spectrahedral lifts (SDPs) and quantum learning

James R. Lee
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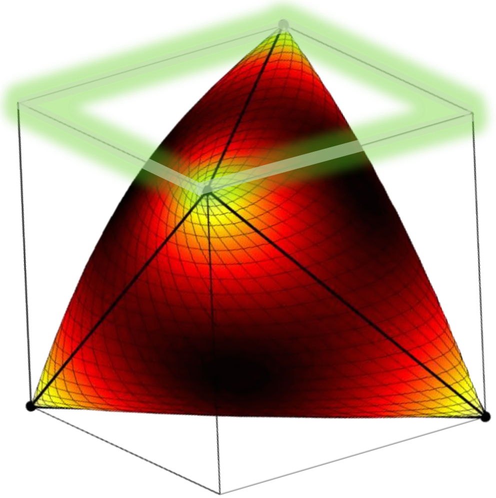

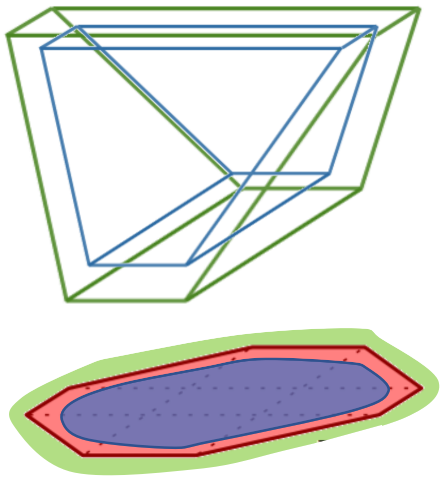

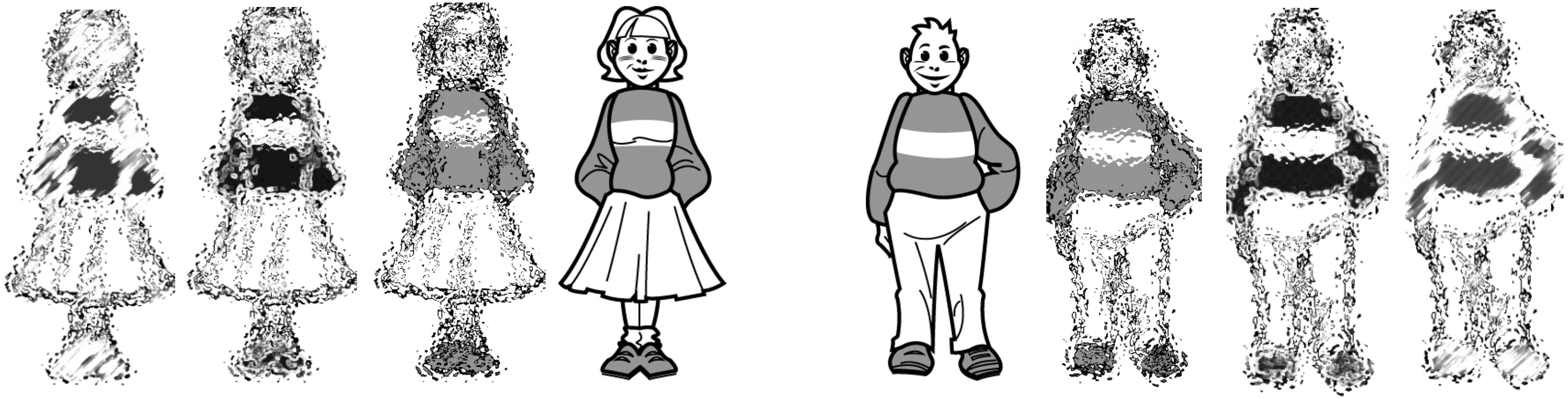

PSPACE 2017

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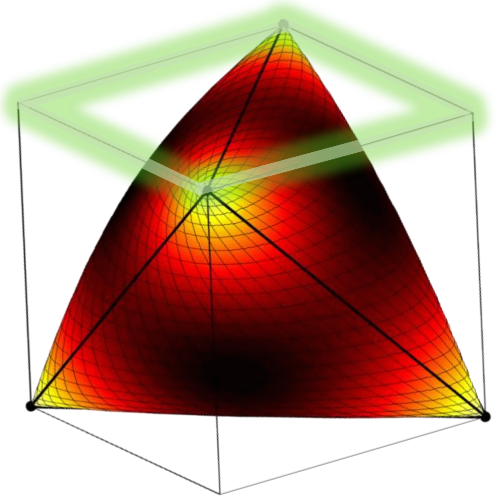
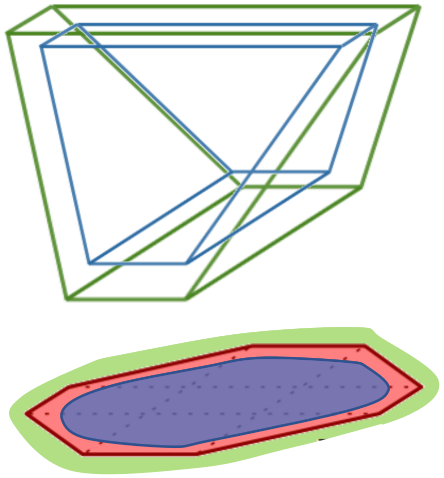
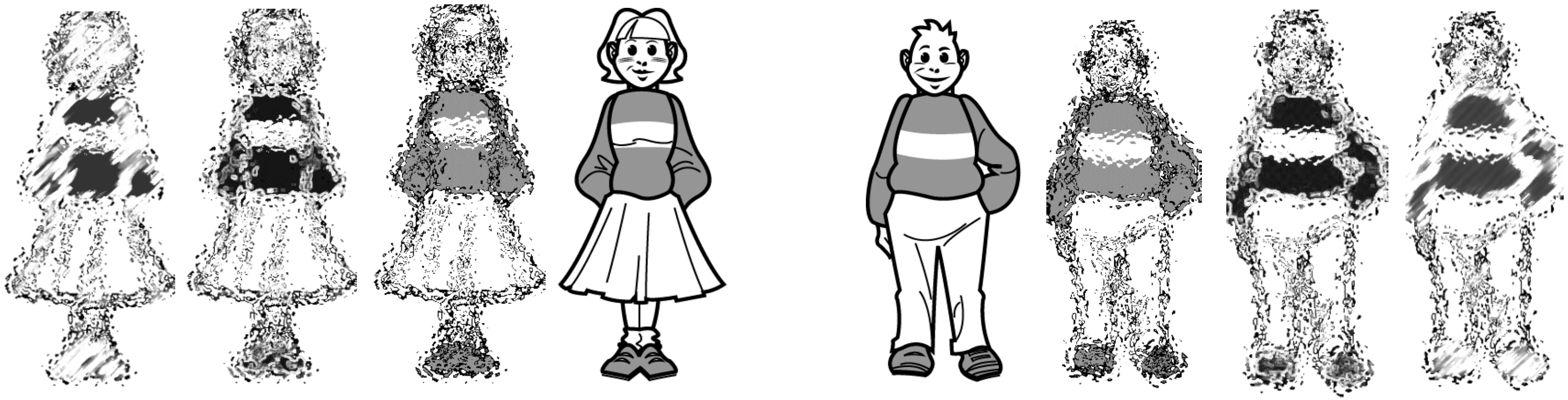




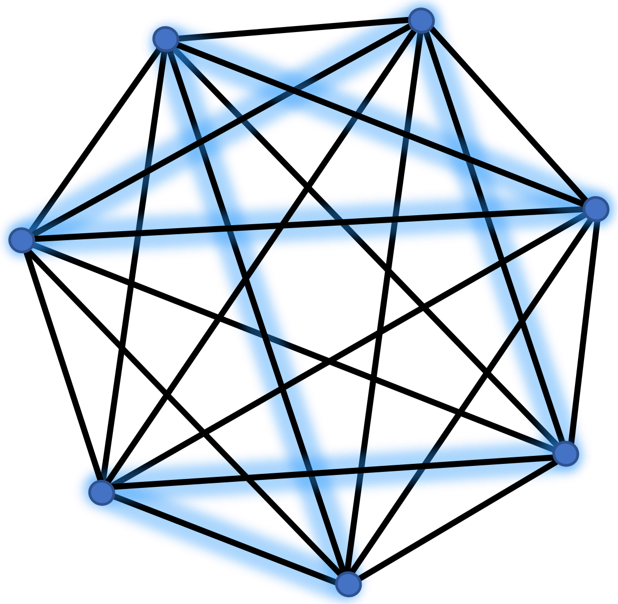


QIP 2017

Traveling Salesman Problem:

Given n cities $\{1,2,\ldots,n\}$ and costs $c_{ij} \geq 0$ for traveling between cities i and j, find the permutation π of $\{1,2,\ldots,n\}$ that minimizes

$$c_{\pi(1)\pi(2)} + c_{\pi(2)\pi(3)} + \cdots + c_{\pi(n)\pi(1)}$$



> Attempts to solve the traveling salesman problem and related problems of discrete minimization have led to a revival and a great development of the theory of polyhedra in spaces of *n* dimensions, which lay practically untouched—except for isolated results—since Archimedes. Recent work has created a field of unsuspected beauty and power, which is far from being exhausted.
>
> Gian Carlo Rota, 1969

combinatorial optimization

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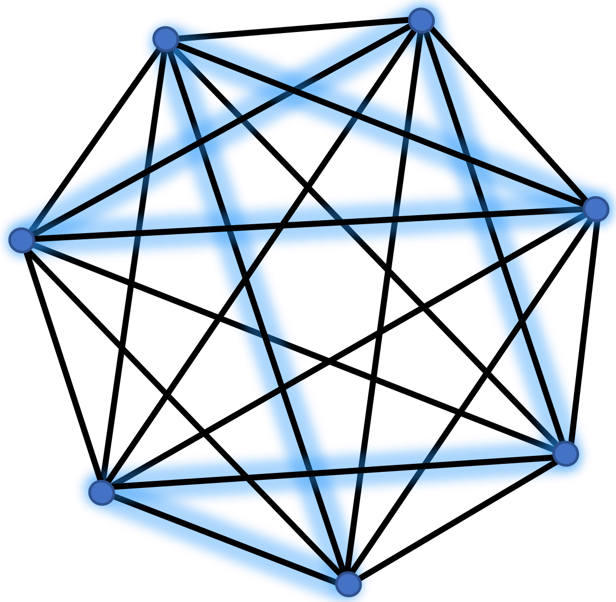

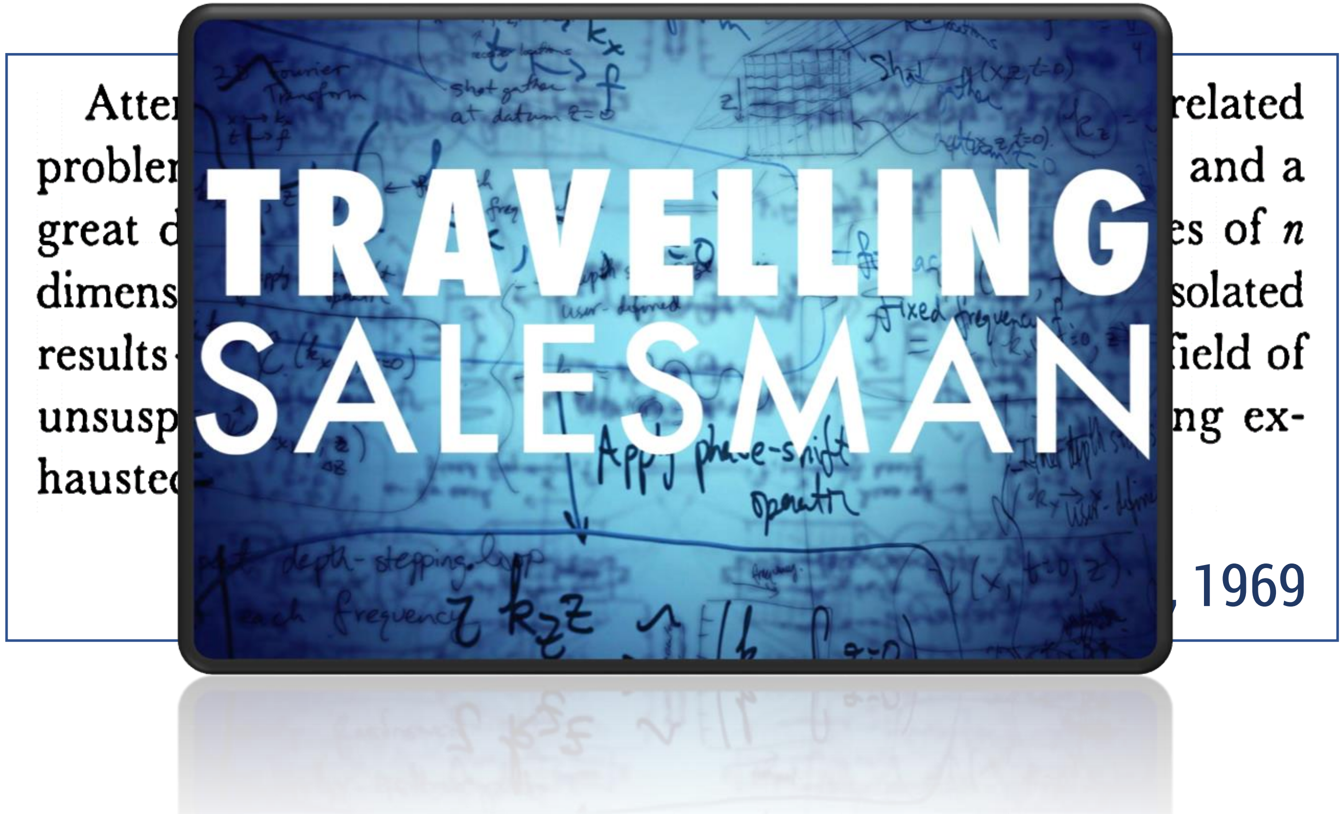

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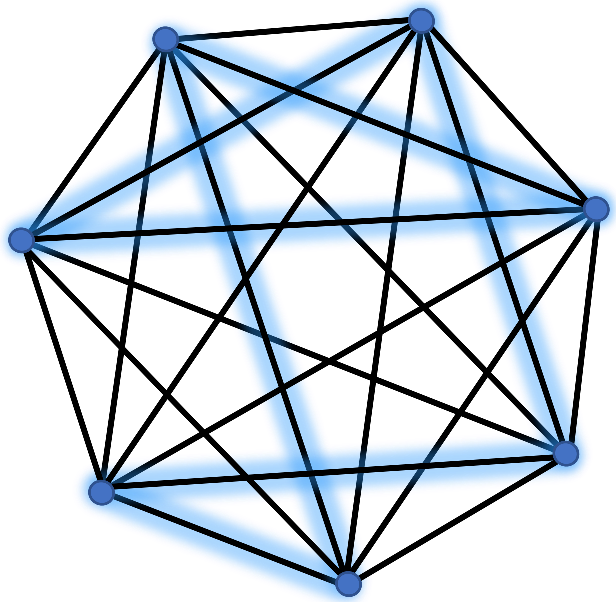
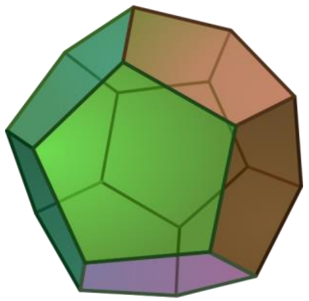


$$\mathrm{TSP}_n = \mathrm{conv}\left(\left\{1_\tau \in \{0,1\}^{\binom{n}{2}} : \tau \text{ is a tour}\right\}\right) \subseteq \mathbb{R}^{\binom{n}{2}}$$

Can find an optimal tour by minimizing a linear function over TSP_n: $\min \{ \langle c, x\rangle : x \in \mathrm{TSP}_n\}$

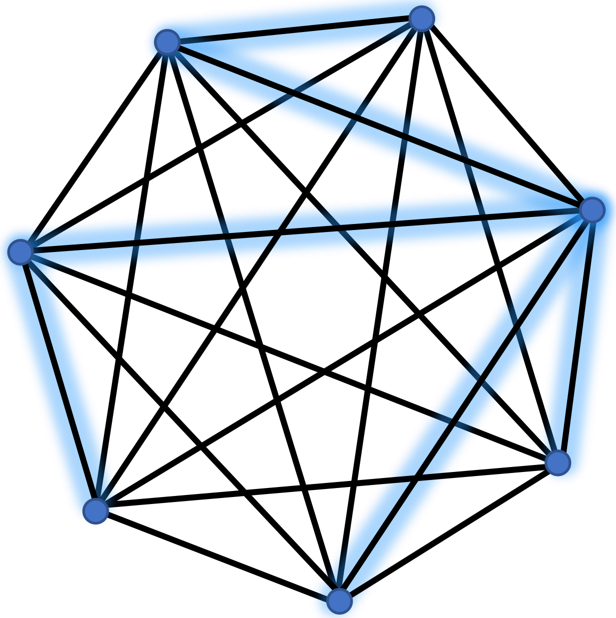


Problem: TSP_n has exponentially many facets!

One can tell the same (short) story for many polytopes associated to NP-complete problems.

Minimum Spanning Tree:

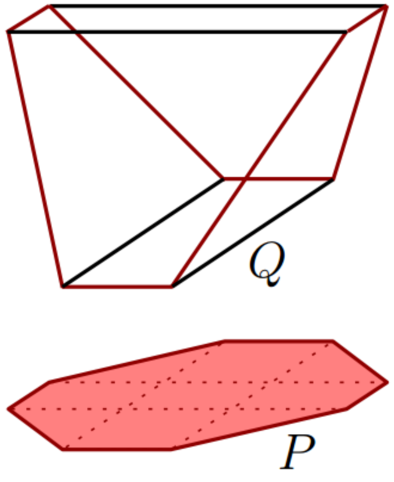
Given n cities $\{1,2,\ldots,n\}$ and costs $c_{ij} \geq 0$ between cities i and j, find a spanning tree of minimum cost.



$$\mathrm{ST}_n = \mathrm{conv}\left(\left\{1_\tau \in \{0,1\}^{\binom{n}{2}} : \tau \text{ is a spanning tree}\right\}\right)$$

Again, has exponentially many facets.

There is a **lift** of ST_n in n^3 dimensions with only $O(n^3)$ facets.
[Martin 1991]



a general model of (small) linear programs

Lifts of polytopes:

A **lift** Q of a polytope $P \subseteq \mathbb{R}^d$ is a polytope $Q \subseteq \mathbb{R}^N$ for $N \geq d$ such that Q linearly projects to P. If we can optimize linear functions over Q, then we can optimize over P.

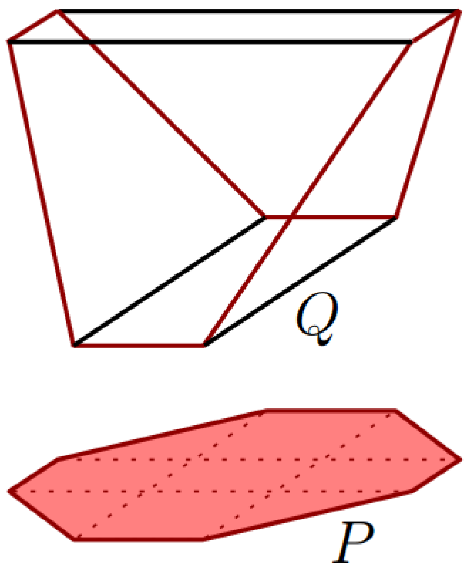
LP design point of view:

A lift corresponds to introducing (arbitrary) new variables and inequalities.

facets in the lift ⇔ # inequality constraints in the LP

Extension complexity:

The **extension complexity** of P is the minimal # of facets in a lift of P.



Examples with exponential savings:

Spanning trees, s-t flows, the permutahedron, …

a general model of (small) linear programs

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Powerful model of computation.

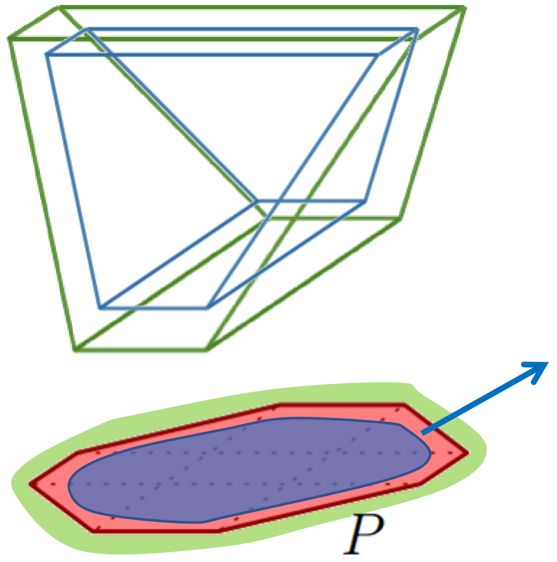
Even more powerful when we allow **approximation**.

Indication of power:

Dominant technique in the design of approximation algorithms.
Integrality gaps for LPs often lead to NP-hardness of approximation.

On the other hand:

Polynomial-size LPs for NP-hard problems would show that
$\mathrm{NP} \subseteq \mathrm{P/poly}$. [Rothvoss 2013]



a brief history of extended formulations

1980s: Fellow tries to prove that $P = NP$ by giving a linear program for TSP.

1989: Yannakakis (the referee) shows that every *symmetric* LP for TSP must have exponential size.

. . .

2011: Different set of fellows try to prove that $P = NP$ by giving an **asymmetric** LP for TSP

2012: Fiorini, Massar, Pokutta, Tiwary, de Wolf show that *every* LP for TSP must have exponential size.

2013: Chan, L, Raghavendra, Steurer show that no polynomial-size LP can approximate MAX-CUT within a factor better than 2.
[Goemans-Williamson 1998: SDPs can do factor ≈ 1.139.]

2014: Rothvoss shows that every LP for the matching polytope must have exponential size.

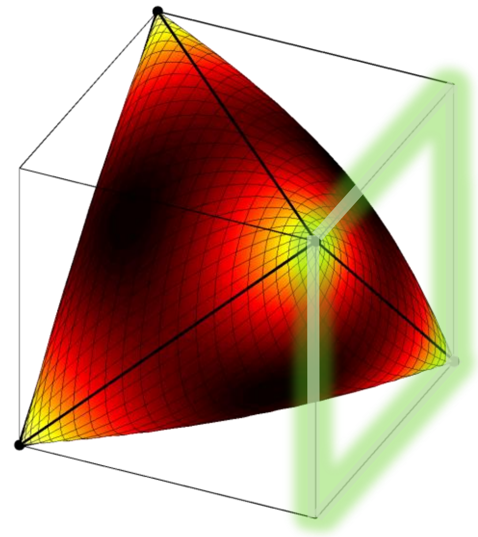
What about semidefinite programs?

semidefinite programs aka spectrahedral lifts

Let $\mathcal{S}_k^+$ denote the cone of $k \times k$ symmetric, positive semidefinite matrices.

A **spectrahedron** is the intersection $\mathcal{S}_k^+ \cap \mathcal{L}$ for some affine subspace $\mathcal{L}$.

This is precisely the feasible region of an SDP.



Definition: A polytope P admits a **PSD lift** of size k if P is a linear projection of a spectrahedron $\mathcal{S}_k^+ \cap \mathcal{L}$.

- Easy to see that minimal size of PSD lift is ≤ minimal size of polyhedral lift
- Assuming the Unique Games Conjecture, integrality gaps for SDPs translate mechanically into NP-hardness of approximation results.
  [Khot-Kindler-Mossel-O'Donnell 2004, Austrin 2007, Raghavendra 2008]
- Sometimes PSD lifts of polytopes are smaller than any polyhedral lift
  Gap of $O(d \log d)$ vs $\Omega(d^2)$ [Fawzi-Saunderson-Parrilo 2015]
  Exponential gaps known for approximation problems like MAX-CUT [Kothari-Meka-Raghavendra 2017]

SDP lifts cannot prove that $P = NP$

From [L-Raghavendra-Steurer 2015]:

Lower bounds on PSD lift size

The TSP_n, CUT_n, and $\mathrm{STAB}(G_n)$ polytopes do not admit PSD lifts of size $c^{n^{2/11}}$ (for some constant $c > 1$ and some family $\{G_n\}$ of n-vertex graphs)

Approximation hardness for constraint satisfaction problems

For max-constraint satisfaction problems, SDPs of polynomial size are equivalent in power to those arising from degree-$O(1)$ SoS relaxations.

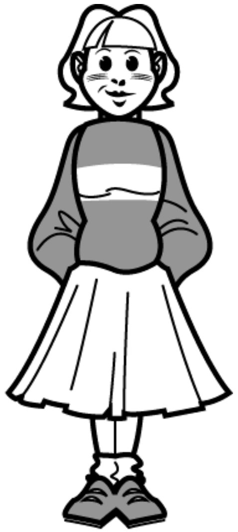
For instance, no family of polynomial-size SDP relaxations can achieve better than a $7/8$-approximation for MAX 3-SAT.

High level: Starting with a small SDP for some problem, we **quantum learn** a roughly equivalent sum-of-squares SDP on a subset of the variables.

communication (in expectation) model

A function $F: \mathcal{A} \times \mathcal{B} \to \mathbb{R}_+$

$a \in \mathcal{A}$



Alice

$\boldsymbol{M}_a$

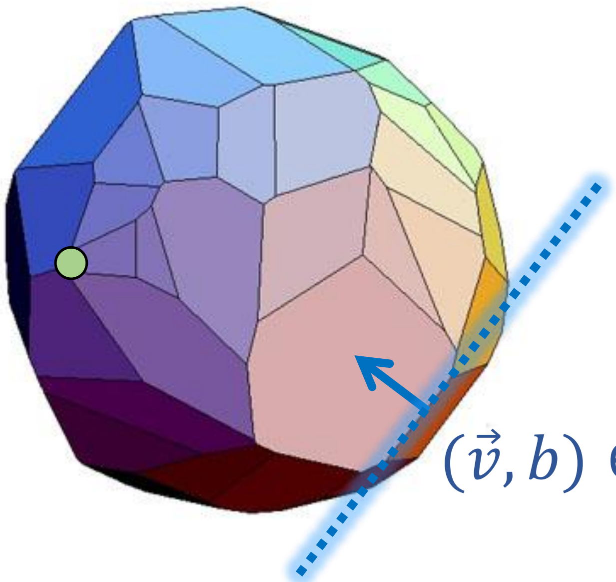
$b \in \mathcal{B}$

Bob

Successful protocol:
Bob outputs a random number $\boldsymbol{B}(b, \boldsymbol{M}_a) \geq 0$ such that for all $a \in \mathcal{A}, b \in \mathcal{B}$,

$$\mathbb{E}[\boldsymbol{B}(b, \boldsymbol{M}_a)] = F(a, b)$$

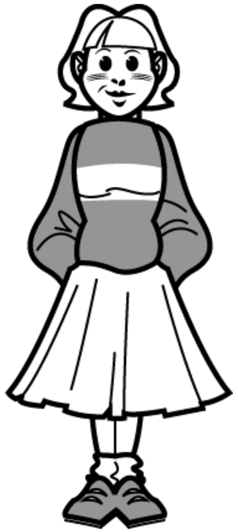
Polytope $P \subseteq \mathbb{R}^n$



$x \in V(P)$

$(\vec{v}, b) \in \mathbb{R}^{n+1}$

$a \in \mathcal{A}$



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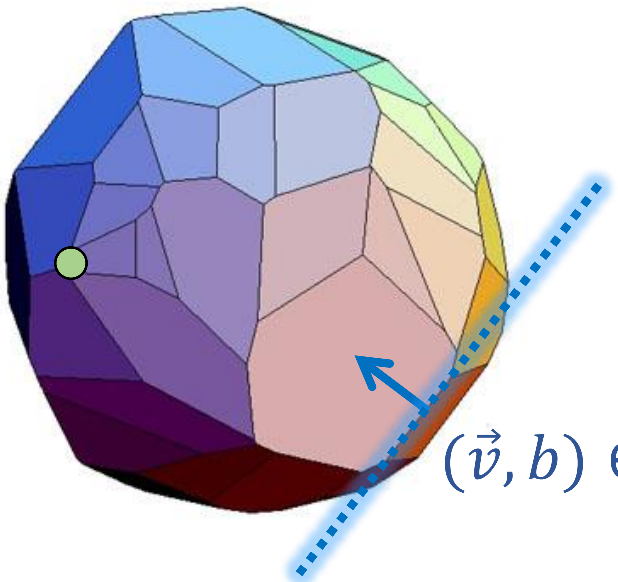

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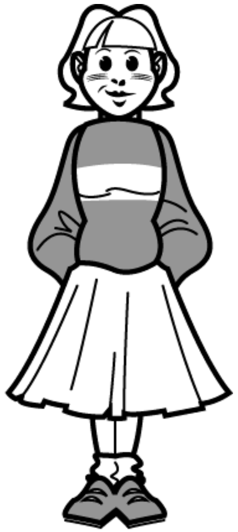

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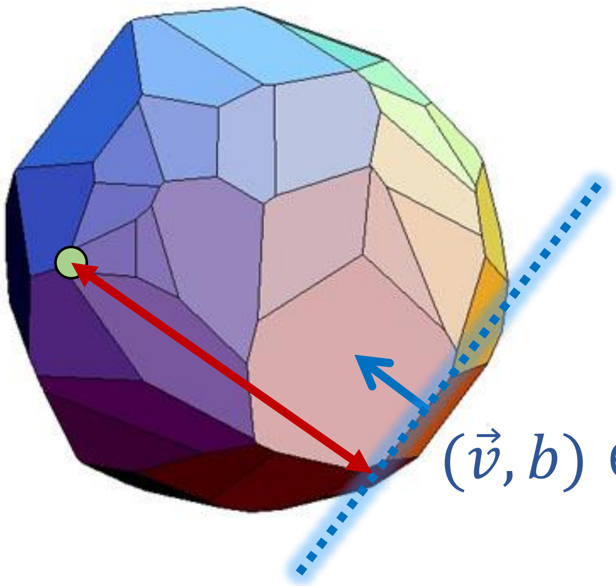
Polytope $P \subseteq \mathbb{R}^n$

$F\big((\vec{v}, b), x\big) = b - \langle \vec{v}, x \rangle$

$x \in \{0,1\}^n$



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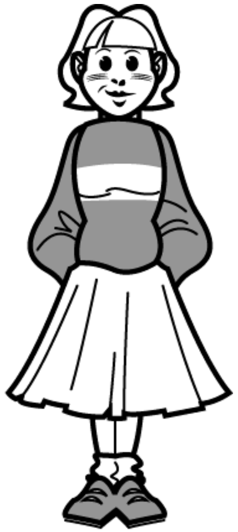

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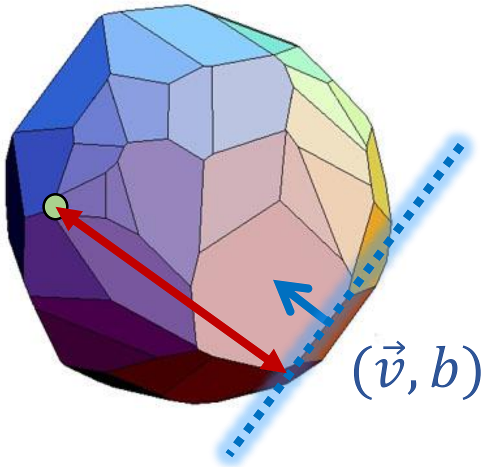

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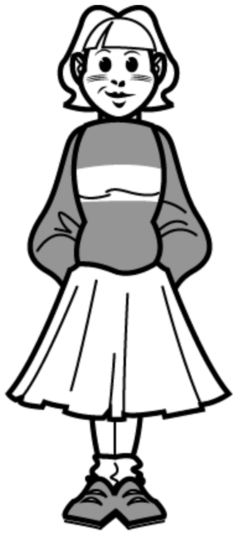

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$\boldsymbol{M}$ m-bit classical message $\Leftrightarrow$
P has a polyhedral lift of size 2^m

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Yannakakis factorization theorem
[+ Fiorini-Massar-Pokutta-Tiwary-de Wolf]

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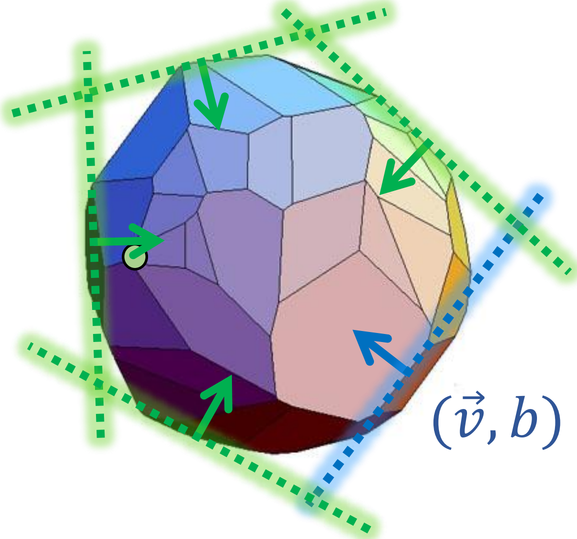

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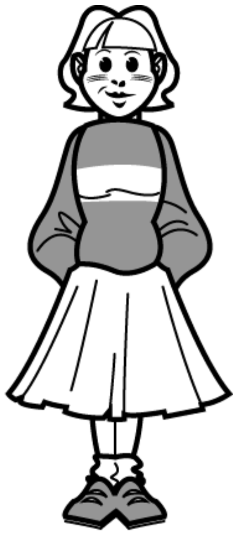

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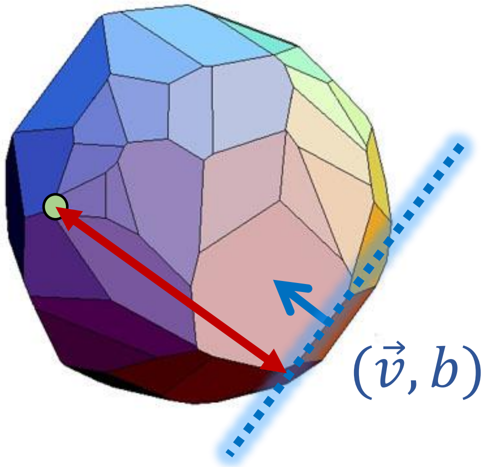

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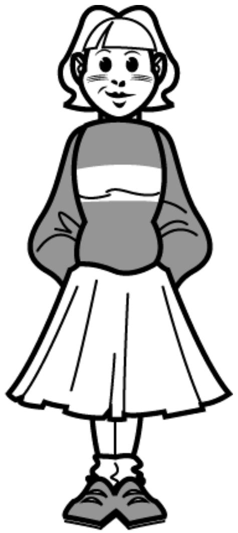

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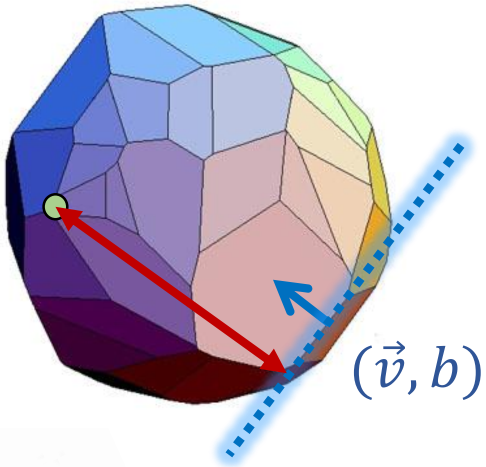
In this model, one-way communication and arbitrary communication are equivalent.
[Kaniewski-T. Lee-de Wolf 2014]

query protocols (k-hapless Bob)

$P \subseteq \mathbb{R}^n$



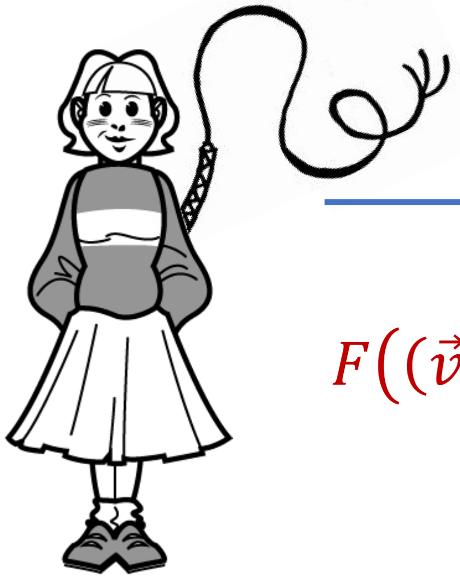
$x \in \{0,1\}^n$

$(\vec{v}, b) \in \mathbb{R}^{n+1}$

Alice specifies k bits
$\Leftrightarrow$ P has a "Sherali Adams" lift of size $\binom{n}{k} 2^k$

Alice specifies k bits in **superposition** + Bob measures
$\Leftrightarrow$ P has a "sum of squares" lift of degree $2k$

[Kaniewski-T. Lee-de Wolf 2014]



$(\vec{v}, b) \in \mathbb{R}^{n+1}$

$\boldsymbol{M}_{(\vec{v},b)}$ →

$F((\vec{v}, b), x) = b - \langle \vec{v}, x \rangle$

$x \in \{0,1\}^n$

Suppose Alice's message says:
Look at k bits of your input in positions:
$i_1, i_2, \ldots, i_k$ and output 1 if you see values
$y_1, y_2, \ldots, y_k$

	1		0	0				1			0
1	1	0	0	0	1	0	1	0	1	1	0

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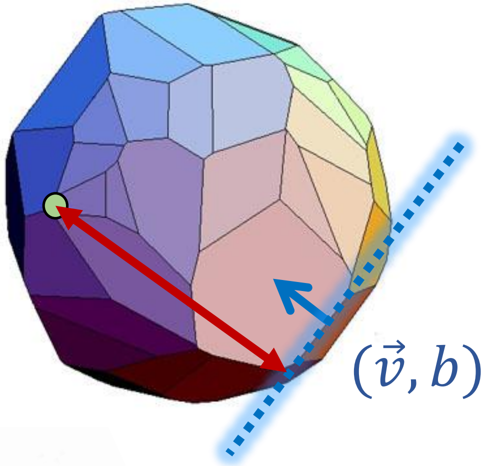

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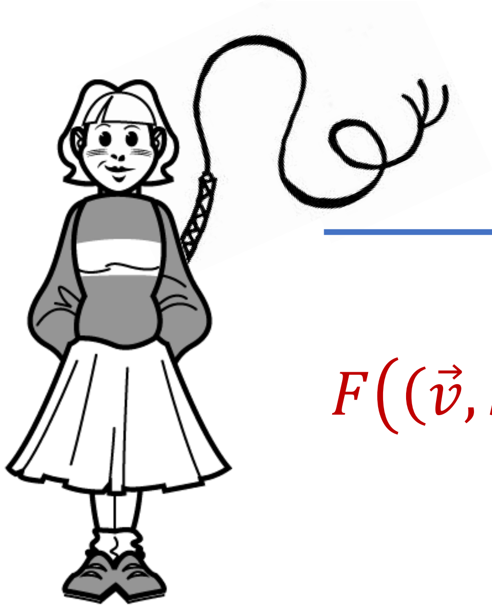

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$x \in \{0,1\}^n$

Alice sends

$$\sum_{|S|=k} \sum_{y \in \{0,1\}^k} a_{S,y} \, |S, y\rangle$$

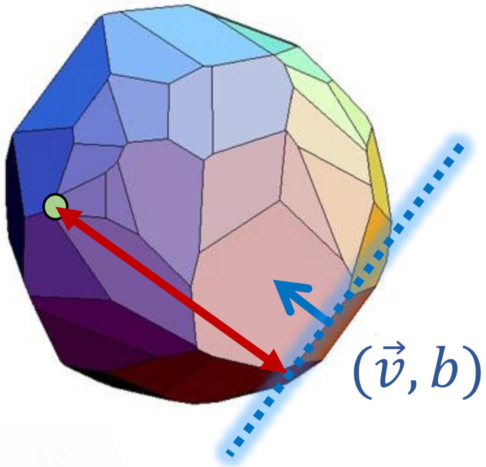
Bob computes

$$\sum_{|S|=k} \sum_{y \in \{0,1\}^k} a_{S,y} \, |S, y\rangle |\mathbf{1}_{x|_S = y}\rangle$$

and then does a computation + measurement (independent of x).

query protocols (k-hapless Bob)

$P \subseteq \mathbb{R}^n$



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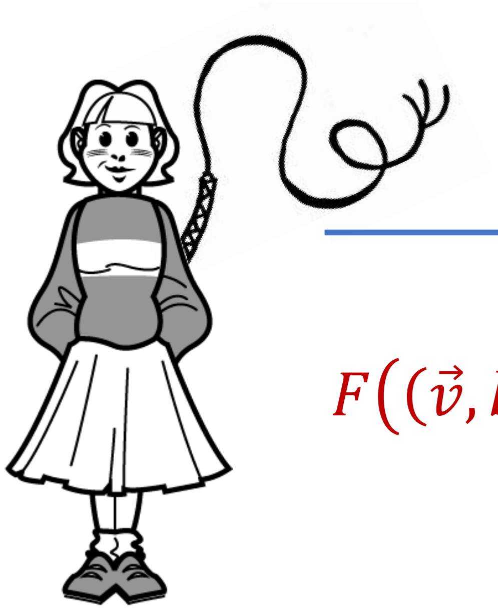

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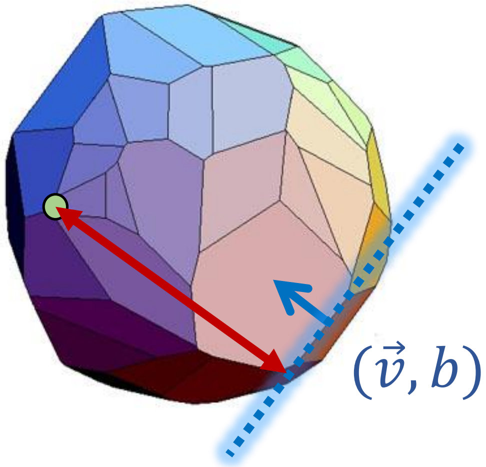
Low-degree sum of squares lifts are well-studied objects in optimization and proof complexity.

Goal:
Relate complexity of arbitrary quantum protocol to complexity of query protocols

simulation by a query protocol

$P \subseteq \mathbb{R}^n$

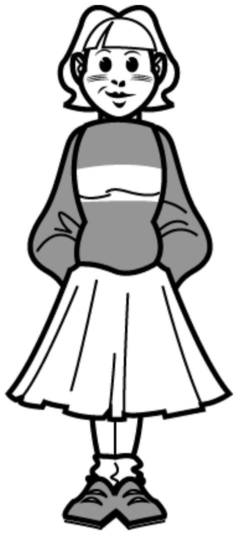
$x \in \{0,1\}^n$



$(\vec{v}, b) \in \mathbb{R}^{n+1}$

Translator converts Alice's message into a "query protocol" message

If a good enough translator exists, then
general protocols ≈ query protocols
⇒ general PSD lifts ≈ SoS lifts



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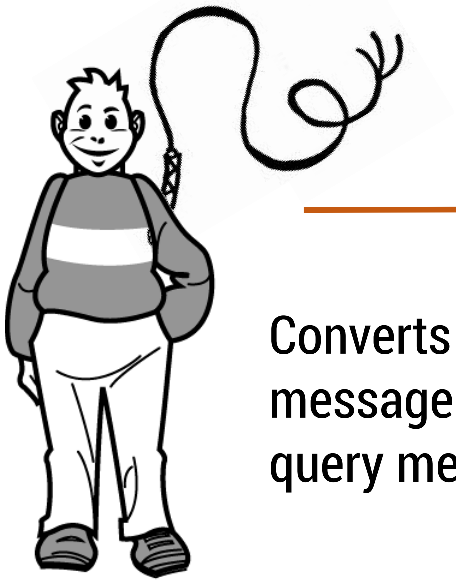


Translator Bob

$\widetilde{\boldsymbol{M}}$

Converts Alice's m-bit message to an $\approx m$-bit query message

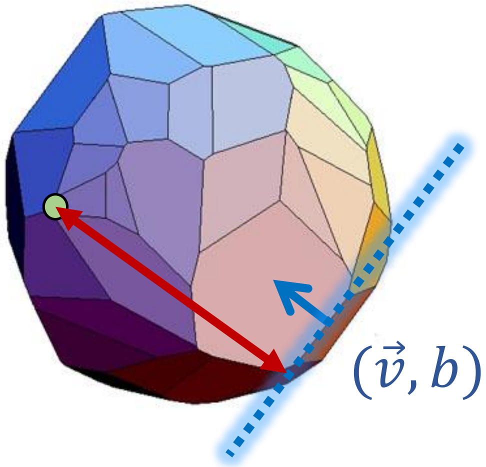
1	1	0	0	0	1	0	1	0	1	1	0

$x \in \{0,1\}^n$

simulation by a query protocol

$P \subseteq \mathbb{R}^n$

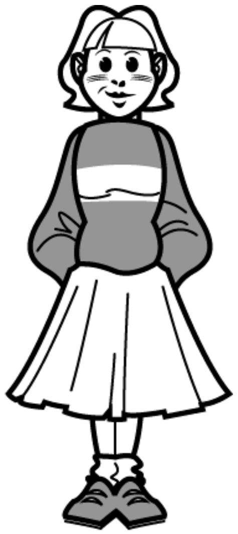
$x \in \{0,1\}^n$



$(\vec{v}, b) \in \mathbb{R}^{n+1}$

Strategy 1: Be smart

Look at Alice's message, make a judgement about what "bits" are most influential (now in a quantum sense), query them, condition(?), measure(?), recurse on the conditional(?) quantum state.



$(\vec{v}, b) \in \mathbb{R}^{n+1}$

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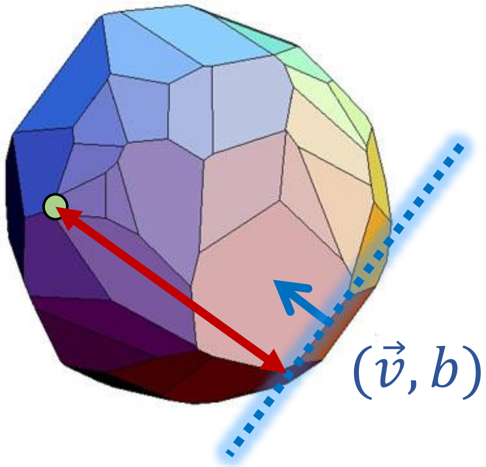
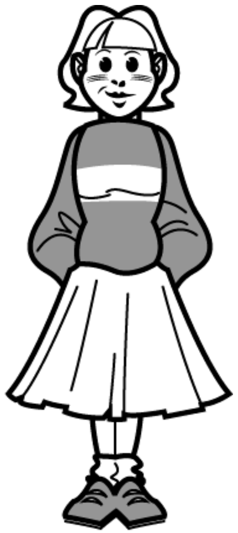


$(\vec{v}, b) \in \mathbb{R}^{n+1}$

Strategy 2: Uhh... machine learning?

AlphaGo



$\boldsymbol{M}_{(\vec{v},b)}$

$$F\big((\vec{v}, b), x\big) = b - \langle \vec{v}, x \rangle$$

$(\vec{v}, b) \in \mathbb{R}^{n+1}$

Translator Bob

Converts Alice's m-bit message to an $\approx m$-bit query message

$\widetilde{\boldsymbol{M}}$

1	1	0	0	0	1	0	1	0	1	1	0

$x \in \{0,1\}^n$

$\approx$

Represent Bob as a QC state:

$$\Phi_{\text{Bob}} = \sum_{x \in \{0,1\}^n} |x\rangle\langle x| \otimes \rho_B^x$$

Key property:

$$\mathcal{D}(\Phi_{\text{Bob}} \,||\, \mathcal{U}) \leq O(m)$$

$\mathcal{D}$ is the relative von Neumann entropy,
$\mathcal{U}$ is the maximally mixed state

Recall: m is the # of qubits in Alice's message.

This holds when F (the function they are computing) is (mildly) reasonable.

the approximation of Blob

$$\Phi_{\text{Bob}} = \sum_{x\in\{0,1\}^n} |x\rangle\langle x| \otimes \rho_B^x$$

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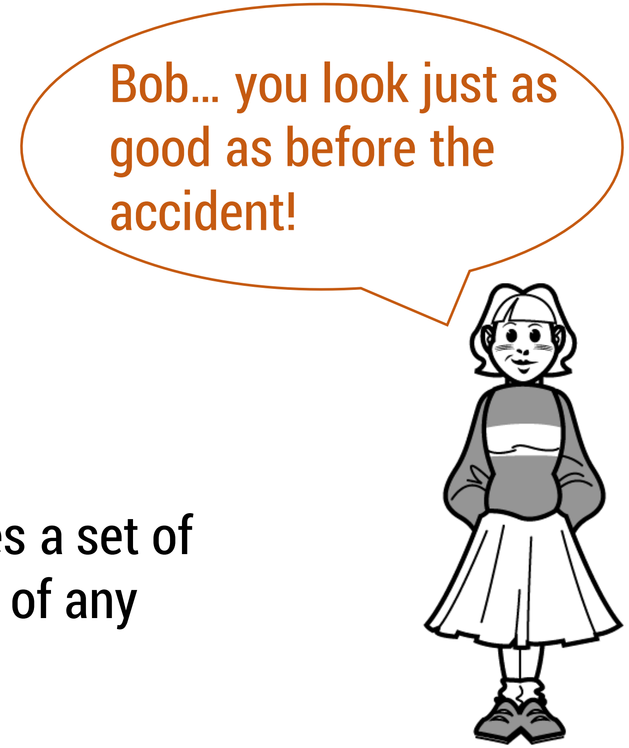

Notion of approximation:

It's enough that Alice cannot distinguish Bob from $\widetilde{\text{Bob}}$.

By the min-max theorem, Alice encodes a set of QC measurements that ensure validity of any potential Bob Φ:

$$\mathrm{Tr}(A_1\Phi) \approx \epsilon_1$$
$$\mathrm{Tr}(A_2\Phi) \approx \epsilon_2$$
$$\mathrm{Tr}(A_3\Phi) \approx \epsilon_3$$
$$\dots$$

$$A = \sum_{x\in\{0,1\}^n} |x\rangle\langle x| \sum_{(\vec{v},b)} \alpha_{(\vec{v},b)}(x)\ \rho_A^{(\vec{v},b)}$$

Jaynes' principle of maximum entropy

$$\Phi_{\text{Bob}} = \sum_{x\in\{0,1\}^n} |x\rangle\langle x| \otimes \rho_B^x$$

$$\mathcal{D}(\Phi_{\text{Bob}} \,\|\, \mathcal{U}) \le O(m)$$

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[validity tests for potential Bob Φ]

Find the "simplest" Bob that passes all the tests:

Minimize $\mathcal{D}(\Phi \,\|\, \mathcal{U})$

subject to:

$$\mathrm{Tr}(A_1\Phi) \approx \epsilon_1$$
$$\mathrm{Tr}(A_2\Phi) \approx \epsilon_2$$
$$\mathrm{Tr}(A_3\Phi) \approx \epsilon_3$$
$$\ldots$$

$$\mathrm{Tr}(\Phi) = 1$$
$$\Phi \succcurlyeq 0$$

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[validity tests for potential Bob Φ]

Find the "simplest" $\widetilde{\text{Bob}}$ that passes all the tests:

Minimize $\mathcal{D}(\Phi \,\|\, \mathcal{U})$

subject to:

$$\mathrm{Tr}(A_1\Phi) \approx \mathrm{Tr}(A_1\Phi_{\text{Bob}})$$
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$$\mathrm{Tr}(A_3\Phi) \approx \mathrm{Tr}(A_3\Phi_{\text{Bob}})$$
$$\dots$$

$$\mathrm{Tr}(\Phi) = 1$$
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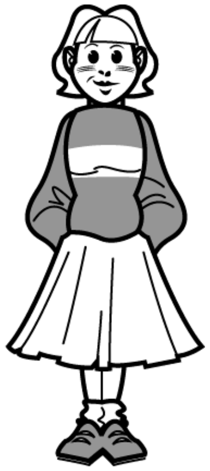
… and just hope?



Jaynes' principle of maximum entropy

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$\mathcal{D}$ is the relative von Neumann entropy,
$\mathcal{U}$ is the maximally mixed state

$\approx$

$$\tilde{A} = \sum_{x\in\{0,1\}^n} |x\rangle\langle x| \sum_{(\vec{v},b)} \tilde{\alpha}_{(\vec{v},b)}(x)\, \rho_A^{(\vec{v},b)}$$

[validity tests for potential Bob Φ]

Find the "simplest" $\widetilde{\text{Bob}}$ that passes all the tests:

Minimize $\mathcal{D}(\Phi \,\|\, \mathcal{U})$

subject to:

$$\text{Tr}(\widetilde{A_1}\Phi) \approx \text{Tr}(\widetilde{A_1}\Phi_{\text{Bob}})$$
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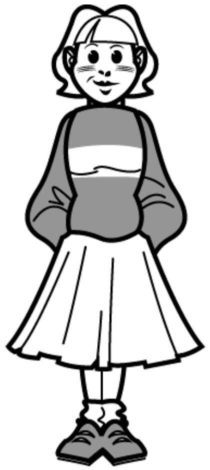
$$\cdots$$

$$\text{Tr}(\Phi) = 1$$
$$\Phi \succcurlyeq 0$$

… and just hope?

If we hope to learn a k-hapless Bob, then any such Bob is **orthogonal** to the Fourier expansion of $\alpha_{(\vec{v},b)}$ above degree k.



Jaynes' principle of maximum entropy

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[validity tests for potential Bob Φ]

Find the "simplest" $\widetilde{\text{Bob}}$ that passes all the tests:

Minimize $\mathcal{D}(\Phi \,\|\, \mathcal{U})$

subject to:

$$\mathrm{Tr}(\widetilde{A_1}\Phi) \approx \mathrm{Tr}(\widetilde{A_1}\Phi_{\text{Bob}})$$
$$\mathrm{Tr}(\widetilde{A_2}\Phi) \approx \mathrm{Tr}(\widetilde{A_2}\Phi_{\text{Bob}})$$
$$\mathrm{Tr}(\widetilde{A_3}\Phi) \approx \mathrm{Tr}(\widetilde{A_3}\Phi_{\text{Bob}})$$
$$\ldots$$

$$\mathrm{Tr}(\Phi) = 1$$
$$\Phi \succcurlyeq 0$$

Optimal solution:

$$\Phi^* \propto \exp\left(\sum_{j\geq 1} \lambda_j \widetilde{A_j}\right)$$

$$= \exp\left(\sum_{j\geq 1} \lambda_j \frac{\widetilde{A_j}}{2}\right)^2$$

$$\approx \mathrm{poly}\left(\sum_{j\geq 1} \lambda_j \widetilde{A_j}\right)^2$$

Jaynes' principle of maximum entropy

$$\Phi_{\text{Bob}} = \sum_{x\in\{0,1\}^n} |x\rangle\langle x| \otimes \rho_B^x$$

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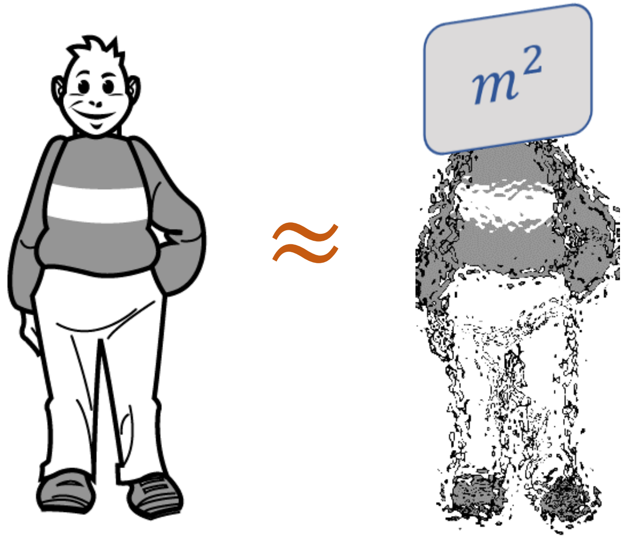

$$\tilde{A} = \sum_{x\in\{0,1\}^n} |x\rangle\langle x| \sum_{(\vec{v},b)} \tilde{\alpha}_{(\vec{v},b)}(x)\, \rho_A^{(\vec{v},b)}$$

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Find the "simplest" $\widetilde{\text{Bob}}$ that passes all the tests:

Minimize $\mathcal{D}(\Phi \,\|\, \mathcal{U})$

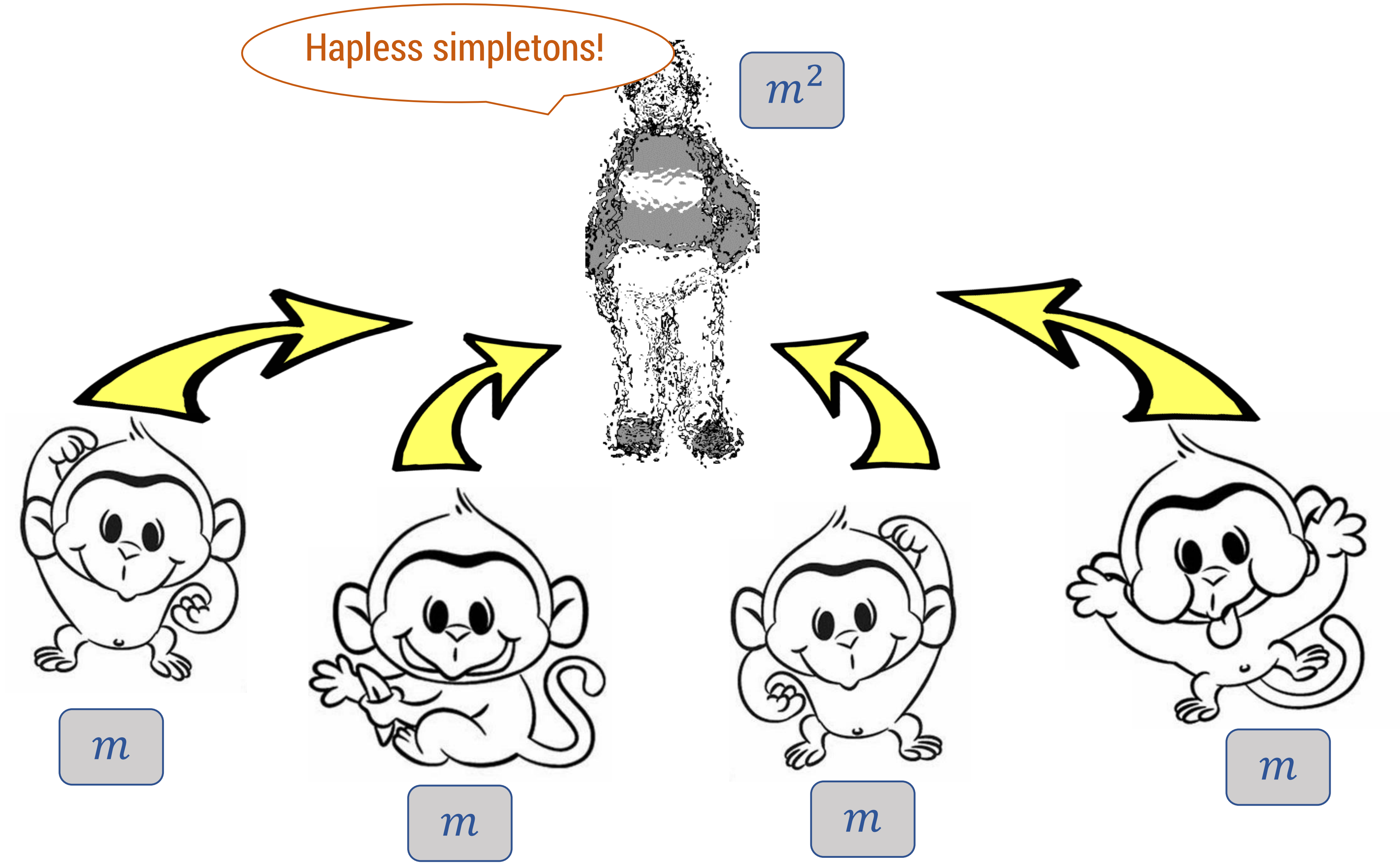
Approximation is an $O(m^2)$-hapless Bob.

subject to:

$$\mathrm{Tr}(\widetilde{A_1}\Phi) \approx \mathrm{Tr}(\widetilde{A_1}\Phi_{\text{Bob}})$$
$$\mathrm{Tr}(\widetilde{A_2}\Phi) \approx \mathrm{Tr}(\widetilde{A_2}\Phi_{\text{Bob}})$$
$$\mathrm{Tr}(\widetilde{A_3}\Phi) \approx \mathrm{Tr}(\widetilde{A_3}\Phi_{\text{Bob}})$$
$$\dots$$

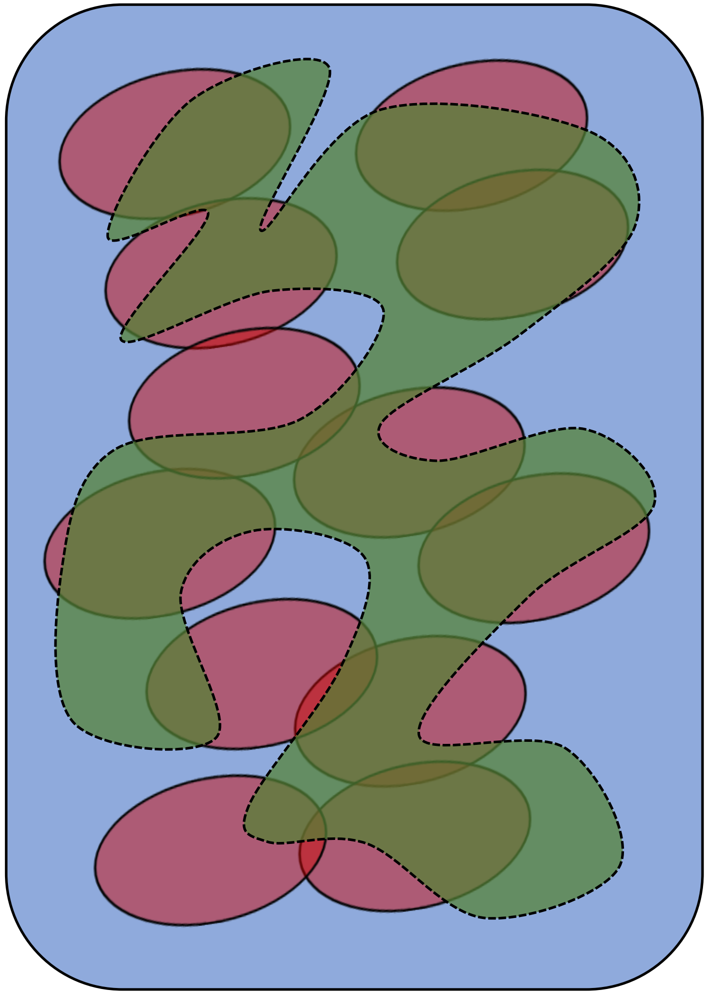
$$\mathrm{Tr}(\Phi) = 1$$
$$\Phi \succcurlyeq 0$$

model is more complex than the training data



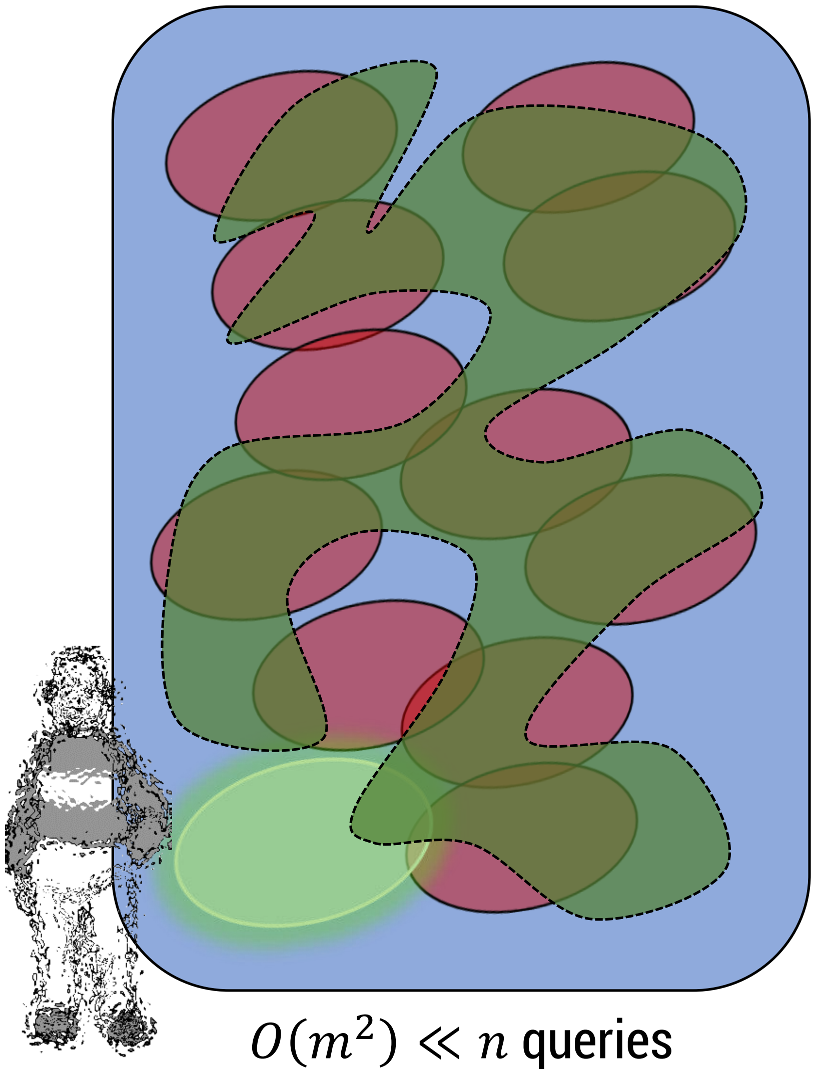
the use of symmetry

n variables



$O(m^2) \ll n$ queries

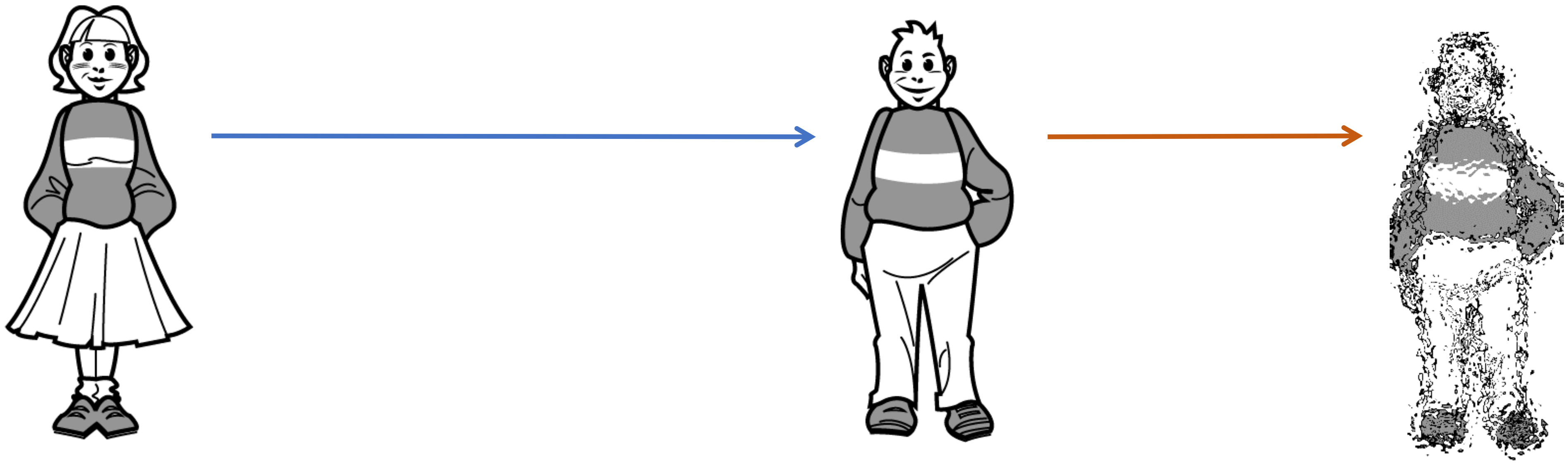
n variables



$O(m^2) \ll n$ queries

Application from yesterday:
"Small SDPs are bad at recognizing separable states"
[Harrow, Natarajan, Wu 2016]

Open problem: Are there small SDP lifts of the perfect matching polytope?
Exponential lower bounds for approximating CSPs?



into the future

Application from yesterday:

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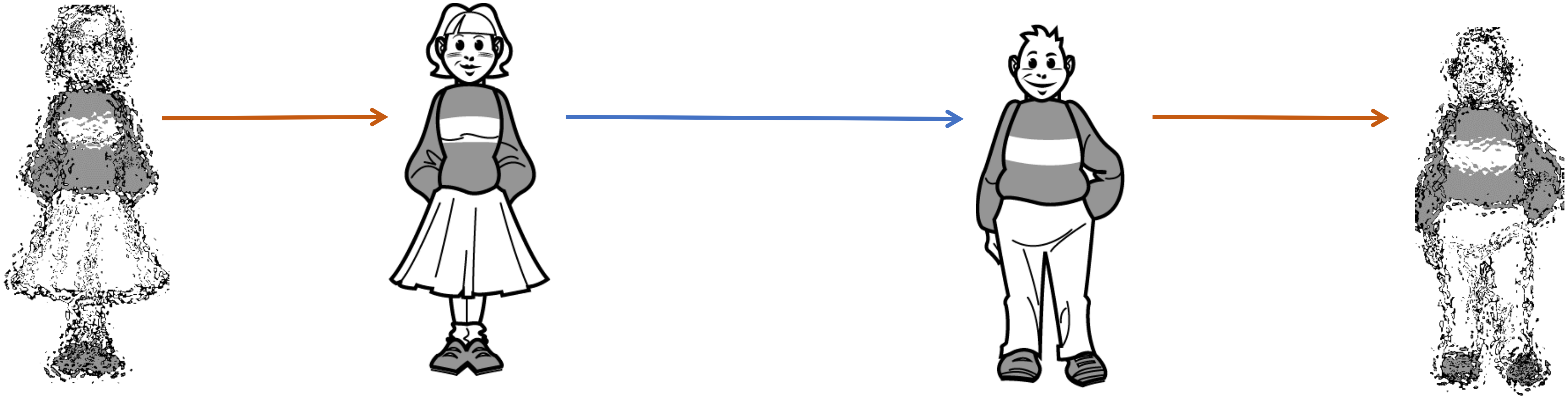

[Kothari-Meka-Raghavendra 2017] do this in the LP setting.

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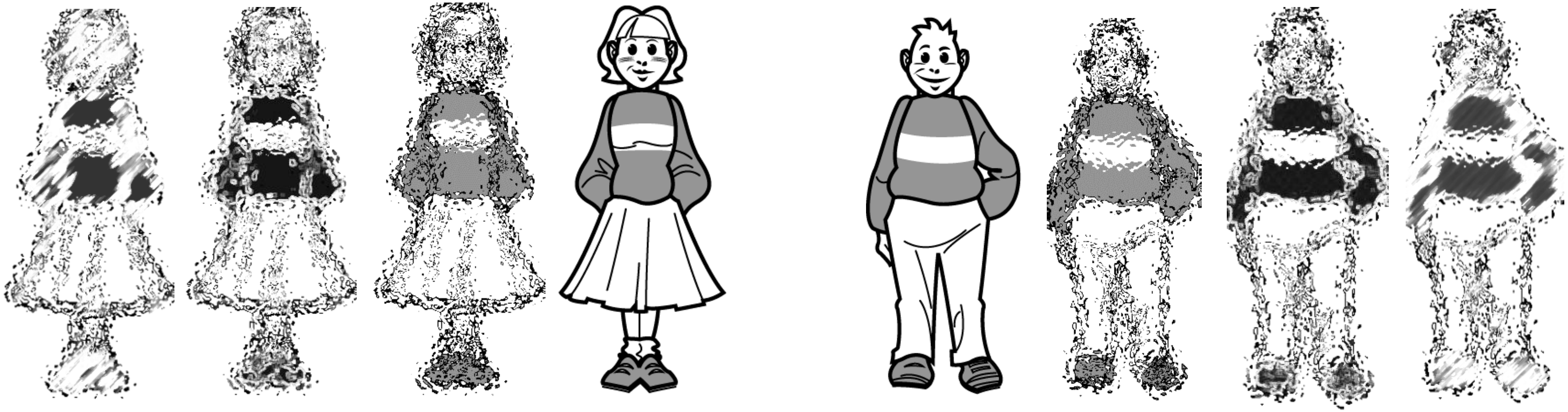

Open problem: Are there small SDP lifts of the perfect matching polytope?
Exponential lower bounds for approximating CSPs?

Log rank conjecture:

[FMPTW'12] observed that for **Boolean** communication problems, this is equivalent to classical-quantum simulation with polynomial overhead.