Information-Theoretic Tools for Interactive Quantum Protocols, and Applications: Flow of Information, Augmented Index, and DYCK(2)

MATHIEU LAURIÈRE, ASHWIN NAYAK, AND DAVE TOUCHETTE

QIP 2017, Seattle

16 January 2017

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Quantum Advantage for Disjointness

- Disjointness: $x, y \subseteq \{1,2,\dots,n\}$, is $x \cap y = \emptyset$?
- $x = x_1 \cdots x_n, y = y_1 \cdots y_n \in \{0,1\}^n$, looking for i such that $x_i = y_i = 1$
- Quantum Protocol [BCW98]: distributed version of Grover search
- QCC(Disj) = $\Theta(\sqrt{n})$ [BCW98, Razb03, AA03]
- CC(Disj) = $\Omega(n)$ [KS92]

Input: x

Initialize: $\frac{1}{n}\sum_i |\, i \,\rangle$

Oracle call: $\frac{1}{n}\sum_i |\, i \,\rangle |\, x_i \rangle \longrightarrow \frac{1}{n}\sum_i (-1)^{x_i \wedge y_i} |\, i \,\rangle |\, x_i \rangle$

$\frac{1}{n}\sum_i (-1)^{x_i \wedge y_i} |\, i \,\rangle \longleftarrow$

Inversion about the mean
Repeat $\approx \sqrt{n}$ times
Measure to get desired i if intersection

Input: y

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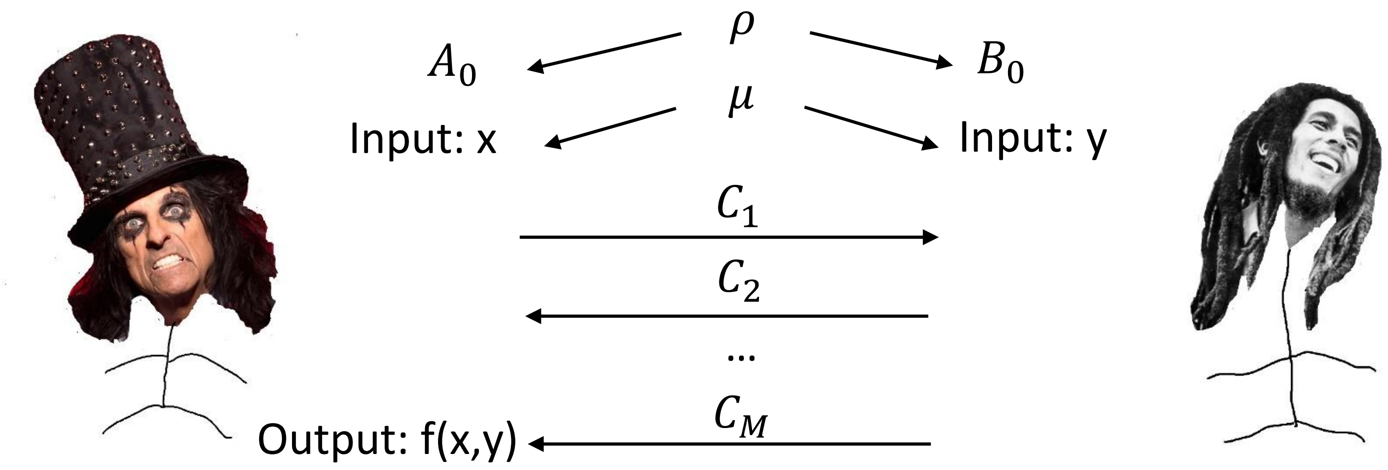
- Quantum Protocol [BCW98]: distributed version of Grover search
- QCC(Disj) = $\Theta(\sqrt{n})$ [BCW98, Razb03, AA03]
- CC(Disj) = $\Omega(n)$ [KS92]
- How does information flow in this protocol?
- Can we avoid transmitting back/forgetting information?

Interactive Communication

- Communication Complexity setting:



- How much **communication** to compute f on $(\mathrm{x},\mathrm{y}) \sim \mu$
- Take information-theoretic view: Information Complexity
  - How much **information** to compute f on $(\mathrm{x},\mathrm{y}) \sim \mu$
- Information content of interactive protocols?
- Classical vs. Quantum?

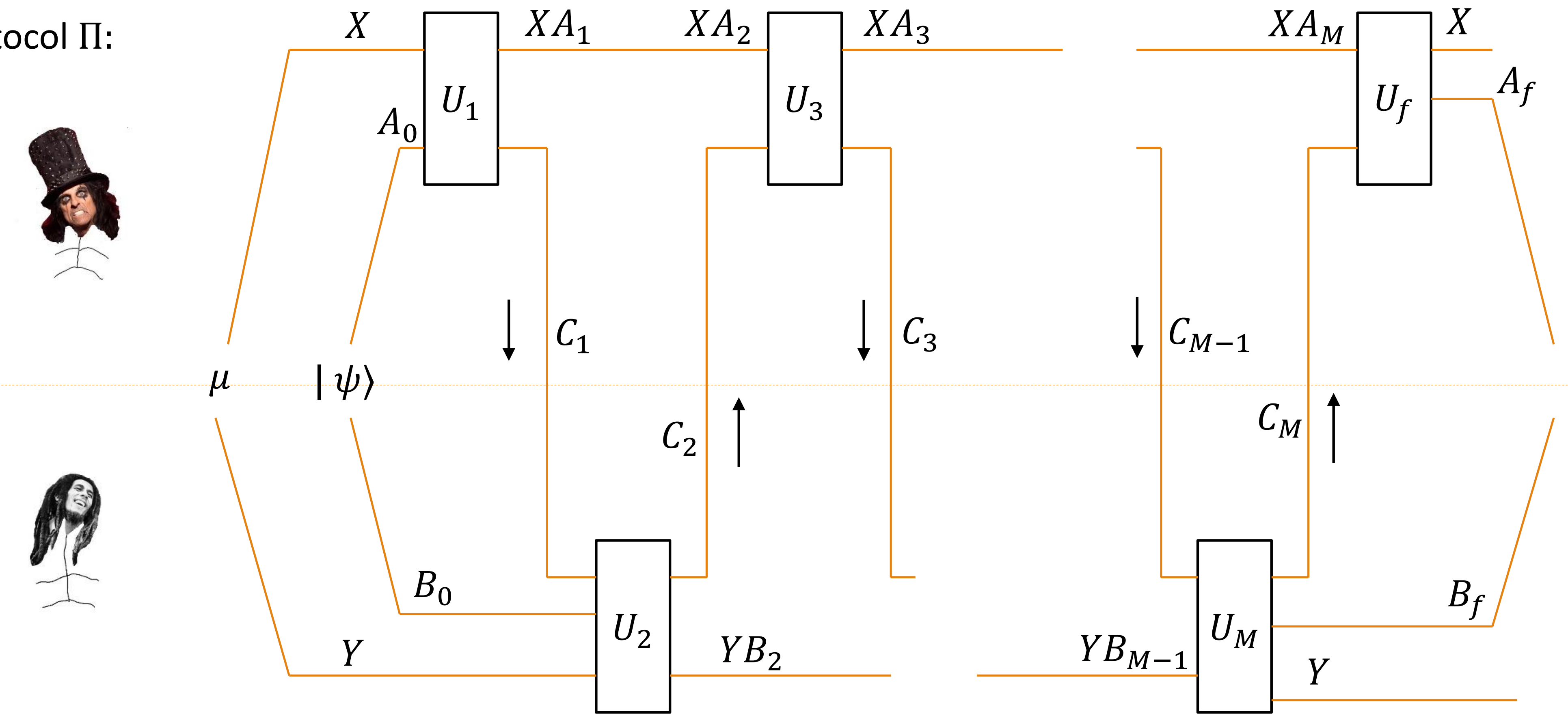
Overview

Based on 2 papers

- 1701.02062: ML & DT, Info. Flow & Cost of Forgetting
  - Th 1: HIC = CIC – CRIC, QIC = CIC + CRIC
    - Tool 1 : Information Flow Lemma
  - Th 2: Π not forgetting for Disjointness => QCC(Π) $\in \Omega(n)$
  - Th 3: Can maintain IC for quantum simulation of classical protocols, and then IC(f_{rdm}) = n (1 - o(1))
- 1610.04937: AN & DT, Aug. Index & Streaming algo. for DYCK(2)
  - Th 4: Any T-pass one-way qu. Streaming algorithm for DYCK(2) requires space $s(N) \in \Omega(\frac{\sqrt{N}}{T^3})$ on length N inputs
  - Th 5: Any t-round protocol for Augmented Index satisfies a QIC trade-off $QIC_{A\rightarrow B}(\Pi, \mu_0) \in \Omega\left(\frac{n}{t^2}\right)$ or $QIC_{B\rightarrow A}(\Pi, \mu_0) \in \Omega\left(\frac{1}{t^2}\right)$
    - Tool 2: Superposition-Average Encoding Theorem
    - Tool 3: Quantum Cut-and-Paste
    - Application of Tool 1

Quantum Communication Complexity

Protocol Π:

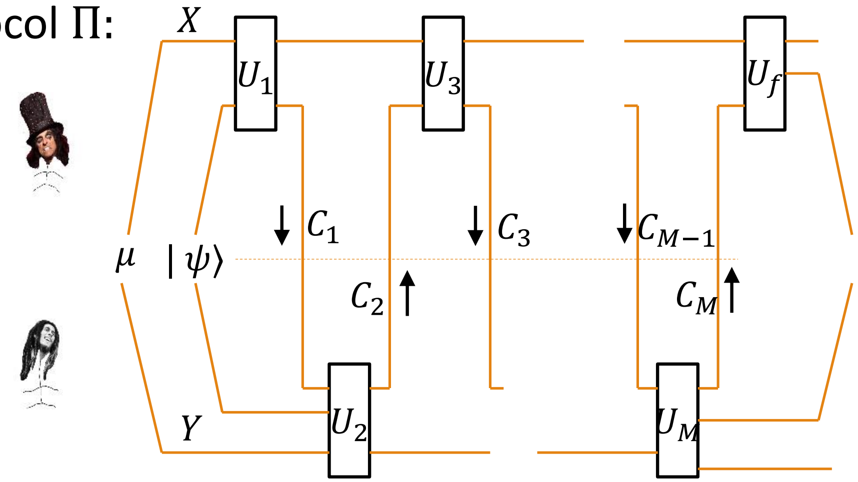


Output: f(X,Y)

Quantum Communication Complexity

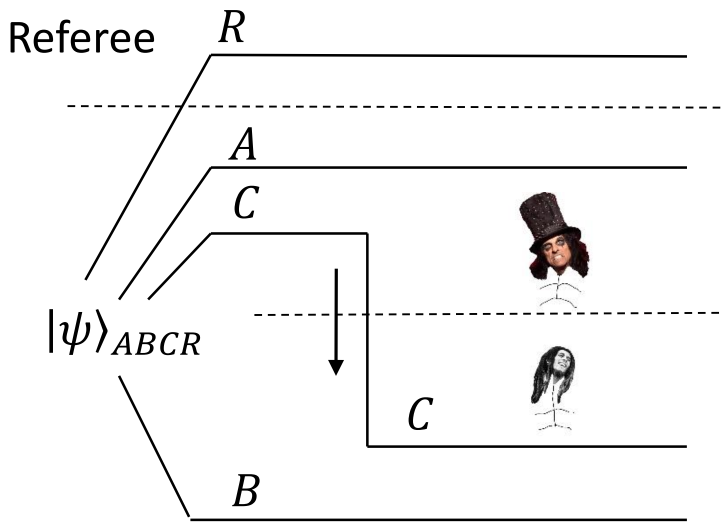
- QCC(f) = $\min_{\Pi}$ QCC(Π)
- Minimization over all Π computing f
- QCC(Π) =$\sum_i \log(\dim(C_i))$; total number of qubits exchanged

Protocol Π:



Quantum Information Theory

- Conditional Quantum Mutual Information
  - $I(R:C|B) = I(R:BC) - I(R:B) = H(R|B) - H(R|BC) = H(RB) + H(BC) - H(B) - H(RBC)$
  - Non-negativity: $I(R:C|B) \geq 0$ [LR73]
  - Chain rule: $I(A:BD|C) = I(A:B|C) + I(A:D|BC)$
  - Invariance under local isometry, satisfies a data processing inequality…
  - Operational interpretation [DY08, YD09]: Quantum state redistribution, optimal communication rate $I(R:C|B) = I(R:C|A)$



[Lieb and Ruskai 1973; Devetak and Yard 2008; Yard and Devetak 2009]

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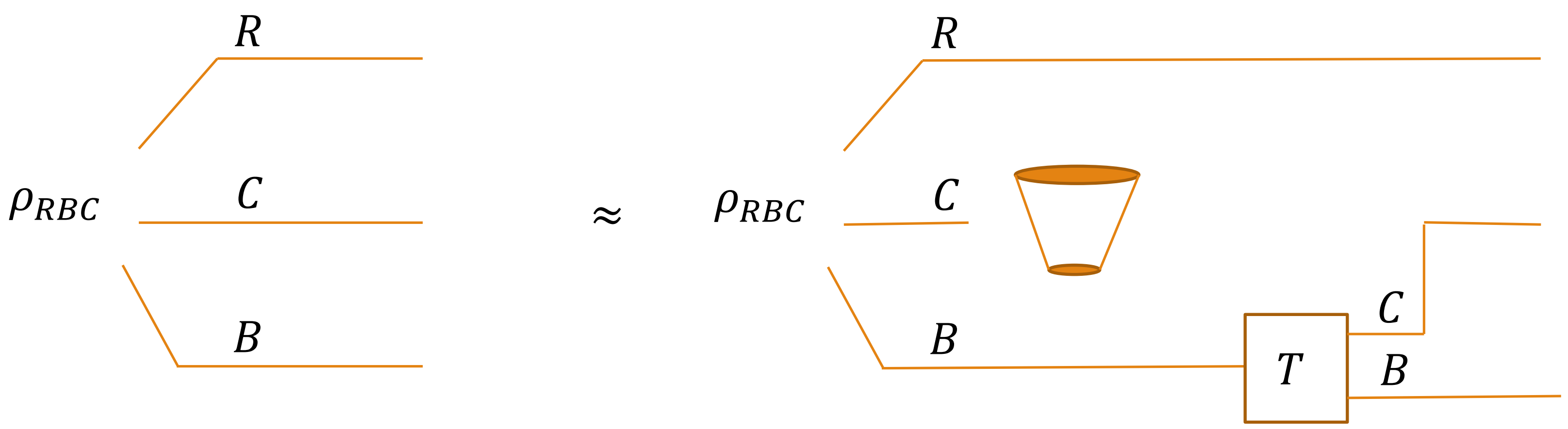
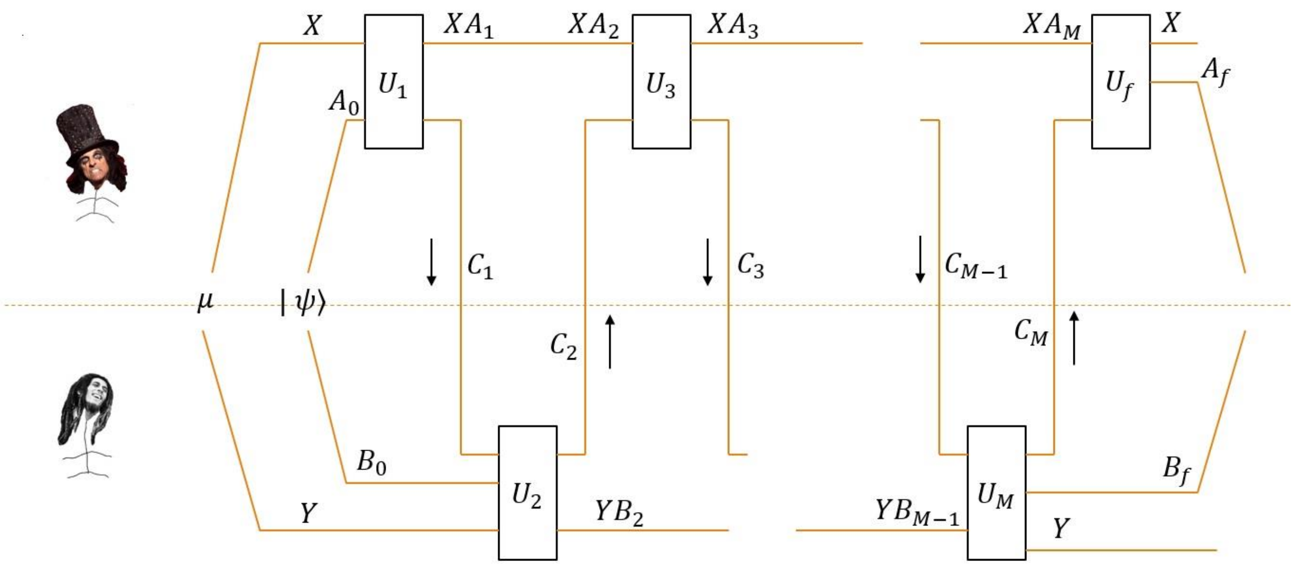
  - Operational interpretation [DY08, YD09]: Quantum state redistribution, optimal communication rate $I(R:C|B) = I(R:C|A)$
  - Recoverability [FR15]
    - There exists a recovery map $T_{B\to BC}$ such that $-\lg F\left(\rho_{RBC}, T_{B\to BC}(\rho_{RB})\right) \leq I(R:C|B)_\rho$

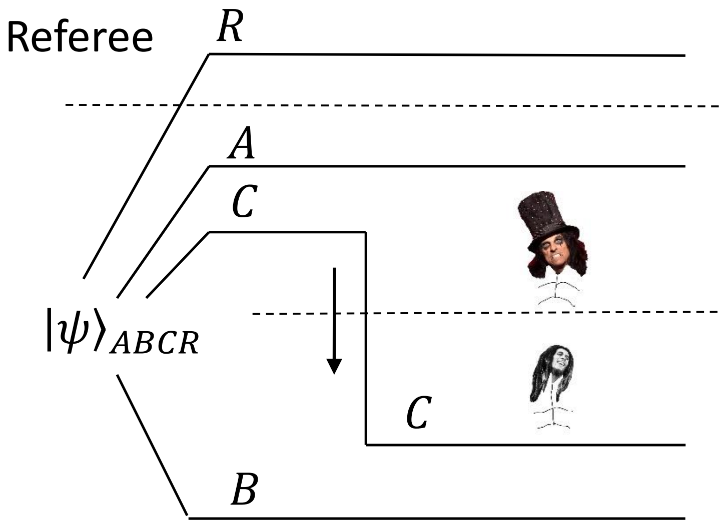


[Lieb and Ruskai 1973; Devetak and Yard 2008; Yard and Devetak 2009; Fawzi and Renner 2015]

Quantum Information Complexity (QIC)

- QIC(f,μ) = $\inf_{\Pi}$ QIC(Π, μ)
- Optimization over all Π computing f
- QIC(Π, μ) = $\sum_{i\ odd} I(R_X R_Y : C_i | Y B_i) + \sum_{i\ even} I(R_X R_Y : C_i | X A_i)$
  - Motivated by quantum state redistribution, with $R_X R_Y$ purifying the XY input registers: $|\rho_\mu\rangle_{R_X X R_Y Y} = \sum_{x,y} \sqrt{\mu(x,y)}\, |xxyy\rangle_{R_X X R_Y Y}$





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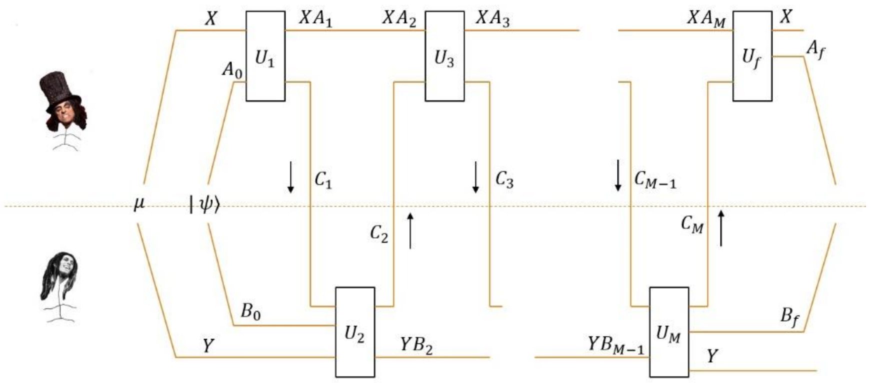
- QIC(Π, μ) = $\sum_{i\ odd} I(R_X R_Y : C_i | Y B_i) + \sum_{i\ even} I(R_X R_Y : C_i | X A_i)$
- Properties [T15]:
  - Information equals amortized communication
  - Additivity
  - QIC $\leq$ QCC
  - Continuity, …

[T. 2015]

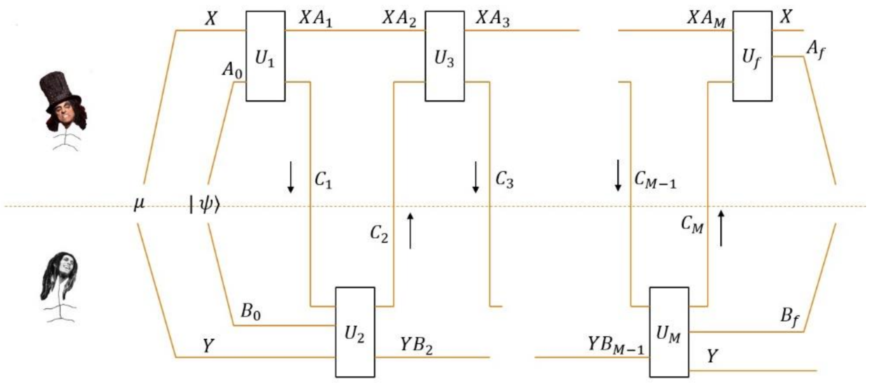
Alternative Notions of QIC

- QIC measures information about what?
  - Satisfies Information equals amortized communication
  - What about these purification registers for classical inputs?



Alternative Notions of QIC

- QIC measures information about what?
  - Satisfies Information equals amortized communication
  - What about these purification registers for classical inputs?
- Can we simply measure the final information?
  - HIC(Π, μ) = $I\left(X\colon B_f \middle| Y\right) + I(Y\colon A_f|X)$
  - Compare to classical IC(Π_C, μ) = $I(X\colon \Pi_C|Y) + I(Y\colon \Pi_C|X)$, with $\Pi_\mathrm{C} = \mathrm{M}_1\mathrm{M}_2 \cdots$ the transcript of messages
  - But reversible computing makes HIC(f) trivial…



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  - HIC(Π, μ) = $I\left(X : B_f \middle| Y\right) + I(Y : A_f | X)$
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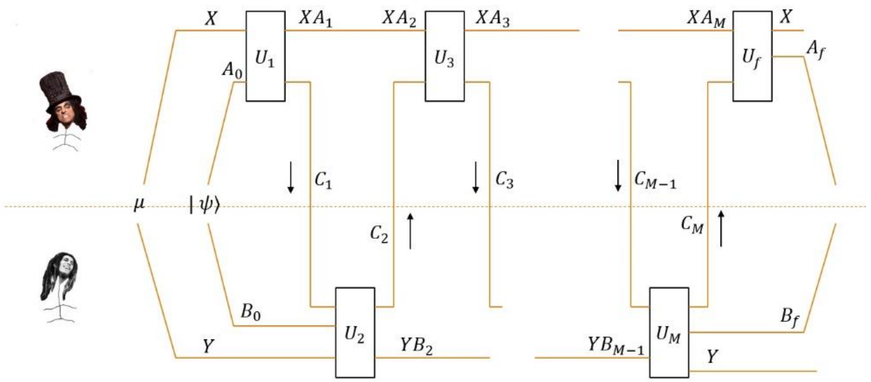
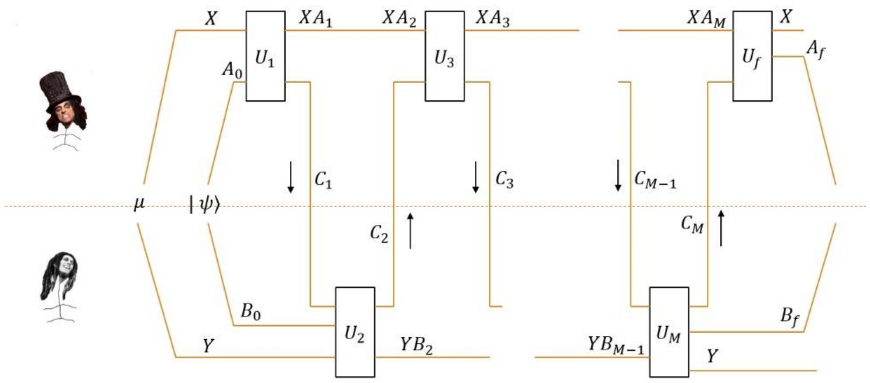
  - But reversible computing makes HIC(f) trivial…
- Can we measure only new classical information?
  - CIC(Π, μ) = $\sum_{i\ odd} I(X : C_i | Y B_i) + \sum_{i\ even} I(Y : C_i | X A_i)$ [KLLGR16]
  - Compare to classical IC(Π_C, μ) = $\sum_{i\ odd} I(X : M_i | Y M_{<i}) + \sum_{i\ even} I(Y : M_i | X M_{<i})$
  - Motivated by privacy concerns



[Kerenidis, Lauriere, Le Gall and Rennela 2016]

Alternative Notions of QIC

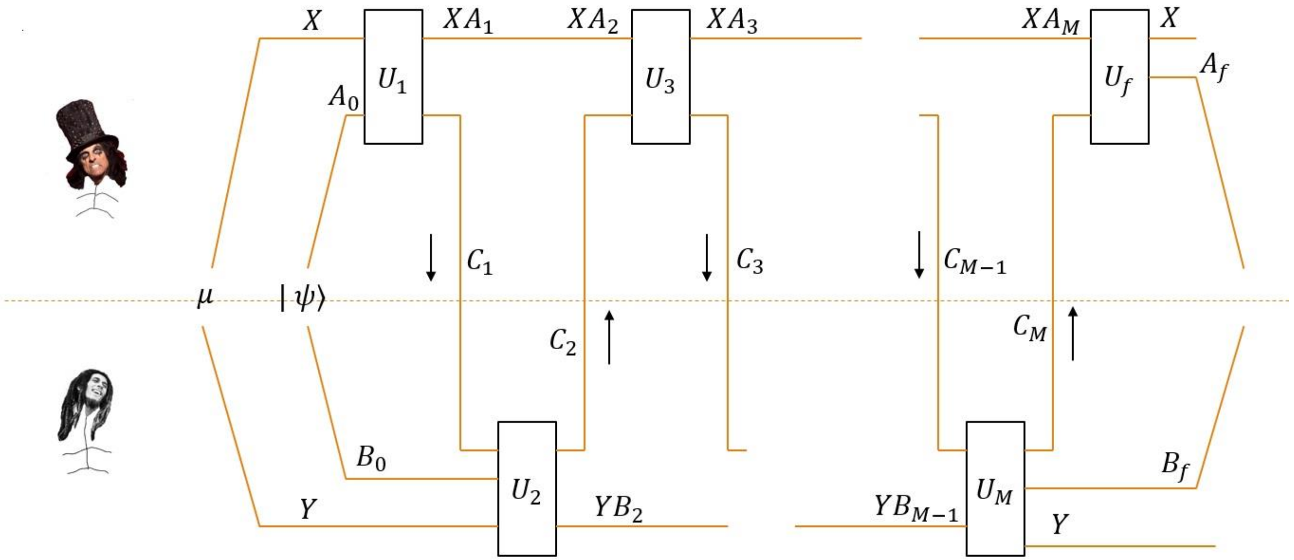
- QIC measures information about what?
  - Satisfies Information equals amortized communication
  - What about these purification registers for classical inputs?
- Can we simply measure the final information?
  - HIC(Π, μ) = $I\big(X\colon B_f \big| Y\big) + I(Y\colon A_f | X)$
  - Compare to classical IC(Π_C, μ) = $I(X\colon \Pi_C | Y) + I(Y\colon \Pi_C | X)$, with $\Pi_C = M_1 M_2 \cdots$ the transcript of messages
  - But reversible computing makes HIC(f) trivial…
- Can we simply measure new classical information?
  - CIC(Π, μ) = $\sum_{i\ odd} I(X\colon C_i | Y B_i) + \sum_{i\ even} I(Y\colon C_i | X A_i)$ [KLLGR16]
  - Compare to classical IC(Π_C, μ) = $\sum_{i\ odd} I(X\colon M_i | Y M_{<i}) + \sum_{i\ even} I(Y\colon M_i | X M_{<i})$
  - Motivated by privacy concerns
- HIC$(\Pi, \mu) \leq$ CIC$(\Pi, \mu) \leq$ QIC(Π, μ)
  - Is there a deeper relationship?



[Kerenidis, Lauriere, Le Gall and Rennela 2016]

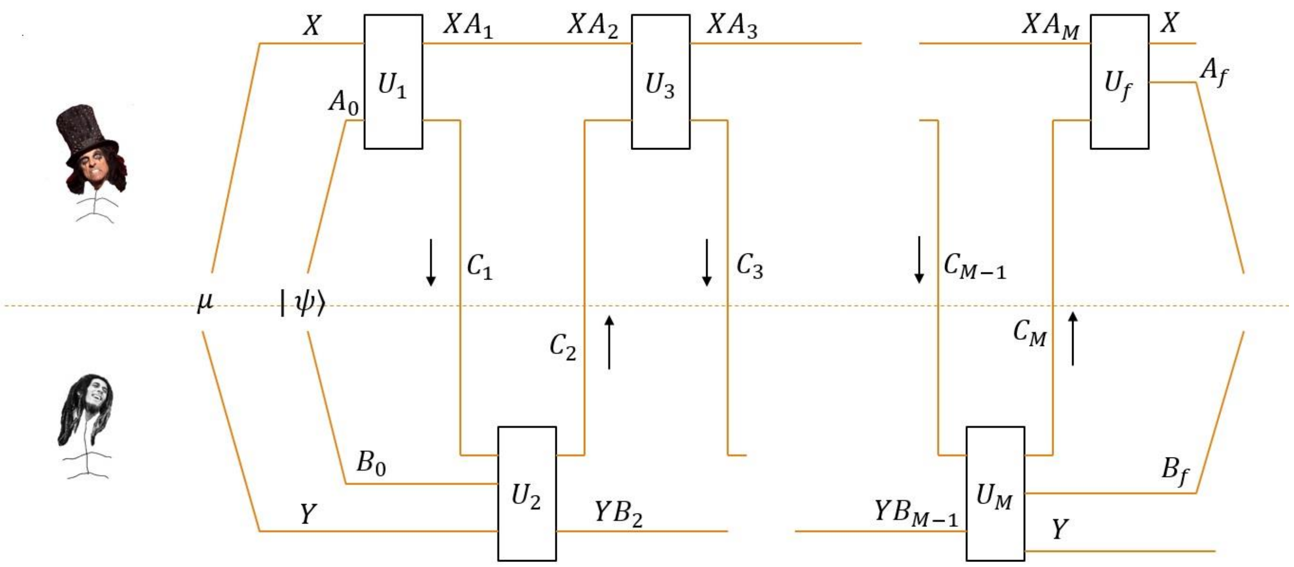
Tool 1: Information Flow Lemma

- Lemma: $I(X\!:\!YB_f) - I(X\!:\!Y) = I(X\!:\!B_f|Y) = \sum_{i\ odd} I(X\!:\!C_i|YB_i) \ - \sum_{i\ even} I(X\!:\!C_i|YB_i)$
  - Can also handle fully quantum processes and arbitrary extension of inputs



Th. 1: Cost of Forgetting

- Rewrite QIC(Π, μ) = $\sum_i I(X: C_i|YB_i) + I(Y: C_i|XA_i)$
  - What are those extra terms compared to CIC?
  - CRIC(Π, μ) = $\sum_{i\ even} I(X: C_i|YB_i) + \sum_{i\ odd} I(Y: C_i|XA_i)$



Th. 1: Cost of Forgetting

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  - CRIC(Π, μ) = $\sum_{i\ even} I(X: C_i | YB_i) + \sum_{i\ odd} I(Y: C_i | XA_i)$
- Using Info. Flow Lemma, rewrite
  - Th. 1.1: HIC(Π, μ) = CIC(Π, μ) - CRIC(Π, μ)
  - QIC(Π, μ) = CIC(Π, μ) + CRIC(Π, μ)
- CRIC corresponds to cost of forgetting
  - Exactly assess back-flow of information
  - No need to introduce purification registers $R_X R_Y$ to define QIC (for classical tasks)

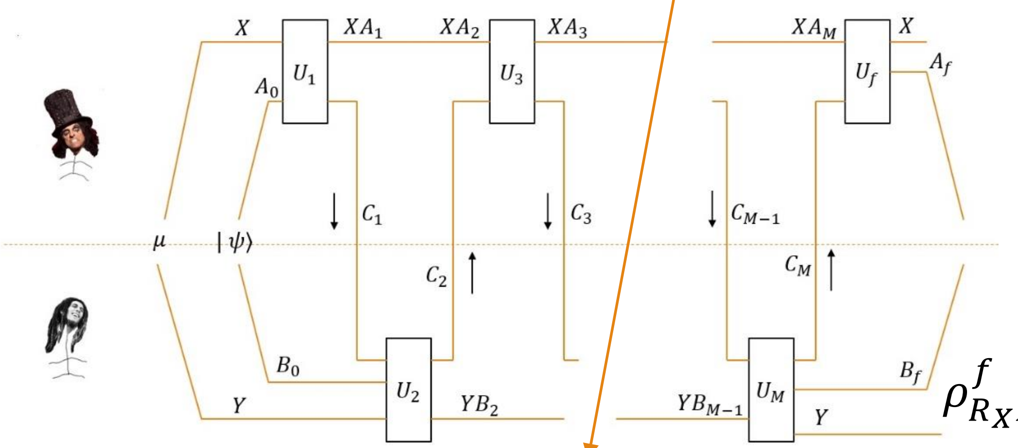
Tool 2: Superposition-Average Encoding Th.

- Average encoding theorem [KNTZ07]: $\mathbb{E}_X[h^2(\rho_B^X, \rho_B)] \leq I(X:B)_\rho$
  - $\rho_{XB} = \sum_x p_X(x)|x\rangle\langle x|_X \otimes \rho_B^x$
  - $\rho_B = \mathbb{E}_X[\rho_B^X]$, average state
  - $\mathrm{h}^2(\sigma, \theta) = 1 - F(\sigma, \theta)$, Bures distance, with $F(\sigma, \theta) = ||\sqrt{\sigma}\sqrt{\theta}||_1$
  - Follows from Pinsker's inequality
  - Many applications, e.g. together with a round-by-round variant of HIC [JRS03]

[Klauck, Nayak, Ta-Shma and Zuckerman 2007; Jain, Radhakrishnan and Sen 2003]

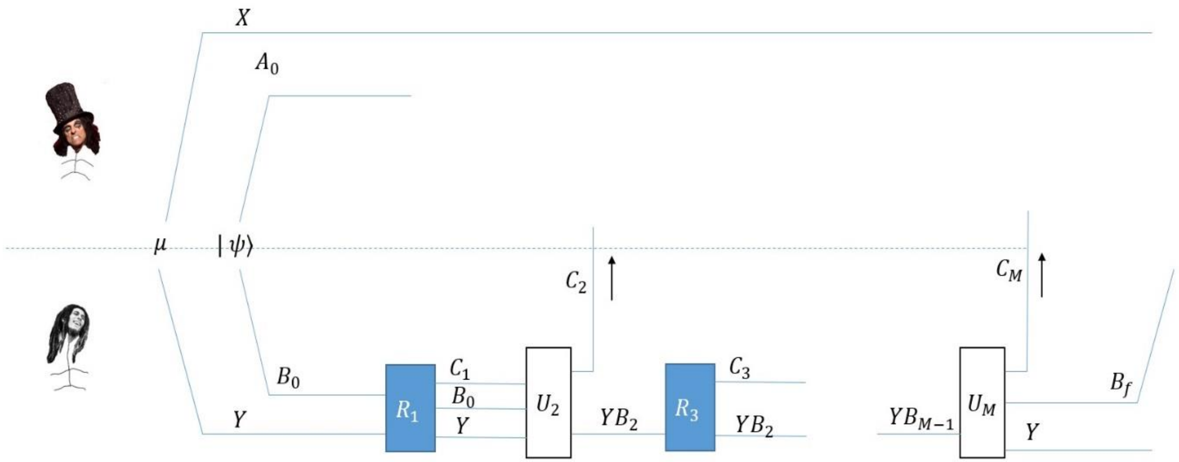
Tool 2: Superposition-Average Encoding Th.

- Average encoding theorem [KNTZ07]: $\mathbb{E}_X[h^2(\rho_B^X, \rho_B)] \leq I(X\!:B)_\rho$
- What about superposition over (part of) X?
- Recall F-R theorem (stated in terms of h)
  - There exists a recovery map $T_{B\to BC}$ such that $h^2(\rho_{RBC}, T_{B\to BC}(\rho_{RB})) \leq I(R\!:C|B)_\rho$
- Theorem: If for odd i then $h^2\left(\rho^f_{R_X R_Y Y B_f}, \sigma^f_{R_X R_Y Y B_f}\right) \leq M\ \sum_i \varepsilon_i$



$\rho^f_{R_X R_Y Y B_f}$

$I(R_X R_Y\!: C_i | Y\ B_i) = \varepsilon_i$



R_i: F-R maps

$\sigma^f_{R_X R_Y Y B_f}$

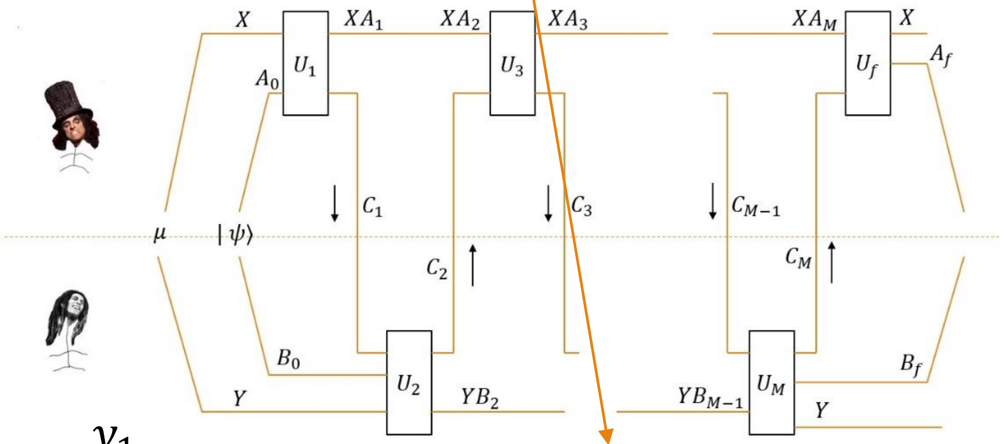
[Klauck, Nayak, Ta-Shma and Zuckerman 2007]

Tool 3: Quantum Cut-and-Paste Lemma

- Variant of a tool developed in [JRS03, JN14]
- Consider input subset $\{x_1, x_2\} \times \{y_1, y_2\}$
- Lemma: If for odd i and for even i, then $h\left(V_{B_t}^{y_1 \to y_2}(\rho^{t\ \ x_2 y_1}_{A_t B_t C_t})\rho^{t,x_2 y_2}_{A_t B_t C_t}\right) \leq 2\sum_{j \leq i} \delta_j$

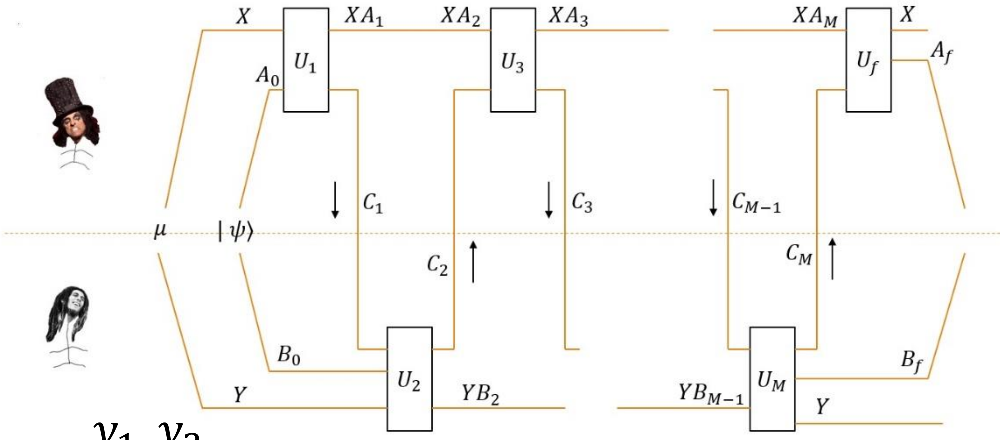
$$h\left(\rho^{i,\,x_1 y_1}_{A_i C_i,}\,\rho^{i,\,x_1 y_2}_{A_i C_i}\right) = \delta_i$$

x_1, x_2



y_1

x_1



y_1, y_2

$$h\left(\rho^{i,\,x_1 y_1}_{B_i C_i,}\,\rho^{i,\,x_2 y_1}_{B_i C_i}\right) = \delta_i$$

[Jain, Radhakrishnan and Sen 2003; Jain and Nayak 2014]

Applications

Th. 2: Disjointness

- Recall Disjointness: $x, y \subseteq [n]$, $Disj_n(x, y) =^? [x \cap y = \emptyset]$
- $\mathrm{CC}(Disj_n) \in \Omega(n)$, $\mathrm{QCC}(Disj_n) \in \Omega(\sqrt{n})$
- For r rounds, $\mathrm{QCC}^r(Disj_n) \in \widetilde{\Omega}(\frac{n}{r})$ [BGKMT15]
- Number of rounds r appears only through a continuity argument
  - Not there for classical protocols
  - Due to possibility of forgetting and retransmitting in quantum protocols
- With no-forgetting (NF), $\mathrm{QCC}^{NF}(Disj_n) \in \Omega\,(n)$

[Braverman, Garg, Kun Ko, Mao and T. 2015]

Th. 3: QIC and IC of Random functions

- Can we simulate classical protocols with quantum ones?
  - Of course!
  - What about maintaining IC?
  - Must be careful with private randomness
  - Bring Π_C in canonical form first
  - Then QIC looks classical… almost!

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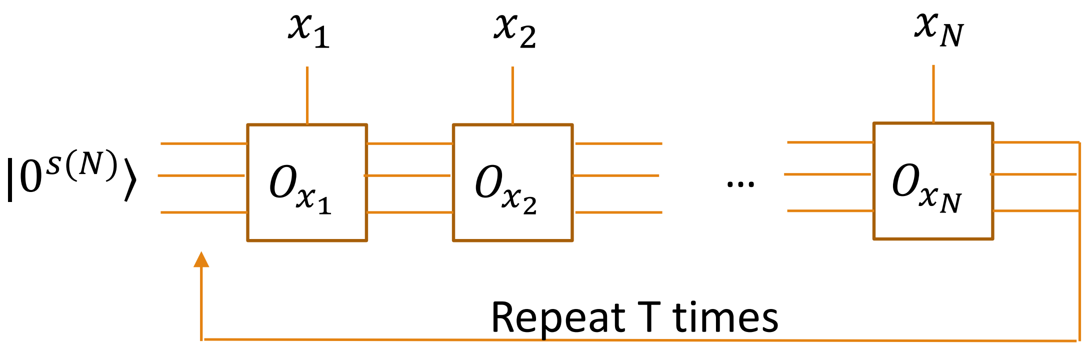
- Can we simulate classical protocols with quantum ones?
  - Of course!
  - What about maintaining IC?
  - Must be careful with private randomness
  - Bring Π_C in canonical form first
  - Then QIC looks classical… almost!
- Known: $QCC(IP_n) = n$ [CDNT99], $QCC(f_{rdm}) = n(1 - o(1))$ [MW07]
  - $IP_n(x, y) = \bigoplus_i x_i \wedge y_i$, f_{rdm} random function on n + n bits
  - Using Info. Flow Lemma, QCC lower bound transfers to QIC lower bound (at zero error)
  - Already known: $IC(IP_n) = n$ [BGPW], $IC(f_{rdm}) = \Omega(n)$ [BW]
- By above simulation, $IC(f_{rdm}) = n(1 - o(1))$

[Cleve, van Dam, Nielsen and Tapp 1999; Montanaro and Winter 2007;
Braverman, Garg, Pankratov and Weinstein 2013; Braverman and Weinstein 2012]

Th. 4: Streaming Algorithms for DYCK(2)

- $\mathrm{DYCK}(2) = \epsilon + [DYCK(2)] + \left(DYCK(2)\right) + DYCK(2) \cdot DYCK(2)$
- Reduction from multi-party QCC to streaming algorithm to DYCK(2) [MMN14]
  - Consider T-pass, one-way quantum streaming algorithms
  - Space s(N) in algorithm corresponds to communication between parties
  - Multi-party problem consists of OR of multiple instances of two-party problem



[Magniez, Mathieu and Nayak 2014]

Th. 4: Streaming Algorithms for DYCK(2)

- $\mathrm{DYCK}(2) = \epsilon + [DYCK(2)] + \left(DYCK(2)\right) + DYCK(2) \cdot DYCK(2)$
- Reduction from multi-party QCC to streaming algorithm to DYCK(2) [MMN14]
  - Consider T-pass, one-way quantum streaming algorithms
  - Space s(N) in algorithm corresponds to communication between parties
  - Multi-party problem consists of OR of multiple instances of two-party problem
- Direct sum argument allows to reduce from a two-party problem
  - Multi-party QCC lower bounds requires two-party QIC lower bound on "easy distribution"
- Th. 2.1: Any T-pass 1-way qu. streaming algo. for DYCK(2) needs space $s(N) \in \Omega(\frac{\sqrt{N}}{T^3})$ on length N inputs

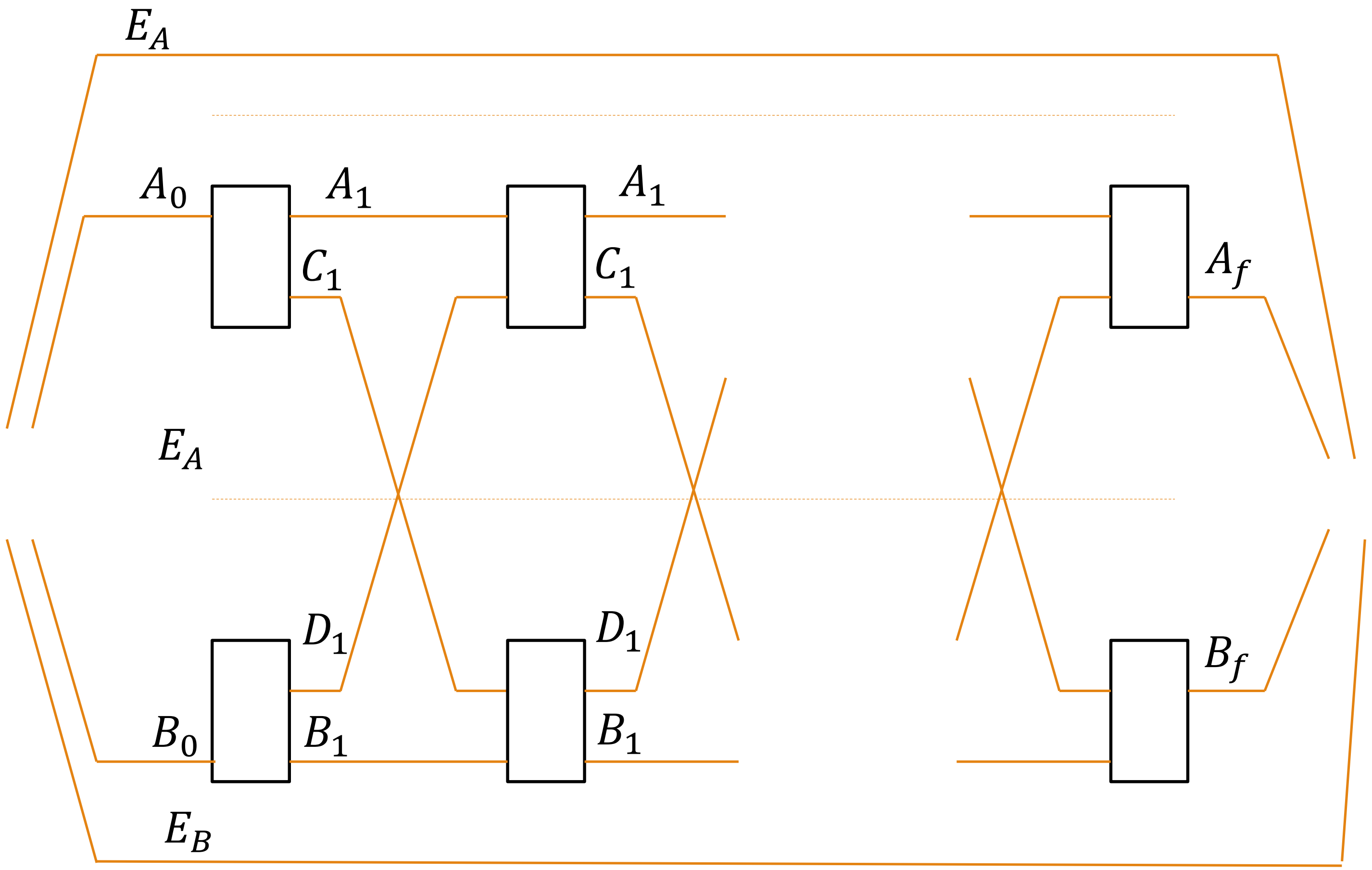
Th. 5: Augmented Index

- Index$(x_1 \dots x_i \dots x_n, i) = x_i$
- Augmented Index: $AI_n\big(x_1 \dots x_n, (i, x_1 \dots x_{<i}, b)\big) = x_i \oplus b$
- Th. 2.2: For any r-round protocol Π for AI_n, either
  - $QIC_{A \to B}(\Pi, \mu_0) \in \Omega\left(\frac{n}{r^2}\right)$ or
  - $QIC_{B \to A}(\Pi, \mu_0) \in \Omega\left(\frac{1}{r^2}\right)$ with
  - μ_0 the uniform distribution on zeros of AI_n ("easy distribution")
- Builds on direct sum approach of [JN14]
- General approach uses Tools 2, 3 (Sup.-Average Encoding Th., Qu. Cut-and-Paste)
- More specialized approach uses Tool 1 (Info. Flow Lemma)

Outlook

- Information-Theoretic Tools for Interactive Quantum Protocols
  - Information Flow Lemma
  - Superposition-average encoding theorem
  - Quantum Cut-and-Paste Lemma
- Applications
  - Intuitive interpretation of QIC, links with CIC, HIC (and other notions)
  - Forgetting an essential feature of quantum protocols for Disjointness
  - Quantum simulation of classical protocols leads to n(1-o(1)) lower bound on IC of random functions
  - Space lower bound on quantum streaming algorithms for DYCK(2)
  - Quantum information trade-off for Augmented Index
  - Further applications..?

V2: Information Flow Lemma

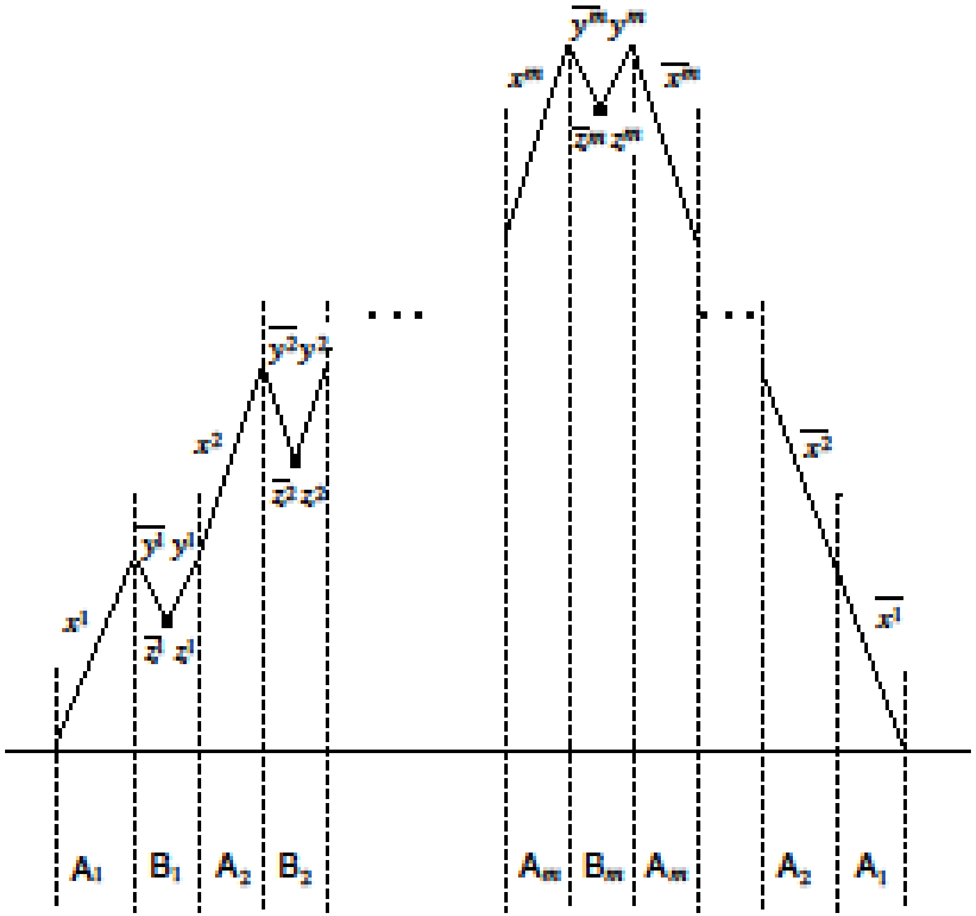


$$I\left(E_A{:}\,B_f|E_B\right) - I(E_A{:}\,B_0|E_B) =$$

$$\sum_i I(E_A{:}\,C_i|E_B B_i)$$

$$-\sum_i I(E_A{:}\,D_i|E_B B_i)$$

ASCENSION



[MMN14]

[Magniez, Mathieu and Nayak 2014]