A GENERAL METHOD FOR EVALUATING OUTAGE PROBABILITIES USING PADÉ APPROXIMATIONS

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Abstract

Many different methods have been derived for evaluating the outage probability given fading, shadowing, or a combination of fading and shadowing. However, each of these methods imposes restrictions on the parameters or combination of fading and shadowing distributions. In this paper, we develop a general method to evaluate outage probabilities without restrictions based on Padé approximations (PAs). Results show that imposing restrictions on the channel model, such as the Ricean specular parameter, can lead to large errors in the outage probability. The PA method is applicable to both the forward link and the reverse link and is extremely efficient due to the use of residues or saddle point integration. Padé approximations allow the use of any fading or shadowing models given the moments of each distribution. In addition, this new method includes the effects of complex AWGN. For picocellular environments operating at low values of carrier SNR, we show that the noise provides a significant contribution to the outage probability.

1 Introduction

The evaluation of outage probabilities (P_o) has been an important and active area of research. Previous works have modeled systems where the channel from the base station (BS) to the mobile station (MS) is distorted by either fading, shadowing, or a combination of fading and shadowing [1]. For the combined case of fading and shadowing, Linnartz [2] requires all channels to exhibit Rayleigh fading. Prasad and Kegel [3] restricts the model to a Ricean faded desired signal and Rayleigh faded interference signals. In Austin and Stüber [4], at least one of the desired or interference signals must exhibit Rayleigh fading. Tjhung, et.al. [5] is valid assuming that the Ricean factors for each Ricean faded channels are identical. In addition, the mean powers of the shadowing must have identical lognormal statistics.

This paper solves the general outage probability problem using Padé approximations (PAs) [6] [7]. The PA method evaluates the P_o by integration of the moment generating function (MGF) in the complex plane using residues or saddle point integration. Any MGF can be represented by a Padé approximation to the truncated power series of $\exp(ux)$. This method does not impose any restrictions on the parameters or combination of Rayleigh and/or Ricean fading channels or the lognormal shadowing. Although this paper discusses the forward link, this method is applicable to the reverse link as well. In addition, the model includes the effects of complex AWGN. We present results for channels with mean square-envelope, lognormal shadowing and Rayleigh and Ricean flat fading, but the method is easily extended to other fading and shadowing models given the moments of the fading and shadowing distributions.

Padé approximations have been used to compute the error probabilities resulting from Direct Sequence/Code Division Multiple Access (DS/CDMA) communication systems [8]. The PA method is computationally efficient due to the evaluation of a product of L+1 rational functions where L is the number of interference base stations. For the case of Rayleigh fading channels without shadowing, the order of the numerator polynomial is 0 while the order of the denominator polynomial is 1 for each of the rational expressions. For the examples presented in this paper, the maximum orders for the PA numerator and denominator are 3 and 5, respectively. This paper is organized as follows. In section 2, we present the system model and derive an expression for the P_o based on the PA for each BS's MGF. We provide numerical results in section 3.

2 System Model

For the forward link, the outage probability is defined as the probability that the ratio of the received power from the reference BS to the sum of the received powers from the interference BSs and complex AWGN is less than the protection ratio, λ_{th}

$$P_o = Pr[\frac{P_{R,0}}{\sum_{l=1}^{L} P_{R,l} + N} < \lambda_{th}] \qquad (1)$$

where N represents the power from complex AWGN. The power received from each BS is

$$P_{R,l} = R_l^2 \xi_l r_l^{-\beta} \hat{P}_l = R_l^2 \xi_l P_l. \qquad (2)$$

In the model, fading and shadowing are represented by the parameters R_l and ξ_l, respectively. We choose to include distance attenuation as a deterministic quantity r_l, which when combined with the transmitted base station power, $\hat{P}_l$, is represented by the power parameter P_l. The capture probability, P_c is

$$P_c = 1 - P_o. \qquad (3)$$

Thus, the outage probability is

$$\begin{aligned} P_o &= Pr[s = P_{R,0} - \lambda_{th} \sum_{l=1}^{L} P_{R,l} - \lambda_{th} N < 0] \\ &= \int_0^{\infty} p(s) ds \qquad (4) \end{aligned}$$

where $p(s)$ is the probability density function (PDF) of the test statistic s. Instead of directly integrating (4), the PA method evaluates the P_o by the complex integration of the moment generating function (MGF) of s. Since $P_{R,0}$, $P_{R,l}$, and N in (4) are independent, the MGF of s is

$$\begin{aligned} h(u) &= E\{\exp(-su)\} = \int_{-\infty}^{\infty} \exp(-su) p(s) ds \\ &= h_0(u) \Pi_{l=1}^{L} h_l(-\lambda_{th} u) h_N(-\lambda_{th} u) \qquad (5) \end{aligned}$$

where $h_l(u)$ and $h_N(u)$ represent the MGFs of the l^{th} source and the AWGN, respectively. The outage and capture probabilities can be recovered by the inverse Laplace transform of the MGF

$$\begin{aligned} P_{c,o} &= \int_{C\pm} u^{-1} h(u) \frac{du}{2\pi j} \qquad (6) \\ &+ \begin{cases} 0 & \text{outage probability along } C_+ \\ 1 & \text{capture probability along } C_- \end{cases} \end{aligned}$$

where C+(-) is a vertical contour in the complex u-plane that crosses the real-u axis in the right(left) half plane. The evaluation of the contour integral along C_- produces a negative value so $0 \leq P_c \leq 1$.

The MGF of the complex AWGN is

$$h_N(u) = E\{\exp(-uN)\} = 1/(1 + 2\sigma_N^2 u). \quad (7)$$

For each BS, the MGF is

$$\begin{aligned} h_l(u) &= E\{\exp(-u R_l^2 \xi_l P_l)\} \\ &= \int_0^{\infty} \int_0^{\infty} \exp(-u R_l^2 \xi_l P_l) p(R_l) p(\xi_l) dR_l d\xi_l \end{aligned} \qquad (8)$$

A closed form expression for the integral in (8) does not exist unless the shadowing is neglected and the channel exhibits Rayleigh fading. By substituting a Padé approximation (PA) of the truncated series representing the average MGF, we obtain a generalized method for computing the outage probability for any channel with fading or shadowing. A $[M_N/M_D]$ PA is a rational approximation with numerator order M_N and denominator order M_D obtained by the moment matching approach [6].

The PA is produced by expanding the exponential in (8) as a Taylor series and approximating the infinite summation as a rational approximation. Substituting gives

$$
\begin{aligned}
h_l(u) &= \sum_{k=0}^{\infty} \frac{(-uP_l)^k}{k!} \times \\
&\quad \int_0^{\infty} R_l^{2k} p(R_l) dR_l \int_0^{\infty} \xi_l^k p(\xi_l) d\xi_l \\
&= \sum_{k=0}^{\infty} \frac{(-uP_l)^k}{k!} \mu_{R_l}^{(2k)} \mu_{\xi_l}^{(k)} \qquad (9)
\end{aligned}
$$

where $\mu_{R_l}^{(2k)}$ are the even moments of fading distribution, and $\mu_{\xi_l}^{(k)}$ are the moments of shadowing distribution for the l^{th} source. Thus,

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(-uP_l)^k}{k!} \mu_{R_l}^{(2k)} \mu_{\xi_l}^{(k)} \\
&\quad = P_l(u) + O(u^{M_N+M_D+1}) \qquad (10)
\end{aligned}
$$

where the PA, $[M_N/M_D]$ $P_l(u)$, is

$$
P_l(u) = \frac{g_l \prod_{i=1}^{M_N}(u - z_{l,i})}{\prod_{j=1}^{M_D}(u - p_{l,j})}. \qquad (11)
$$

In (11), we neglect the terms of order M_N+M_D+1 and higher. The infinite summation in (10) is multiplied by the denominator of (11). Noting that the numerator of (11) only affects terms up to order M_N of the infinite summation, terms of order M_N+1 to M_N+M_D determine a set of linear equations which yield the denominator coefficients. Once the denominator coefficients are determined, we can solve for the numerator coefficients.

2.1 Fading

The evaluation of the MGF in (8) requires the even moments of the fading distribution. For Rayleigh Fading, the PDF is

$$
p(R_l) = \frac{R_l}{\sigma_{R_l}^2} \exp(-R_l^2/2\sigma_{R_l}^2) \qquad (12)
$$

and the moments are

$$
\mu_{R_l}^{(k)} = (2\sigma_{R_l}^2)^{\frac{k}{2}} \Gamma(1 + \frac{k}{2}). \qquad (13)
$$

Likewise for Ricean fading with specular parameter, s_l, the PDF is

$$
p(R_l) = \frac{R_l}{\sigma_{R_l}^2} \exp(\frac{-(R_l^2 + s_l^2)}{2\sigma_{R_l}^2}) I_0(\frac{R_l s_l}{\sigma_{R_l}^2}). \qquad (14)
$$

We compute the even moments for Ricean fading, $\mu_{R_l}^{(2k)}$, using the efficient recursion in Helstrom [10, p.524].

2.2 Shadowing

For the mean square-envelope, lognormal shadowing, the PDF and moments are

$$
p(\xi_l) = \frac{1}{\sqrt{2\pi\sigma_{\xi_l}^2}\,\xi_l} \exp(-(\ln(\xi_l) - m_{\xi_l})^2/2\sigma_{\xi_l}^2) \qquad (15)
$$

and

$$
\begin{aligned}
\mu_{\xi_l}^{(k)} &= E\{\exp(\xi_l k)\} = \int_0^{\infty} \exp(\xi_l k) p(\xi_l) d\xi_l \\
&= \exp(k m_{\xi_l} + 0.5k^2\sigma_{\xi_l}^2). \qquad (16)
\end{aligned}
$$

Some authors model log normal shadowing using the mean envelope instead of the mean square-envelope [1, p.86]. By simply using the half moments from (16) instead of the full moments, we can model mean envelope, log normal shadowing.

2.3 Residues

The evaluation of (8) requires numerical contour integration along the Bromwich contour. The results presented in section 3 have been evaluated using saddle point integration (SPI) [8]. However, the efficiency of the PA method can be further improved by evaluating the contour integral using residue theory by closing the contour found from SPI. By including the MGF of the noise (7), or by requiring the $M_N < M_D$ for at least one of the PAs and $M_N \leq M_D$ for the remaining PAs, the analytic function representing the overall MGF is guaranteed to have zero contribution at $c \pm j\infty$. Thus, we can close the contour and evaluate the outage probability using residues. The outage probability can be found from the Cauchy principle value

$$
\begin{aligned}
P_o &= \text{p.v.} \int_{c-j\infty}^{c+j\infty} u^{-1} h(u) \frac{du}{2\pi j} \\
&= \lim_{\rho\to\infty} \int_{c-j\rho}^{c+j\rho} u^{-1} h(u) \frac{du}{2\pi j} \qquad (17)
\end{aligned}
$$

3 Results

In this section, we use the PA method to investigate the outage probabilities for models with either Rayleigh or Ricean fading and with or without lognormal shadowing. The carrier signal-to-noise ratio (SNR) is defined to be

$$SNR_{dB} = 10\log_{10}(E\{R_0^2\}E\{\xi_0\}P_0/\sigma_n^2). \quad (18)$$

The signal-to-interference ratio (SIR), the ratio of the received power from the reference BS to an individual interference BS, is

$$SIR_{dB} = 10\log_{10}(\frac{(E\{R_0^2\}E\{\xi_0\}P_0)}{(E\{R_l^2\}E\{\xi_l\}P_l)}). \quad (19)$$

In all our examples, the SIR for each of the interference BSs is 10 dB. We normalize the fading attenuation by $E\{R_l^2\} = 1$ for all Rayleigh and Ricean channels. This corresponds to a Ricean factor of $K_l = s_l^2/(2\sigma_{R_l}^2) = 0$ dB. For each of the models including shadowing, the shadowing intensity is 6 dB. For the models which do not include shadowing, we neglect the $E\{\xi_l\}$ terms in (18) and (19). The protection ratio is set to $\lambda_{th} = 0$ dB for each example. The numerator and denominator orders for each of the PAs are given in table 1 for the various system models. The orders are chosen to match Monte Carlo simulations.

Fading	Lognormal Shadowing	Complex AWGN	PA Order $[M_N/M_D]$
Rayleigh	Not Incl.	Not Incl.	[0 / 1]
Rayleigh	Not Incl.	Included	[0 / 1]
Rayleigh	Included	Not Incl.	[2 / 3]
Rayleigh	Included	Included	[3 / 5]
Ricean	Not Incl.	Not Incl.	[2 / 3]
Ricean	Not Incl.	Included	[3 / 4]
Ricean	Included	Not Incl.	[3 / 4]
Ricean	Included	Included	[3 / 5]

Table 1: Padé approximation orders

In figure 1, we demonstrate that a general method for evaluating outage probabilities is critical because imposing restrictions on the channel parameters can lead to large errors. In this figure, we evaluate the outage probabilities for L=5 interference base stations with Ricean fading channels and no shadowing. In the P_o curve with averaged Ricean specular parameters, all channels assume a Ricean specular parameter of 6 dB. In the other curve, the channel from the reference BS has a specular parameter of 6 dB and the specular parameter of the five interference channels range from 4 to 8 dB in 1 dB steps. The results from figure 1 show an error of almost an order of magnitude when forced to average the channel statistics of all independent channels.

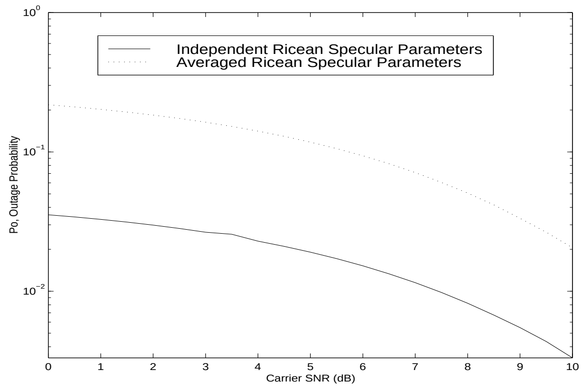

Figure 1: P_o vs. carrier SNR for Ricean fading channels with and independent and dependent specular parameters.

In figure 2, we compare the outage probabilities for a single interference BS. All model combinations of fading, shadowing, and noise are considered. Cellular radio systems operate at high SNR rates relative to the receiver noise. Thus, the interference from adjacent BSs masks the receiver noise. However, for pico-cellular environments operating at much lower values of SNR, figure 2 illustrates that the noise cannot be neglected. In addition, the figure shows that P_o is increased by 0.02 by including shadowing with an intensity of 6 dB. For the two cases of Rayleigh and Ricean fading without shadowing or noise, $P_o = 0.0909$ (Rayleigh) and $P_o = 0.0727$ (Ricean) which matches the closed form results given in [1, p.130].

Next in figure 3, we evaluate each of the models in the previous example but increase the number of interference BSs to L=3. As in figure 2, we see that the complex AWGN has less of an effect on the P_o for three interference sources than for a single interference source at lower values of SNR.

Finally, we consider the capture probability relative to the number of BSs in figure 4. For this example, the carrier SNR is 10 dB. As ob-

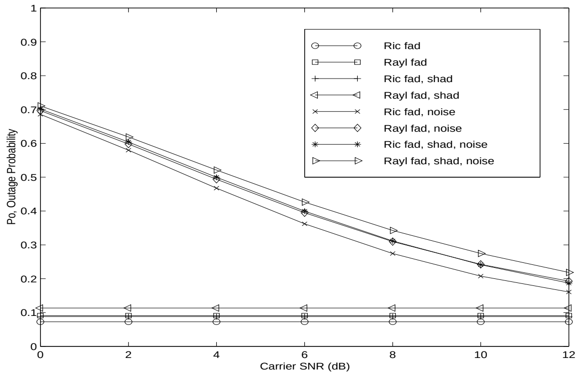

Figure 2: P_o vs. carrier SNR for Rayleigh and Ricean fading channels with and without lognormal shadowing, L=1 interference source.

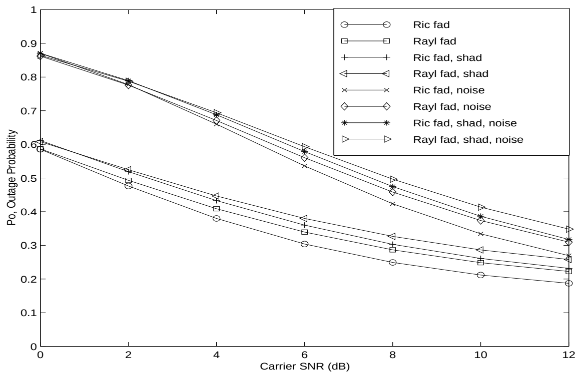

Figure 3: P_o vs. carrier SNR for Rayleigh and Ricean fading channels with and without lognormal shadowing, L=3 interference sources.

served in figures 2 and 3, we see that Ricean fading channel without shadowing offers the best performance followed by Rayleigh fading, Ricean fading with shadowing, and Rayleigh fading with shadowing. However, near L=14 interference BSs, we see that the capture probability curves cross with the channel exhibiting Ricean fading only performing worse than any of the other channels.

4 Conclusion

The outage probabilities for all cases of Rayleigh and Ricean fading with and without lognormal shadowing are evaluated using the Padé approximation method. We approximate the truncated series representing each BS's MGF with a Padé approximation based on the moments of the fading and shadowing distributions. In addition to providing a new general method for evaluating the P_o which does not impose restrictions on the parameters or combinations of the fading and shadowing distributions, the computation time is significantly reduced since the PA method uses residues or saddle point integration. The PA method is valid for both the forward and reverse links and does not introduce potentially large errors due to channel parameter restrictions. By including the effects of complex AWGN, we show that the noise cannot be neglected for pico-cellular environments operating at low values of carrier SNR.

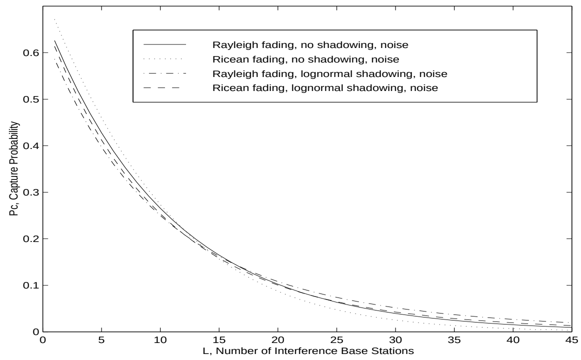

Figure 4: P_c vs. number of interference BSs.

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