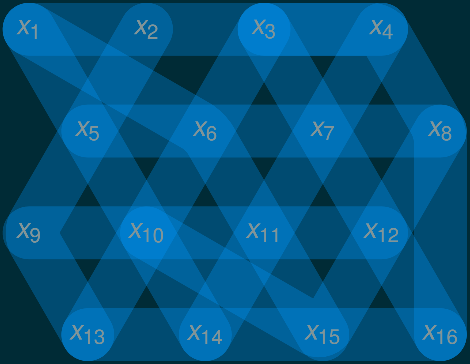

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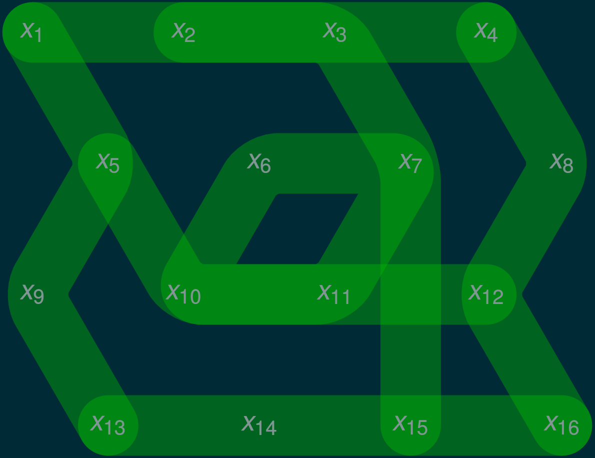

The system is always frustration-free

Overview of results

		Classical	Quantum
$\exists$	Orig.	Lovász & Erdős ('75)	Ambainis et al. ('09)
	Best	Shearer ('85)	Sattath et al. ('16)
🔧	Orig.	Moser & Tardos ('09)	Schwarz et al.; Arad et al. ('13) (only for commutative case)
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Finding happiness: Classical

Classical: finding a “happy" assignment

The Moser-Tardos resampling algorithm (2009)

init uniform random assignment
for all $i \in [m]$:
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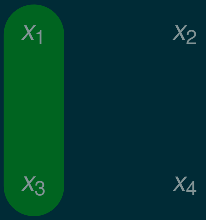

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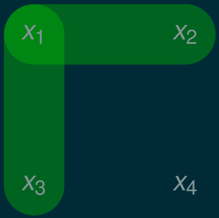

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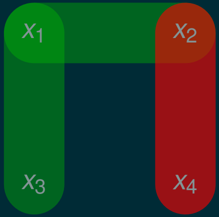

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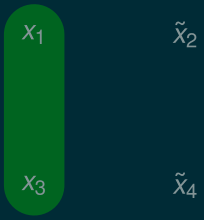

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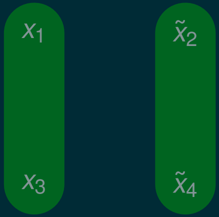

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Commutative quantum: finding a "happy" state

The commutative quantum resampling algorithm

---

init uniform random qubits
for all $i \in [m]$:
  fix(Π_i)

fix(Π_i):
  measure Π_i
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Schwarz et al.; Arad et al. (2013)

Our simplified analysis

Our key lemma

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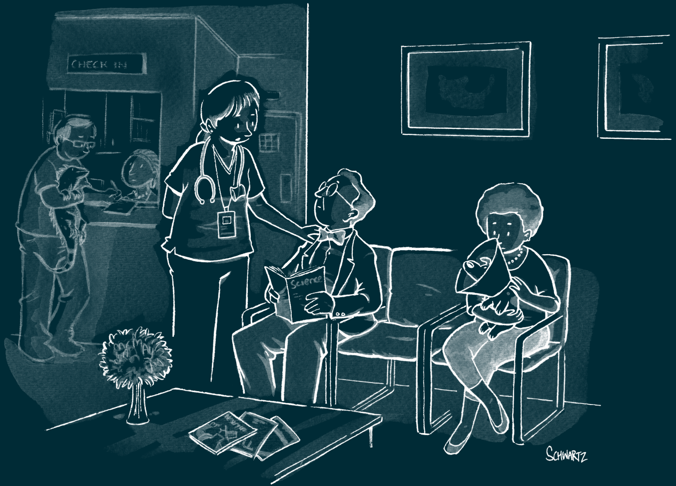
When does this algorithm terminate quickly?

- The number of length-$3m$ resample sequences is $\ll (ed)^{3m}$ (easy)

$\Rightarrow$ The probability of seeing a length-$3m$ resample seq. $\ll (p \cdot d \cdot e)^{3m}$

If $p \cdot d \cdot e \leq 1$ then w.h.p. the alg. performs $< 3m$ resamplings

Finding happiness: Quantum



"About your cat, Mr. Schrödinger – I have good news and bad news."

Issues with non-commutativity

Becoming “unhappy" after seeing others “happy"

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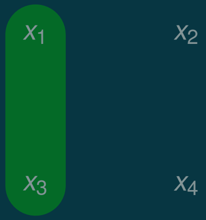

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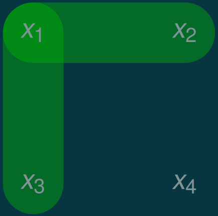

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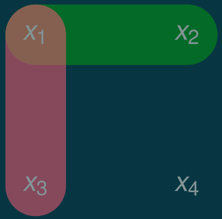

Non-commutative quantum: finding a “happy" state

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Our key lemma

Probability of doing a specific length-ℓ resample sequence is $\leq p^{\ell}$

Measuring joint happiness

Perfect ground space projections of subsystems

F : set of already fixed projectors.
Define Π_F via $\ker(\Pi_F) = \bigcap_{j \in F} \ker(\Pi_j)$.
(In the commuting case $\Pi_F = \prod_{j \in F} \Pi_j$.)

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Generalised measurement procedure $\mathcal{M}$ – for our key lemma

If $\Pi_F|\psi\rangle = 0$ (i.e. F is “happy") and we measure it using $\mathcal{M}_{F,i}$ returning result

- “happy", then

$$\Pi_{F \cup \{i\}} \mathcal{M}_{F,i}(|\psi\rangle) = 0$$

- “unhappy", then

$$\Pi_i \mathcal{M}_{F,i}(|\psi\rangle) = \mathcal{M}_{F,i}(|\psi\rangle)$$

(while preserving “happiness" of non-neighbour projectors.)

Weak measurement

Weak measurement of Π_i

To weakly measure $\{\Pi_i, \mathrm{Id} - \Pi_i\}$ use an ancilla and a Π_i-controlled rotation:

$$\Pi_i^\theta = \Pi_i \otimes R^\theta + (\mathrm{Id} - \Pi_i) \otimes \mathrm{Id} \text{ , where } R^\theta = \begin{pmatrix} \sqrt{1-\theta} & -\sqrt{\theta} \\ \sqrt{\theta} & \sqrt{1-\theta} \end{pmatrix}.$$

Apply Π_i^θ on $|\psi\rangle \otimes |0\rangle$ and measure the ancilla qubit (in the $|0\rangle, |1\rangle$ basis).

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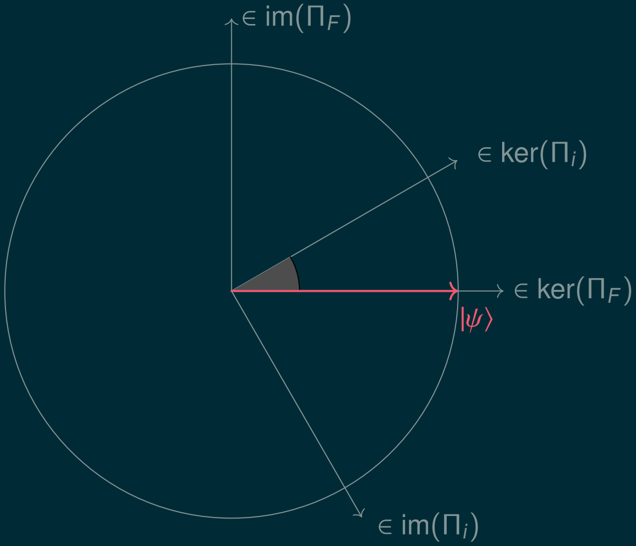
The outcomes of a weak measurement

$$\text{Outcome 1 : } \left|\psi_1^\theta\right\rangle = \sqrt{\theta}\Pi_i|\psi\rangle \quad \text{(unnormalised)}$$

$$\text{Outcome 0 : } \left|\psi_0^\theta\right\rangle = (\mathrm{Id} - \Pi_i)|\psi\rangle + \sqrt{1-\theta}\Pi_i|\psi\rangle \approx |\psi\rangle - (\theta/2)\Pi_i|\psi\rangle$$

Problem with strong measurement



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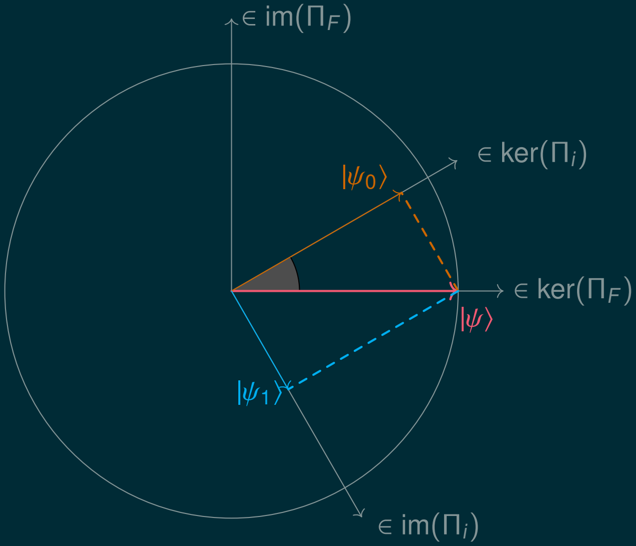

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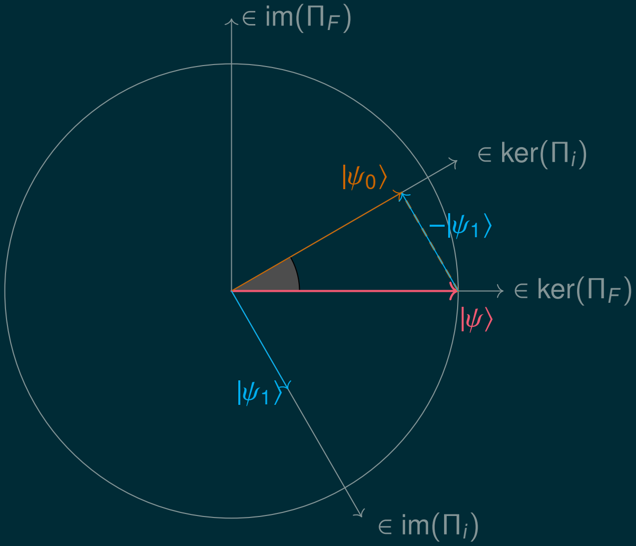

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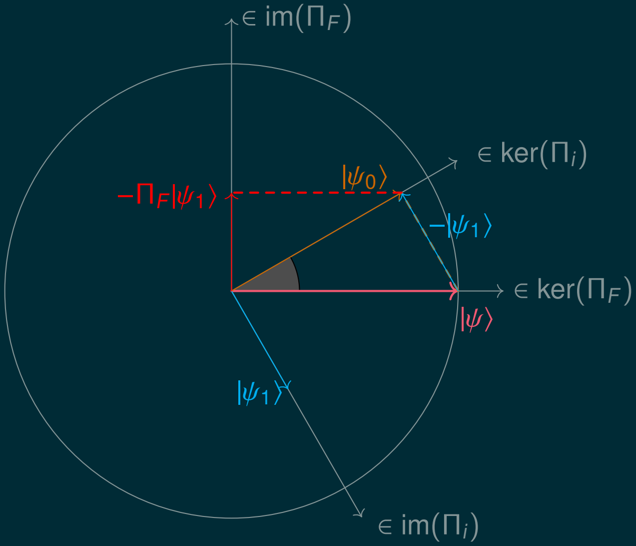
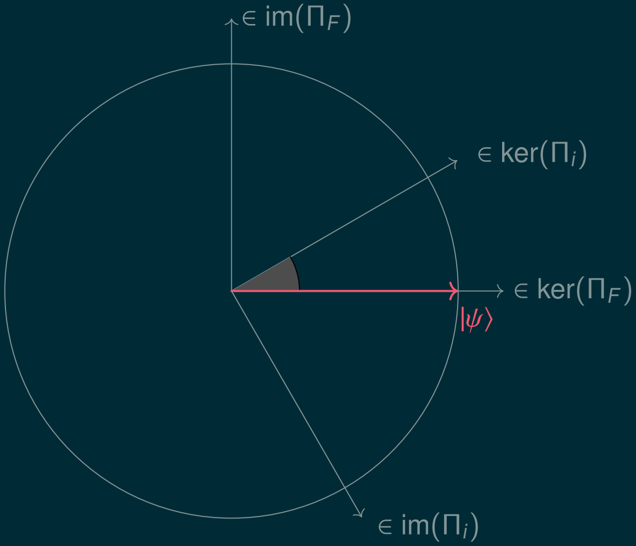


Weak measurement + quantum Zeno effect



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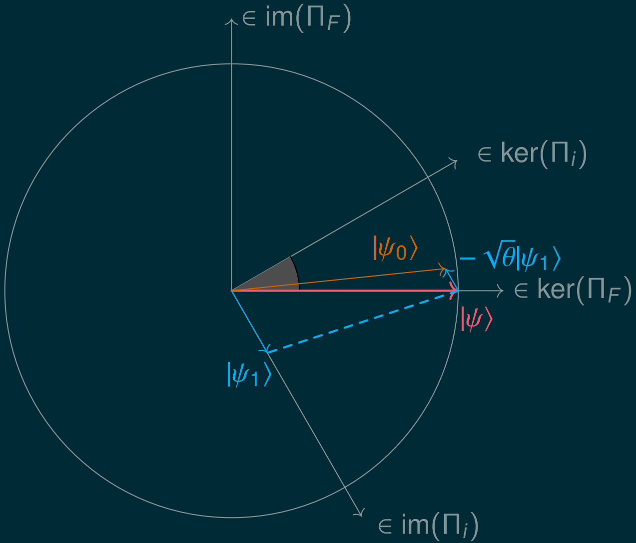

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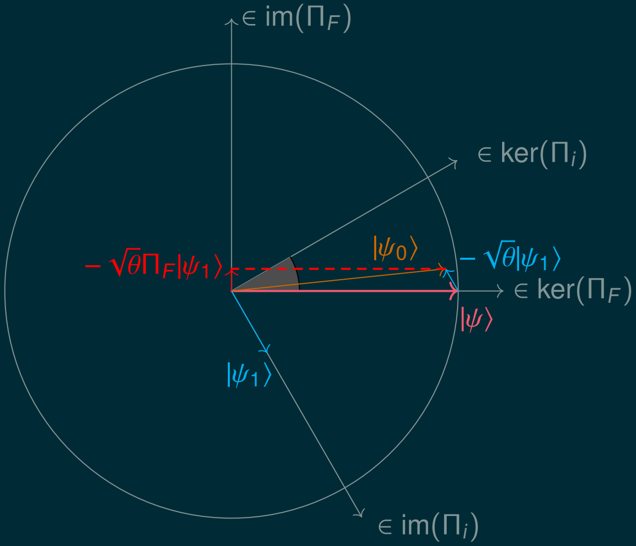

Implementation of $\mathcal{M}$

Generalised measurement $\mathcal{M}_{F,i}$

repeat T times **do**

measure Π_i weakly **if** Π_i was detected **then return** i is "unhappy"

measure Π_F (for quantum Zeno effect)

end repeat and return $F \cup \{i\}$ is "happy"

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Let γ be the energy gap (smallest non-zero. energy) of $H_{F\cup\{i\}} = \Pi_i + \sum_{j\in F} \Pi_j$.

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We "know in advance" the outcome of all Π_F measurement!

$\Rightarrow$ Π_F can be simulated by meas. $\sim \frac{|F|}{\gamma}$ times a randomly chosen $(\Pi_j)_{j\in F}$

Runtime

The uniform gap

For $H = \sum_{i\in[m]} \Pi_i$ we define the uniform gap of H as

$$\gamma(H) := \min_{F\subseteq[m]} \operatorname{gap}\left(\sum_{i\in F} \Pi_i\right).$$

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The overall runtime of the quantum algorithm using $\mathcal{M}$

$$\text{total number of measurements} = \tilde{O}\left(\frac{m^3 \cdot d}{\gamma^2} \cdot \log^2\left(\frac{1}{\delta}\right)\right)$$

- m: number of projectors
- d: maximum number of neighbours of a projector
- γ: uniform gap
- δ: maximum trace distance of the output from a density operator supported on the ground space

Discussion

Benefits of the algorithm

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- Can prepare the ground state of a 50 qubit system using 51 qubits!
- Due to quantum Zeno effect it probably does not need error correction

Open questions

- Is there a variant which can prepare low-energy states without gap promise?
- Physically motivated examples? (quantum chemistry, spin systems, ...)
- Getting speed ups for some interesting classical problem?
- Can this result be used for showing quantum supremacy?

Without a promise on the gap

What can we do without knowing the size of the gap?

For any input $(\Pi_i)_{i\in[m]}$ satisfying the Lovász (or Shearer) condition and $\epsilon \in \mathbb{R}_+$ we can do one of the following:

- Prepare a quantum state supported on energy eigenstates with energy below ϵ.

Or Conclude that the uniform gap is below ϵ.

Preparing low-energy quantum states

Let Π_S^δ denote the projection to the subspace of energy eigenstates with energy at least δ, with respect to $H_S = \sum_{i \in S} \Pi_i$.

Generalising the two main properties to low energy subspaces

Suppose $|\psi\rangle$ is such that $\Pi_S^\delta |\psi\rangle = 0$. We need a quantum channel $\mathcal{M}_{S,i}$ with two possible (probabilistic) outcomes:

- "happy": $\Pi_{S \cup \{i\}}^{\delta+\varepsilon} \mathcal{M}_{S,i}(|\psi\rangle) = 0$
- "unhappy": $\left(\Pi_{S \setminus \Gamma(i)}^{\delta+\varepsilon} \preceq \Pi_S^\delta \otimes (\mathrm{Id} - \Pi_i)\right) \mathcal{M}_{S,i}(|\psi\rangle) = 0$.

Main issue

$\Pi_{S \setminus \Gamma(i)}^{\delta+\varepsilon} \preceq \Pi_S^\delta$ does not always hold! (Only if $\delta = 0$.)

Simulation results for the non-commuting case

- Various topologies tested up to 21 qubits, including cycles, grids, octahedron, dodecahedron
- Poor performance even for cycles? 2-SAT easy even classically!

Output of the LIQ$Ui|\rangle$ simulation, on C_{10}

```
0:0000.0/Classical upper bound on the expected number of resamplings : 45.0
0:0003.0/Projectors constructed
0:0003.3/Singular values found: 1022, smallest: 0.039998
0:0003.3/Hamiltoninan constructed
0:0003.7/Kernel Gate constructed
0:0003.7/Run quantum test on a fixed random projector set
0:0017.2/Average resamplings in 100 simulation runs:
 0: M:  0   R: 0   E: 2.6074 P: 0.0010
 1: M: 22.1 R: 4.0 E: 0.4994 P: 0.0204
 2: M: 14.4 R: 1.5 E: 0.1820 P: 0.0364
 3: M: 12.2 R: 0.7 E: 0.1082 P: 0.0413
 4: M: 12.3 R: 0.8 E: 0.1177 P: 0.0516
 5: M: 11.3 R: 0.4 E: 0.0774 P: 0.0514
10: M: 10.6 R: 0.2 E: 0.0406 P: 0.0701
15: M: 10.7 R: 0.2 E: 0.0370 P: 0.0740
20: M: 10.6 R: 0.2 E: 0.0264 P: 0.0716
```