Rigorous RG algorithms and area laws for low energy eigenstates in 1D

THOMAS VIDICK

CALIFORNIA INSTITUTE OF TECHNOLOGY

JOINT WORK WITH

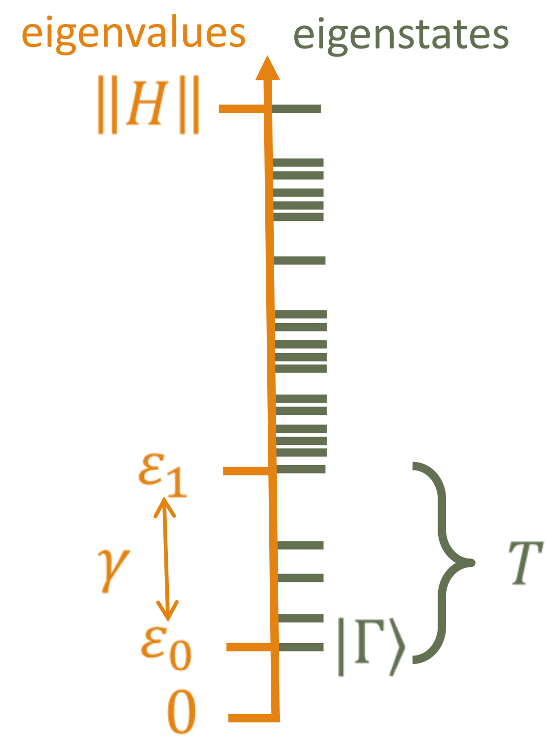
ITAI ARAD (TECHNION),
ZEPH LANDAU AND UMESH VAZIRANI (UC BERKELEY)

Local Hamiltonians

n-qubit 1D local Hamiltonian $H = h_1 + \cdots + h_{n-1}$

$h_i \geq 0,$
$\|h_i\| \leq 1$

Ex: Heisenberg model, AKLT model, etc.



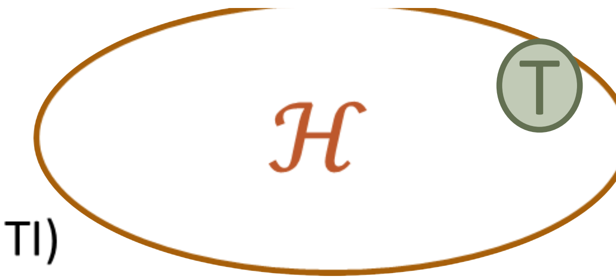
Ground state $|\Gamma\rangle$ has energy $\varepsilon_0 = \langle\Gamma|H|\Gamma\rangle$

Low-energy space $T = \mathrm{H}_{|[\varepsilon_0, \varepsilon_1]}$, $\varepsilon_1 = \varepsilon_0 + \gamma$

Can we "map out" T?

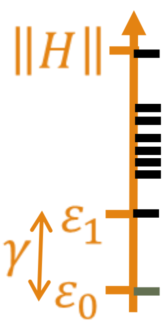
Low-lying states of local Hamiltonians

n-qubit Local Hamiltonian $H = h_1 + \cdots + h_{n-1}$

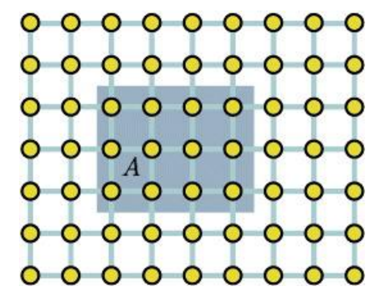


- [Kitaev,Gottesman-Irani] Finding T is QMA-hard

$\longrightarrow$ no efficient description in general (even for 1D, even TI)



[Hastings] *Gapped* 1D systems satisfy *area law*



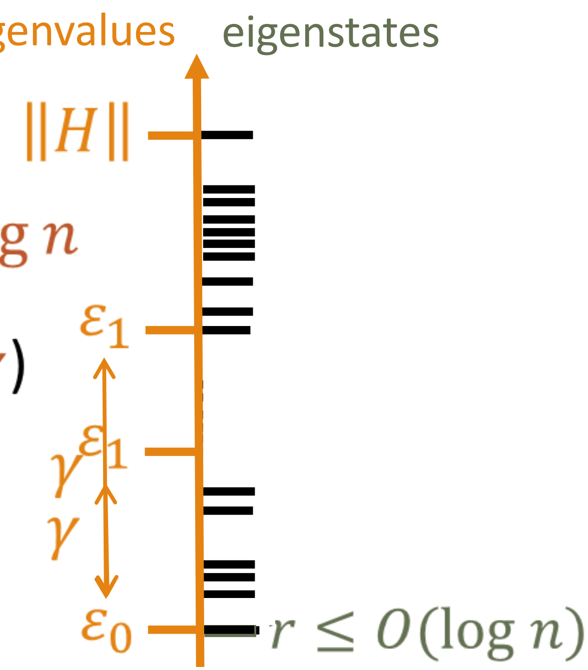
- [Landau-Vazirani-V.] unique g.s. $\longrightarrow$ algorithm in time poly$(n, 2^{2^{\gamma^{-1}}})$
- [Chubb-Flammia] extension to ground space with log. degeneracy

Low-lying states of local Hamiltonians

n-qubit Local Hamiltonian $H = h_1 + \cdots + h_{n-1}$

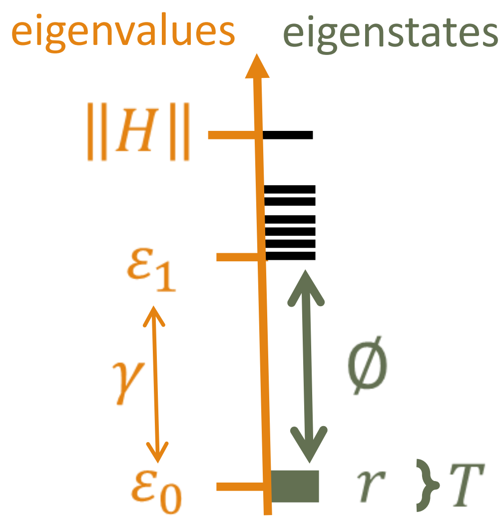
Low-energy space $T = H_{|[\varepsilon_0, \varepsilon_1]}$ is poly-size subspace of 2^n-dim $\mathcal{H}$

- Hardness results apply to $\gamma \leq 1/\text{poly}\, n$
- Area law guarantees efficient desc. for $\gamma \gg 1/\log n$
- Algorithms find basis in poly-time (for constant γ)
- Practical heuristics (DMRG)
  challenged beyond constant degeneracy



Results

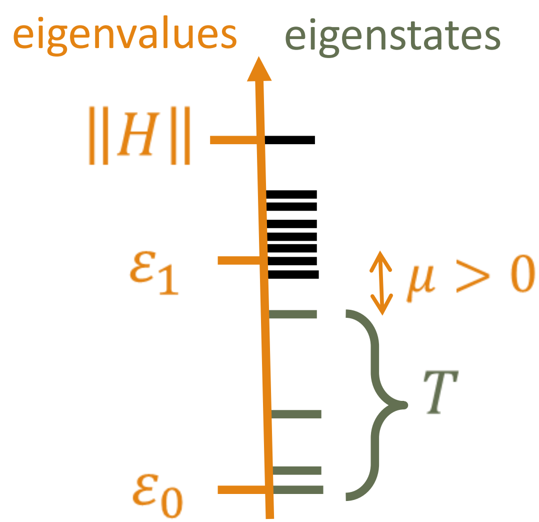
Input: 1D n-qubit local Hamiltonian H



(DG): $T = H_{|\varepsilon_0}$, $\mathrm{r} = \dim(T) = \mathrm{poly}(n)$, gap $\gamma > 0$

→ basis with Schmidt rank $r^2 e^{O(\gamma^{-1})}$

Time $n^{O(\gamma^{-2})}$



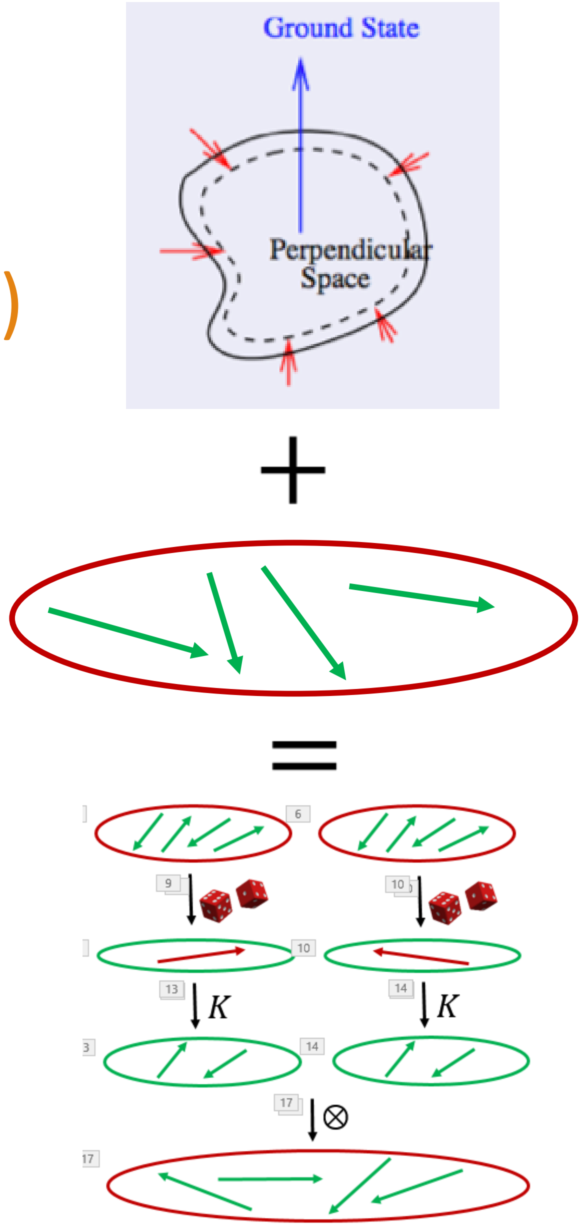
(LD): $T = H_{|[\varepsilon_0,\varepsilon_1]}$, $\mathrm{r} = \dim(T) = \mathrm{poly}(n)$

→ basis with Schmidt rank $r^2 n^{O(\mu^{-1})}$

Time $n^{\mathrm{polylog}(n)}$

Outline

1. Approximate ground state projections (AGSP)
2. Viable sets
3. Algorithm



Approximate Ground State Projections [ALV'12]

n-qubit Local Hamiltonian $H = h_1 + \cdots + h_{n-1}$

Low-energ… $\ldots, \varepsilon_1 = \gamma]$

Only need to squish eigenvalues > γ

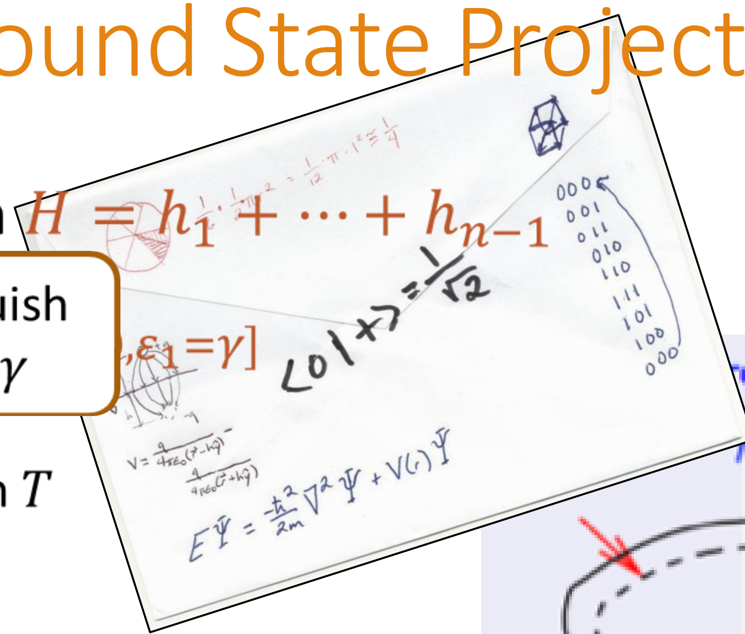
Π_T = orthogona… …ojection on T

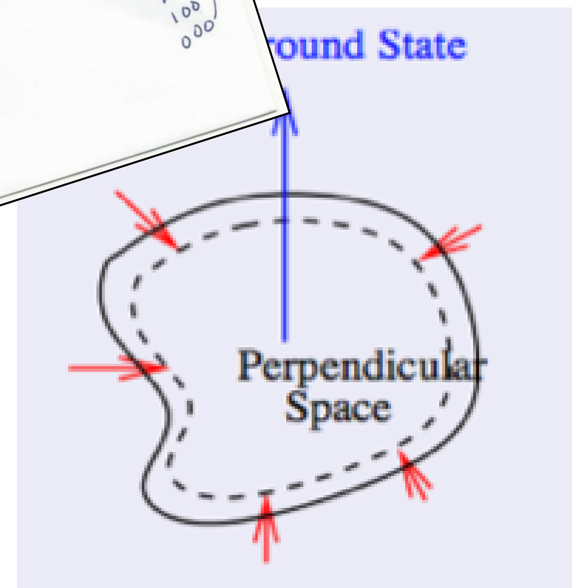
$= \lim_{\beta \to \infty} e^{-\beta H}$

Taylor expansion can be truncated at $\sim \|H\|/\gamma$

K

$\frac{\|\ldots_T K|\psi\rangle\|}{\|K|\psi\rangle\|} \geq 1 - \delta$





Q.E.D. ■

Approximate Ground State Projections [ALV'12]

n-qubit Local Hamiltonian $H = h_1 + \cdots + h_{n-1}$

Low-energy space $T = \mathrm{H}_{|[\varepsilon_0 = 0, \varepsilon_1 = \gamma]}$

$$K \approx \sum_{0 \le k \le \frac{\|H\|}{\gamma}} a_k \prod_{i_{1,\dots,i_k}} h_{i_1} \cdots h_{i_k}$$

- K has exponentially many terms

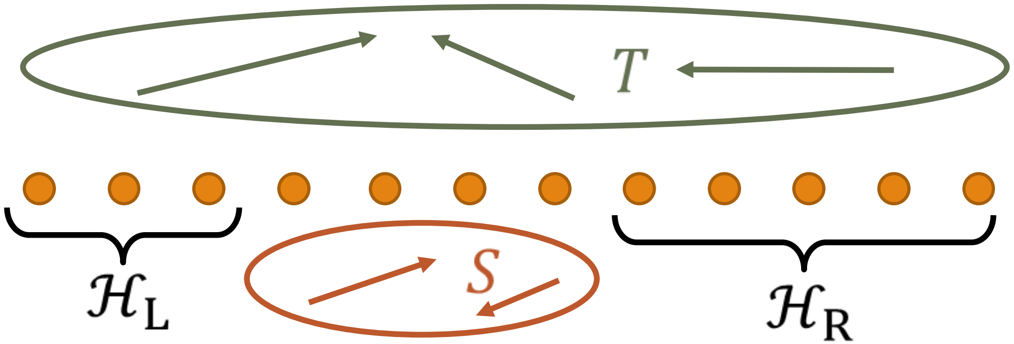
  $\longrightarrow$ can it be applied efficiently?

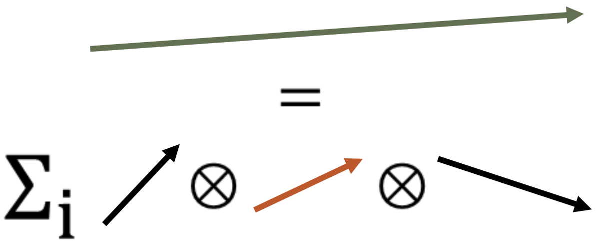
$$\|\Pi_T |\psi\rangle\| \ge \frac{1}{2} \qquad \Rightarrow \qquad \frac{\|\Pi_T K|\psi\rangle\|}{\|K|\psi\rangle\|} \ge 1 - \delta$$

- Need constant approximation to start with!

- Here: iterative merging of states defined on increasing subsets of qubits

Viable sets [LVV'15]





S is δ-viable for T if:

$$P_T\left(\mathrm{Id}_{\mathcal{H}_L} \otimes P_S \otimes \mathrm{Id}_{\mathcal{H}_R}\right)P_T \geq (1-\delta)P_T$$

Goal: construct poly-size viable set, on all n qubits, for $T = \mathrm{H}_{|[\varepsilon_0,\varepsilon_1]}$

Observations: 1) viable sets on constant number of qubits are easy

The algorithm

Initialization: create viable sets on pairs of qubits

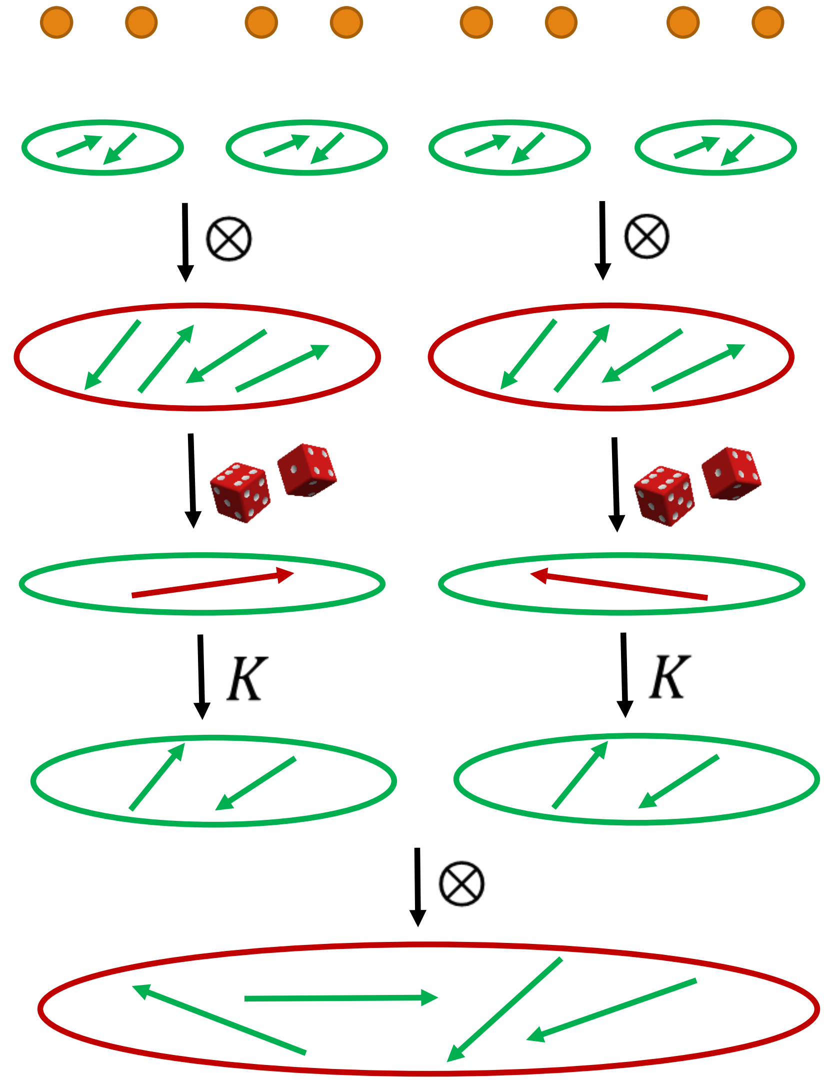
1) Merge: combine two neighboring sets
size $s \to s^2$ error $\delta \to 2\delta$

2) Sample: select random subset
size $s^2 \to \frac{s}{D^2}$ error $2\delta \to 1 - \frac{1}{sD}$

3) Improve: apply AGSP K
size $\frac{s}{D^2} \to s$ error $1 - \frac{1}{sD} \to \delta$

Repeat for $\log n$ steps.

Return r lowest-energy vectors

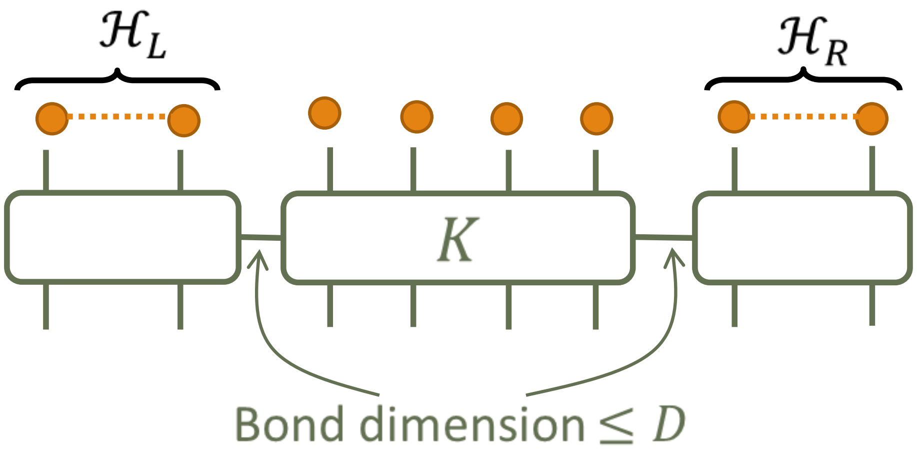


Approximation error remains constant throughout!

AGSP construction

Local operator $K \approx \Pi_T \approx e^{-\frac{c}{\gamma}H}$

- Computable, poly bond dimension
- Tight control of bond dimension D across boundary cuts
- Shrinkage Δ s.t. $D^2\Delta \ll 1$



Size: $\frac{s}{D^2} \to s$ error: $1-\Delta \to 0.1$

- Bond dimension D across boundary cuts determined by degree

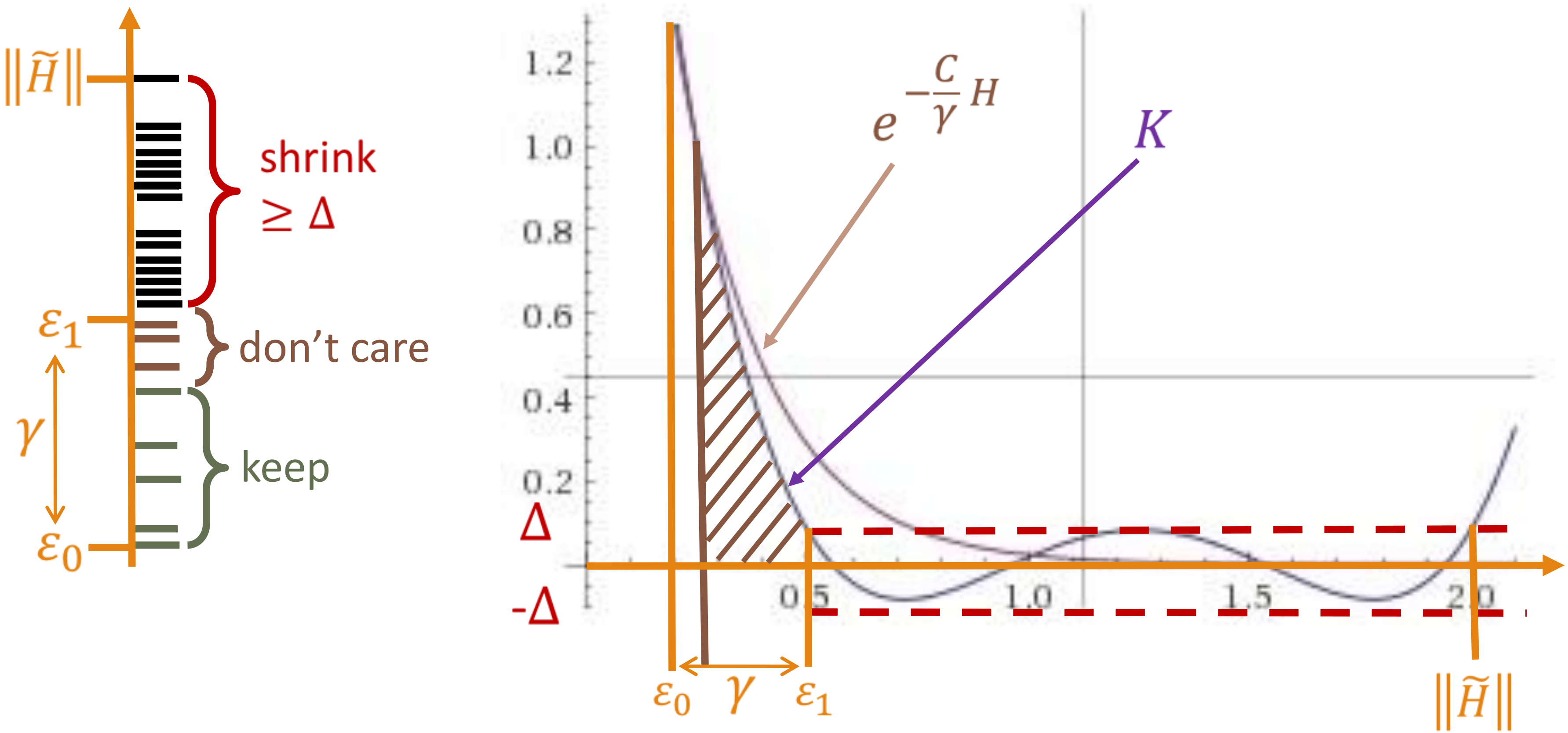
$K = T_k(H_L + H_M + H_R)$, T_k: degree $k \sim \gamma^{-2}$ Chebyshev polynomial

- Norm control: $H_L, H_M, H_R \to$ cst. norm (non-local) $\widetilde{H}_L, \widetilde{H}_M, \widetilde{H}_R$

"soft" truncation gives poly-size MPO

AGSP construction

$K = T_k(\widetilde{H}_L + \widetilde{H}_M + \widetilde{H}_R)$, T_k: degree $k \sim \gamma^{-2}$ Chebyshev polynomial



Summary

Area law + efficient algorithms for:

- (DG) polynomial-size ground space, constant gap $S(|\psi\rangle) \approx 4\log r + O(\gamma^{-1})$
- (LD) polynomial density of states $S(|\psi\rangle) \approx 4\log r + O(\log n)$

Features:

- Iterative tree-like procedure reminiscent of RG
- Elementary structure is viable set rather than state
- Operates in delicate constant-approximation regime, controlled by AGSP

Questions:

- Efficient?? [Roberts-Motrunich-V., in progress: benchmark against DMRG]
- Poly-time algorithm for (LD)? Using MERA?
- Higher dimension!