Multivariate trace inequalities

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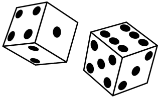
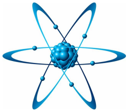
What are trace inequalities and why we should care





1. Main difference between classical and quantum world are complementarity and entanglement
   - Quantum mechanical observables may not be simultaneously measurable (complementarity)
   - Mathematically this means that operators do not need to commute
   - A and B commute if $[A, B] := AB - BA = 0$

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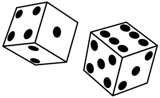
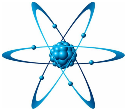

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$\Rightarrow$ Trace inequalities allow us to do that

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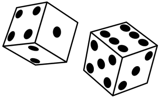
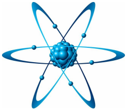

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> To understand QM we need to comprehend the behavior of functions involving matrices that do not commute

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2. Trace inequalities are powerful (mathematical) tools in proofs

Golden-Thompson (GT) inequality (1965)

Golden-Thompson: Let H_1 and H_2 be Hermitian. Then

$$\mathrm{tr}\, \mathrm{e}^{H_1+H_2} \leq \mathrm{tr}\, \mathrm{e}^{H_1} \mathrm{e}^{H_2}$$

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- Not so easy to prove
- If $[H_1, H_2] = 0$ then equality holds (trivial)
- Incredibly useful (wherever matrix exponentials occur)
  - Statistical physics (bound partition function) [Golden-65 & Thompson-65]
  - Random matrix theory (tail bounds via Laplace method) [Ahlswede-Winter-02]
  - Information theory (entropy inequalities) [Lieb-Ruskai-73]
  - Control theory, dynamical systems, $\cdots$
- Does not extend to n matrices (at least not in an obvious way)

GT inequality for more than two matrices

$\operatorname{tr} e^{H_1+H_2} \leq \operatorname{tr} e^{H_1} e^{H_2}$

Extensions to three matrices are not immediate

$\operatorname{tr} e^{H_1+H_2+H_3} \not\leq \operatorname{tr} e^{H_1} e^{H_2} e^{H_3}$

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Lieb's triple matrix inequality (1973)

$$\operatorname{tr} e^{H_1+H_2+H_3} \leq \int_0^\infty \mathrm{d}\lambda \, \operatorname{tr} e^{H_1} \left(e^{-H_2} + \lambda\right)^{-1} e^{H_3} \left(e^{-H_2} + \lambda\right)^{-1}$$

Equivalent to many other interesting statements

- Lieb's concavity theorem: $A \mapsto \operatorname{tr} \exp(H + \log A)$ is concave
- Strong subadditivity of quantum entropy (SSA): $H(AB) + H(BC) - H(ABC) - H(B) \geq 0$

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Open problem: $\exists$ extensions of GT for more than 3 matrices?

Outline for the rest of the talk

1. Understanding GT better (intuitive proof based on pinching)
2. Extending GT to n matrices
3. Tightening the result (using interpolation theory)
4. Application: entropy inequalities via extended GT

The spectral pinching method

Question: How do we force matrices to commute, changing them as little as possible?

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Any positive definite matrix A can be written (spectral decomposition) as

$$A = \sum_{\lambda \in \mathrm{spec}(A)} \lambda P_\lambda$$

The pinching map with respect to A is

$$\mathcal{P}_A \;:\; X \mapsto \sum_{\lambda \in \mathrm{spec}(A)} P_\lambda \, X \, P_\lambda$$

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Properties of pinching maps:

1. $[\mathcal{P}_A(X), A] = 0$ for all $X \geq 0$
2. $\mathrm{tr}\, \mathcal{P}_A(X) A = \mathrm{tr}\, AX$ for all $X \geq 0$
3. $\mathcal{P}_A(X) \geq \frac{1}{|\mathrm{spec}(A)|} X$ for all $X \geq 0$

trace is cyclic, i.e., $\mathrm{tr}\, AB = \mathrm{tr}\, BA$

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Operator inequality
$A \geq B \iff A - B \geq 0$

An intuitive proof of the GT inequality

Golden-Thompson: Let H_1 and H_2 be Hermitian. Then

$$\operatorname{tr} \mathrm{e}^{H_1+H_2} \leq \operatorname{tr} \mathrm{e}^{H_1} \mathrm{e}^{H_2}$$

Any Hermitian matrix H can be written as $\log A$ for some positive definite matrix A

Let $H_k := \log A_k \iff A_k = \mathrm{e}^{H_k}$ for $k \in \{1, 2\}$

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Let A_1 and A_2 be positive definite matrices. Then

$$\operatorname{tr} \exp(\log A_1 + \log A_2) \leq \operatorname{tr} A_1 A_2$$

An intuitive proof of the GT inequality (con't)

To show: $\mathrm{tr}\exp(\log A_1 + \log A_2) \leq \mathrm{tr} A_1 A_2$

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$$\log \operatorname{tr} \exp(\log A_1 + \log A_2)$$
$$= \frac{1}{m} \log \operatorname{tr} \exp(\log A_1^{\otimes m} + \log A_2^{\otimes m})$$

- trace is multiplicative under tensor products, i.e., $\operatorname{tr} B^{\otimes m} = (\operatorname{tr} B)^m$

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- Pinching property 3: $\mathcal{P}_A(X) \geq \frac{1}{|\mathrm{spec}(A)|} X$
- $|\mathrm{spec}(A^{\otimes m})| = \binom{m+d-1}{d-1} = \mathrm{poly}(m)$
- $\log(\cdot)$ is operator monotone, i.e. $X \geq Y \Rightarrow \log X \geq \log Y$
- $\operatorname{tr} \exp(\cdot)$ is operator monotone

If $\mathrm{spec}(A) = \{\lambda_1, \lambda_2\}$ then $\mathrm{spec}(A^{\otimes 2}) = \{\lambda_1^2, \lambda_1\lambda_2, \lambda_2\lambda_1, \lambda_2^2\}$

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- Pinching property 1: $[\mathcal{P}_A(X), A] = 0$
- $\log A + \log B = \log AB$ if $[A, B] = 0$

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- Pinching property 2: $\operatorname{tr}\mathcal{P}_A(X)A = \operatorname{tr} AX$

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- $\lim_{m\to\infty} \frac{\log \mathrm{poly}(m)}{m} = 0$

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Why should this be intuitive?

Extension of GT to n matrices

Same proof technique can be applied (pinch iteratively)

Fact: For any $A > 0$ $\exists$ a probability measure μ on $\mathbb{R}$ such that

$$\mathcal{P}_A(X) = \int_{-\infty}^{\infty} \mu(\mathrm{d}t) A^{\mathrm{i}t} X A^{-\mathrm{i}t}$$

- Note that $A^{\mathrm{i}t}$ is a unitary that commutes with A

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$$\operatorname{tr} \mathrm{e}^{H_1 + H_2 + H_3} \leq \sup_{t \in \mathbb{R}} \operatorname{tr} \mathrm{e}^{H_1} \mathrm{e}^{\frac{1+\mathrm{i}t}{2} H_2} \mathrm{e}^{H_3} \mathrm{e}^{\frac{1-\mathrm{i}t}{2} H_2}$$

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  Example: $n = 4$

$$\operatorname{tr} \mathrm{e}^{H_1+H_2+H_3+H_4} \leq \sup_{t_1,t_2\in\mathbb{R}} \operatorname{tr} \mathrm{e}^{H_1} \mathrm{e}^{\frac{1+\mathrm{i}t_1}{2} H_2} \mathrm{e}^{\frac{1+\mathrm{i}t_2}{2} H_3} \mathrm{e}^{H_4} \mathrm{e}^{\frac{1-\mathrm{i}t_2}{2} H_3} \mathrm{e}^{\frac{1-\mathrm{i}t_1}{2} H_2}$$

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Can we replace the supremum by something independent of H_k?

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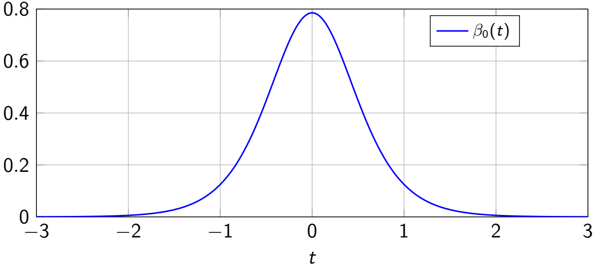
Extension of GT to n matrices (con't)

n matrix extension of GT: Let $p \geq 1$, $n \in \mathbb{N}$ and consider a collection $\{H_k\}_{k=1}^n$ of Hermitian matrices. Then

$$\log \left\| \exp\left(\sum_{k=1}^{n} H_k \right) \right\|_p \leq \int_{-\infty}^{\infty} \mathrm{d}t \, \beta_0(t) \, \log \left\| \prod_{k=1}^{n} \exp\big((1+\mathrm{i}t)H_k\big) \right\|_p$$

where

$$\beta_0(t) := \frac{\pi}{2}\big(\cosh(\pi t) + 1\big)^{-1}$$



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- Let $n = 3$ and $p = 2$

$$\begin{aligned} \operatorname{tr} \mathrm{e}^{H_1+H_2+H_3} &\leq \int_{-\infty}^{\infty} \mathrm{d}t \, \beta_0(t) \operatorname{tr} \mathrm{e}^{H_1} \mathrm{e}^{\frac{1+\mathrm{i}t}{2} H_2} \mathrm{e}^{H_3} \mathrm{e}^{\frac{1-\mathrm{i}t}{2} H_2} \\ &= \int_0^{\infty} \mathrm{d}\lambda \, \operatorname{tr} \mathrm{e}^{H_1} \left(\mathrm{e}^{-H_2} + \lambda\right)^{-1} \mathrm{e}^{H_3} \left(\mathrm{e}^{-H_2} + \lambda\right)^{-1} \end{aligned}$$

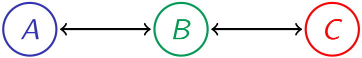
- Reproduces Lieb's triple matrix inequality
- Proof uses complex interpolation theory (Stein-Hirschman — see [Junge-Renner-S-Wilde-Winter-15])
- Complex interpolation theory has been used in QIT recently, e.g., [Beigi-13], [Dupuis-14], [Wilde-15]

Applications

Approximate quantum Markov chains

Strengthened strong subadditivity of entropy

Approximate quantum Markov chains



Definition: A density matrix ρ_{ABC} is a *quantum Markov chain* (QMC) if there exists a recovery map $\mathcal{R}_{B\to BC}$ such that

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Theorem [Petz-88]: ρ_{ABC} is a QMC iff $I(A:C|B) = 0$ with

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Question: What about states such that $I(A:C|B) \leq \epsilon$?

Approximate quantum Markov chains

Definition: A density matrix ρ_{ABC} is a *quantum Markov chain* (QMC) if there exists a recovery map $\mathcal{R}_{B\to BC}$ such that

$$\rho_{ABC} = (\mathcal{I}_A \otimes \mathcal{R}_{B\to BC})(\rho_{AB})$$

Theorem [Petz-88]: ρ_{ABC} is a QMC iff $I(A:C|B) = 0$ with

$$\mathcal{R}_{B\to BC} : X_B \mapsto \rho_{BC}^{\frac{1}{2}}(\rho_B^{-\frac{1}{2}} X_B \rho_B^{-\frac{1}{2}} \otimes \mathrm{id}_C)\rho_{BC}^{\frac{1}{2}}$$

Question: What about states such that $I(A:C|B) \leq \epsilon$?

Theorem [Fawzi-Renner-14]: For any ρ_{ABC} there exists $\mathcal{R}_{B\to BC}$ such that

$$I(A:C|B)_\rho \geq -2\log F\big(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB})\big) \geq 0$$

Why the classical case is easy

Theorem **[Fawzi-Renner-14]:** For any ρ_{ABC} there exists $\mathcal{R}_{B\to BC}$ such that

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Suppose A, B, and C are classical (i.e., ρ_{AB}, ρ_{BC}, and ρ_B are diagonal)

$$\begin{aligned} I(A:C|B)_\rho &= D\big(\rho_{ABC}\,\|\,\exp(\log\rho_{AB} + \log\rho_{BC} - \log\rho_B)\big) \\ &= D\big(\rho_{ABC}\|\rho_{BC}^{\frac{1}{2}}(\rho_B^{-\frac{1}{2}}\rho_{AB}\rho_B^{-\frac{1}{2}} \otimes \mathrm{id}_C)\rho_{BC}^{\frac{1}{2}}\big) \\ &= D\big(\rho_{ABC}\|\mathcal{R}_{B\to BC}(\rho_{AB})\big) \end{aligned}$$

If $[X,Y] := XY - YX = 0$, then $\log XY = \log X + \log Y$

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If A, B, and C are quantum, the density matrices ρ_{AB}, ρ_{BC}, and ρ_B are not diagonal and do not commute

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$$D(\rho\|\sigma) \geq -2\log F(\rho,\sigma)$$

Details about Fawzi-Renner-14

Theorem [Fawzi-Renner-14]: For any ρ_{ABC} there exists $\mathcal{R}_{B\to BC}$ such that

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Measured relative entropy: $D_{\mathbb{M}}(\rho\|\sigma) := \sup_{\mathcal{M}} D\big(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)\big)$

1. $D_{\mathbb{M}}(\rho\|\sigma) \geq -2\log F(\rho,\sigma)$
2. $D_{\mathbb{M}}(\rho\|\sigma) = D(\rho\|\sigma)$ iff $[\rho,\sigma]=0$

There are several generalizations and improvements of the Fawzi-Renner bound (see QIP 2016)

Open question: $\exists$ a bound that is tight in the classical case with an explicit and universal recovery map?

Application: Strenghtened strong subadditivity

Variational formula for relative entropy [Petz-88]:

$$D(\rho\|\sigma) = \sup_{\omega>0} \mathrm{tr}\,\rho \log \omega + 1 - \mathrm{tr}\exp(\log \sigma + \log \omega)$$

Variational formula for measured relative entropy [Berta-Fawzi-Tomamichel-15]:

$$D_{\mathbb{M}}(\rho\|\sigma) = \sup_{\omega>0} \mathrm{tr}\,\rho \log \omega + 1 - \mathrm{tr}\sigma\omega$$

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$I(A:C|B)_\rho$

$= D\big(\rho_{ABC} \| \exp(\log \rho_{AB} + \log \rho_{BC} - \log \rho_B)\big)$

Follows by definition

$D(\rho\|\sigma) := \operatorname{tr}\rho \log \rho - \operatorname{tr}\rho \log \sigma$

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Variational formula for relative entropy

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&I(A:C|B)_\rho \\
&= D\big(\rho_{ABC}\|\exp(\log\rho_{AB} + \log\rho_{BC} - \log\rho_B)\big) \\
&= \sup_{\omega>0} \operatorname{tr}\rho_{ABC}\log\omega + 1 - \operatorname{tr}\exp(\log\rho_{AB} + \log\rho_{BC} - \log\rho_B + \log\omega) \\
&\geq \sup_{\omega>0} \operatorname{tr}\rho_{ABC}\log\omega + 1 - \int_{-\infty}^{\infty}\mathrm{d}t\,\beta_0(t)\operatorname{tr}\rho_{BC}^{\frac{1+it}{2}}\left(\rho_B^{-\frac{1+it}{2}}\rho_{AB}\rho_B^{-\frac{1-it}{2}}\otimes\mathrm{id}_C\right)\rho_{BC}^{\frac{1-it}{2}}\omega
\end{aligned}$$

4 matrix extension of GT ($n = 4$ and $p = 2$)

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&= D_{\mathbb{M}}(\rho_{ABC}\|\mathcal{R}_{B\to BC}(\rho_{AB}))\,, \quad \text{Variational formula for meas. rel. entropy}
\end{aligned}$$

with $\mathcal{R}_{B\to BC}(\cdot) = \int_{-\infty}^{\infty}\mathrm{d}t\,\beta_0(t)\rho_{BC}^{\frac{1+it}{2}}\left(\rho_B^{-\frac{1+it}{2}}(\cdot)\rho_B^{-\frac{1-it}{2}}\otimes\mathrm{id}_C\right)\rho_{BC}^{\frac{1-it}{2}}$

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Strenghtened strong subadditivity (con't)

We just saw that

Theorem: $I(A:C|B)_\rho \geq D_{\mathbb{M}}\big(\rho_{ABC} \| \mathcal{R}_{B\to BC}(\rho_{AB})\big)$

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- Tight for commutative case
- Explicit recovery map that is universal (only depends on ρ_{BC})
- Proof based (only) on 4 matrix extension of GT
- Can be generalized to monotonicity of relative entropy
- Improves Fawzi-Renner and its follow up papers

Conclusions

arXiv:1604.03023
Commun. Math. Phys. 2016

- If matrices do not commute things get complicated
- Trace inequalities are powerful tools expressing relations between matrices that do not commute
- Spectral pinching method is an intuitive approach to prove matrix (trace) inequalities
- Applications:
  - Strengthening of strong subadditivity (FR bound)
  - Hopefully many more (random matrix theory? other entropy inequalities?, ...)

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Thank you

More trace inequalities

Let A and B be positive definite matrices and $q \in \mathbb{R}_+$
$A^q := \exp(q \log A)$ is well-defined

Araki-Lieb-Thirring: Let $r \in [0, 1]$

$$\mathrm{tr}(B^{r/2} A^r B^{r/2})^{\frac{q}{r}} \leq \mathrm{tr}(B^{1/2} A B^{1/2})^q$$

- If $r \geq 1$ the inequality holds in the opposite direction

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- Implies the GT inequality via Lie-Trotter formula

$$\lim_{r \searrow 0} \left(\prod_{k=1}^{n} C_k^r \right)^{\frac{1}{r}} = \exp\left(\sum_{k=1}^{n} \log C_k \right)$$

For $q = 1$ this gives $\mathrm{tr} \exp(\log A + \log B) \leq \mathrm{tr} AB$

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Exercise: Prove ALT via the spectral pinching method

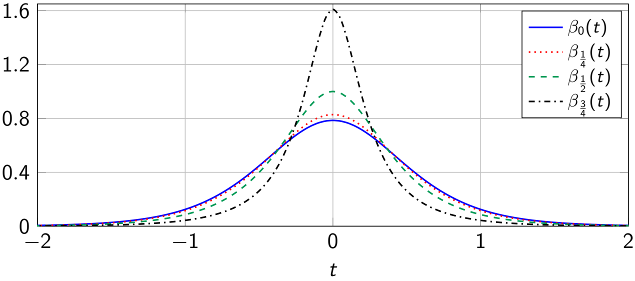
- We can prove extensions to n matrices via pinching or/and interpolation theory

Summary of results

n matrix extension of ALT: Let $p \geq 1$, $r \in (0,1]$, $n \in \mathbb{N}$, and consider a collection $\{A_k\}_{k=1}^n$ of positive semi-definite matrices. Then

$$\log \left\| \left| \prod_{k=1}^{n} A_k^r \right|^{\frac{1}{r}} \right\|_p \leq \int_{-\infty}^{\infty} \mathrm{d}t \, \beta_r(t) \log \left\| \prod_{k=1}^{n} A_k^{1+\mathrm{i}t} \right\|_p$$

- $\beta_r(t) = \frac{\sin(\pi r)}{2r(\cosh(\pi t)+\cos(\pi r))}$ is a probability distribution on $\mathbb{R}$



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- $\beta_r(t) = \frac{\sin(\pi r)}{2r(\cosh(\pi t)+\cos(\pi r))}$ is a probability distribution on $\mathbb{R}$
- Proof uses Stein-Hirschman interpolation theorem
- Using Lie-Trotter (i.e. $r \to 0$) we get as a corollary

n matrix extension of GT: Let $p \geq 1$, $n \in \mathbb{N}$ and consider a collection $\{H_k\}_{k=1}^n$ of Hermitian matrices. Then

$$\log \left\| \exp\left(\sum_{k=1}^n H_k \right) \right\|_p \leq \int_{-\infty}^{\infty} \mathrm{d}t \, \beta_0(t) \log \left\| \prod_{k=1}^n \exp\big((1+\mathrm{i}t)H_k\big) \right\|_p$$

Stein-Hirschman operator interpolation theorem

Strengthening of the Hadamard three lines theorem
see [Junge-Renner-S-Wilde-Winter-15]

- $S := \{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$
- $L(\mathcal{H})$ is the space of bounded linear operators acting on $\mathcal{H}$
- Let $G : \overline{S} \to L(\mathcal{H})$ be
  - uniformly bounded on $\overline{S}$
  - holomorphic on S
  - continuous on the boundary $\partial\overline{S}$
- Let $\theta \in (0,1)$ and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ where $p_0, p_1 \in [1, \infty]$

$$\log \|G(\theta)\|_{p_\theta} \leq \int_{\mathbb{R}} \mathrm{d}t \left(\beta_{1-\theta}(t) \log \|G(\mathrm{i}t)\|_{p_0}^{1-\theta} + \beta_\theta(t) \log \|G(1+\mathrm{i}t)\|_{p_1}^{\theta} \right)$$

with

$$\beta_\theta(t) := \frac{\sin(\pi\theta)}{2\theta\left[\cosh(\pi t) + \cos(\pi\theta)\right]}$$

Proof of n matrix extension of ALT

- Choose $G(z) = \prod_{k=1}^{n} A_k^z$
  - is bounded on $\bar{S}$, holomorphic on S and continuous on ∂S
- Let $\theta = r$, $p_0 = \infty$ and $p_1 = p$
- $\log \|G(1+\mathrm{i}t)\|_{p_1}^{\theta} = r \log \left\| \prod_{k=1}^{n} A_k^{1+\mathrm{i}t} \right\|_p$
- $\log \|G(\mathrm{i}t)\|_{p_0}^{1-\theta} = (1-r) \log \left\| \prod_{k=1}^{n} A_k^{\mathrm{i}t} \right\|_\infty = 0$
- $\log \|G(\theta)\|_{p_\theta} = \log \|\prod_{k=1}^{n} A_k^r\|_{\frac{p}{r}} = r \log \left\| |\prod_{k=1}^{n} A_k^r|^{\frac{1}{r}} \right\|_p$

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Now we apply Stein-Hirschman

$$\log \left\| \left| \prod_{k=1}^{n} A_k^r \right|^{\frac{1}{r}} \right\|_p \leq \int_{-\infty}^{\infty} \beta_r(t) \log \left\| \prod_{k=1}^{n} A_k^{1+\mathrm{i}t} \right\|_p$$

□