

RANDOM SUBGRAPHS OF FINITE GRAPHS. II. THE LACE EXPANSION AND THE TRIANGLE CONDITION

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In a previous paper we defined a version of the percolation triangle condition that is suitable for the analysis of bond percolation on a finite connected transitive graph, and showed that this triangle condition implies that the percolation phase transition has many features in common with the phase transition on the complete graph. In this paper we use a new and simplified approach to the lace expansion to prove quite generally that, for finite graphs that are tori, the triangle condition for percolation is implied by a certain triangle condition for simple random walks on the graph.

The latter is readily verified for several graphs with vertex set $\{0, 1, \dots, r - 1\}^n$, including the Hamming cube on an alphabet of r letters (the n -cube, for $r = 2$), the n -dimensional torus with nearest-neighbor bonds and n sufficiently large, and the n -dimensional torus with $n > 6$ and sufficiently spread-out (long range) bonds. The conclusions of our previous paper thus apply to the percolation phase transition for each of the above examples.

1. Introduction and results.

1.1. *Introduction.* The percolation phase transition on the complete graph is well understood and forms a central part of modern graph theory [4, 6, 21]. In the language of mathematical physics, the phase transition is *mean-field*. It can be expected that the percolation phase transition on many other high-dimensional finite graphs will be similar to that for the complete graph. In other words, mean-field behavior will apply much more generally.

In a previous paper [7] we introduced the finite-graph triangle condition, and proved that it is a sufficient condition for several aspects of the phase transition on a finite connected transitive graph to be mean-field. This triangle condition is an adaptation of the well-known triangle condition of Aizenman and Newman [3] for infinite graphs. In this paper we verify the finite-graph triangle condition for a class of graphs with the structure of high-dimensional tori. Examples include the n -cube, the Hamming cube and periodic approximations to \mathbb{Z}^n for large n .

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Our proof of the triangle condition is based on an adaptation of the percolation lace expansion of Hara and Slade [13] from \mathbb{Z}^n to finite tori. We use the same expansion as [13], but our proof of convergence of the expansion is new and improved. This is the first time that the lace expansion has been applied in a setting where finite-size scaling plays a role. An advance in our application of the lace expansion is that we prove a general theorem that the percolation triangle condition on a finite torus is a consequence of a corresponding condition for *random walks* on the torus. Thus, we are able to verify the percolation triangle condition for our examples by a relatively simple analysis of random walks on these graphs.

1.2. *The triangle condition on infinite graphs.* Let \mathbb{V} be a finite or countably infinite set and let \mathbb{B} be a subset of the set of all two-element subsets $\{x, y\} \subset \mathbb{V}$. Then $\mathbb{G} = (\mathbb{V}, \mathbb{B})$ is a finite or infinite graph with vertex set \mathbb{V} and bond (or edge) set \mathbb{B} . The degree of a vertex $x \in \mathbb{V}$ is defined to be the number of edges containing x . A bijective map $\varphi : \mathbb{V} \rightarrow \mathbb{V}$ is called a graph-isomorphism if $\{\varphi(x), \varphi(y)\} \in \mathbb{B}$ whenever $\{x, y\} \in \mathbb{B}$. We say that \mathbb{G} is *transitive* if, for each pair $x, y \in \mathbb{V}$, there is a graph-isomorphism φ with $\varphi(x) = y$. We will always assume that \mathbb{G} is connected, and usually assume that \mathbb{G} is also transitive. In the latter case, we denote the common degree of each vertex by Ω .

We consider percolation on \mathbb{G} . That is, we associate independent Bernoulli random variables to the edges, taking the value “occupied” with probability p and “vacant” with probability $1 - p$, where $p \in [0, 1]$ is a parameter. Let $x \leftrightarrow y$ denote the event that the vertices x and y are connected by a path in \mathbb{G} consisting of occupied bonds, let $C(x) = \{y \in \mathbb{V} : x \leftrightarrow y\}$ denote the connected cluster of x , and let $|C(x)|$ denote the cardinality of the random set $C(x)$. Let

$$(1.1) \quad \tau_p(x, y) = \mathbb{P}_p(x \leftrightarrow y)$$

denote the *two-point function* and define the *susceptibility* by

$$(1.2) \quad \chi(p) = \mathbb{E}_p |C(0)|.$$

For many infinite graphs, such as \mathbb{Z}^n with $n \geq 2$, or for a regular tree with degree at least three, there is a $p_c = p_c(\mathbb{G}) \in (0, 1)$ such that

$$(1.3) \quad p_c(\mathbb{G}) = \sup\{p : \chi(p) < \infty\} = \inf\{p : \mathbb{P}_p(|C(0)| = \infty) > 0\}.$$

Thus, $\chi(p) < \infty$ and $\mathbb{P}_p(|C(0)| = \infty) = 0$ when $p < p_c$, whereas $\chi(p) = \infty$ and $\mathbb{P}_p(|C(0)| = \infty) > 0$ if $p > p_c$. The equality of the infimum and supremum of (1.3) is a theorem of [2, 23].

Percolation on a tree is well understood ([10], Chapter 10) and infinite graphs whose percolation phase transition is analogous to the transition on a tree are said to exhibit *mean-field* behavior. In 1984 Aizenman and Newman [3] introduced the triangle condition as a sufficient condition for mean-field behavior. The triangle condition is defined in terms of the *triangle diagram*

$$(1.4) \quad \nabla_p(x, y) = \sum_{w, z \in \mathbb{V}} \tau_p(x, w)\tau_p(w, z)\tau_p(z, y),$$

and states that, for all $x \in \mathbb{V}$,

$$(1.5) \quad \nabla_{p_c}(x, x) < \infty.$$

It is predicted that the triangle condition on \mathbb{Z}^n holds for all $n > 6$.

We write $f(p) = \Theta(g(p))$ if $|f(p)/g(p)|$ is bounded away from zero and infinity in an appropriate limit. Aizenman and Newman used a differential inequality for $\chi(p)$ to show that the triangle condition implies that

$$(1.6) \quad \chi(p) = \Theta((p_c - p)^{-\gamma}) \quad \text{uniformly in } p < p_c,$$

with $\gamma = 1$, and Nguyen [24] extended this to show that

$$(1.7) \quad \frac{\mathbb{E}_p[|C(0)|^{t+1}]}{\mathbb{E}_p[|C(0)|^t]} = \Theta((p_c - p)^{-\Delta_{t+1}}) \quad \text{uniformly in } p < p_c,$$

with $\Delta_{t+1} = 2$ for $t = 1, 2, 3, \dots$. Subsequently, Barsky and Aizenman [5] showed, in particular, that the triangle condition also implies that the percolation probability obeys

$$(1.8) \quad \mathbb{P}_p(|C(0)| = \infty) = \Theta((p - p_c)^{\hat{\beta}}) \quad \text{uniformly in } p \geq p_c,$$

with $\hat{\beta} = 1$.

In 1990 Hara and Slade established the triangle condition for nearest-neighbor bond percolation on \mathbb{Z}^n for large n (it is now known that $n \geq 19$ is large enough) and for a wide class of long-range models, called *spread-out* models, for $n > 6$ [13, 14]. Their proof of the triangle condition was based on the *lace expansion*, an adaptation of an expansion introduced in 1985 by Brydges and Spencer [9] to study the self-avoiding walk in high dimensions. Since the late 1980s, lace expansion methods have been used to derive detailed estimates on the critical behavior of several models in high dimensions; see [14, 22, 26] for reviews. Recent extensions of the lace expansion for percolation can be found in [12, 16].

1.3. The triangle condition on finite graphs. On a finite graph, $|C(0)| \leq |\mathbb{V}| < \infty$. Thus, there cannot be a phase transition characterized by the divergence to infinity of the susceptibility or the existence of an infinite cluster. Instead, the phase transition takes place in a small window of p values, below which clusters are typically small in size and above which a single giant cluster coexists with many relatively small clusters. The basic example is the phase transition on the complete graph.

Let \mathbb{G} be a connected transitive finite graph, let $V = |\mathbb{V}| < \infty$ denote its number of vertices, and let Ω denote the common degree of these vertices. The susceptibility $\chi(p) = \mathbb{E}_p|C(0)|$ is an increasing function of p , with $\chi(0) = 1$ and $\chi(1) = V$. In [7] we defined the *critical threshold* $p_c = p_c(\mathbb{G}) = p_c(\mathbb{G}; \lambda)$ to be the unique solution to the equation

$$(1.9) \quad \chi(p_c(\mathbb{G})) = \lambda V^{1/3},$$

where λ is a fixed small parameter. As discussed in more detail in [7], the power $V^{1/3}$ in (1.9) is inspired by the fact that, on the complete graph, the critical susceptibility is proportional to $V^{1/3}$, and we expect (1.9) to be the correct definition only for high-dimensional graphs. The flexibility in the choice of λ in (1.9) is connected with the fact that the phase transition in a finite system is smeared out over a window rather than occurring at a sharply defined threshold, and any value in the window could be chosen as a threshold.

On a finite graph, the triangle diagram (1.4) is bounded above by V^2 , and thus (1.5) is satisfied trivially. In [7] we defined the triangle condition for a finite graph to be the statement that

$$(1.10) \quad \nabla_{p_c(\mathbb{G})}(x, y) \leq \delta_{x,y} + a_0,$$

where a_0 is sufficiently small. In particular, (1.10) implies that $\nabla_{p_c(\mathbb{G})}(x, y)$ is uniformly bounded as $V \rightarrow \infty$. In addition, we defined the *stronger* triangle condition to be the statement that there are constants K_1, K_2 such that, for $p \leq p_c(\mathbb{G})$,

$$(1.11) \quad \nabla_p(x, y) \leq \delta_{x,y} + K_1 \Omega^{-1} + K_2 \frac{\chi(p)^3}{V}.$$

Note that (1.10) is a consequence of (1.11), provided Ω is sufficiently large and λ is sufficiently small. Moreover, since $\sum_y \nabla_{p_c}(x, y) = \chi(p_c)^3 = \lambda^3 V$, (1.10) implies that $\lambda^3 \leq V^{-1} + a_0$ and, hence, λ must be taken to be small for the triangle condition to hold.

As described in more detail below, we showed in [7] that the triangle condition (1.10) implies that the percolation phase transition on a finite graph shares many features with the transition on the complete graph. In this paper we prove (1.11) and, hence, (1.10) for several finite graphs, assuming that λ is a sufficiently small constant and that Ω is sufficiently large. These graphs all have vertex set $\mathbb{V} = \{0, 1, \dots, r-1\}^n$ for some $r \geq 2$ and $n \geq 1$, with periodic boundary conditions. We consider various edge sets, as follows.

1.4. Periodic tori. There are three levels of generality that we will use. First, we use \mathbb{G} to denote a finite connected graph, which in general need not be transitive nor regular. Our derivation of the lace expansion, and much of the diagrammatic estimation of the lace expansion, is valid for general \mathbb{G} . Second, for our analysis of the lace expansion, we restrict \mathbb{G} to have the vertex set of the torus $\mathbb{T} = \mathbb{T}_{r,n} = (\mathbb{Z}_r)^n$, where \mathbb{Z}_r denotes the integers modulo r , for $r = 2, 3, \dots$. The torus $\mathbb{T}_{r,n}$ is an additive group under coordinate-wise addition modulo r , with volume $V = r^n$. We allow any edge set that respects the symmetries of translation and $x \mapsto -x$ reflections. That is, we assume that the edge set is such that $\{0, x\}$ is an edge if and only if $\{y, y \pm x\}$ is an edge for every vertex y . For the torus (or for any regular \mathbb{G}), we denote the vertex degree by Ω . Third, we will verify the stronger percolation triangle condition (1.11) for the following specific edge sets:

1. The nearest-neighbor torus: an edge joins vertices that differ by 1 (modulo r) in exactly one component. For $r = 2$, this is the n -cube. For n fixed and r large, this is a periodic approximation to \mathbb{Z}^n . Here $\Omega = 2n$ for $r \geq 3$ and $\Omega = n$ for $r = 2$. We study the limit in which $V = r^n \rightarrow \infty$, in any fashion, provided that $n \geq 7$ and $r \geq 2$.
2. The Hamming torus: an edge joins vertices that differ in exactly one component (modulo r). Here $\Omega = (r - 1)n$. For $r = 2$, this is again the n -cube. We study the limit in which $V = r^n \rightarrow \infty$, in any fashion, provided that $n \geq 1$ and $r \geq 2$.
3. The spread-out torus: an edge joins vertices $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ if $0 < \max_{i=1, \dots, n} |x_i - y_i| \leq L$ (with $|\cdot|$ the metric on \mathbb{Z}_r). We study the limit $r \rightarrow \infty$, with $n \geq 7$ fixed and L large (depending on n) and fixed. This gives a periodic approximation to range- L percolation on \mathbb{Z}^n . Here $\Omega = (2L + 1)^n - 1$, provided that $r \geq 2L + 1$, which we will always assume.

1.5. *Fourier analysis on a torus.* Our method relies heavily on Fourier analysis. We denote the Fourier dual of the torus $\mathbb{T}_{r,n}$ by $\mathbb{T}_{r,n}^* = \frac{2\pi}{r} \mathbb{T}_{r,n}$. We will *always* identify the dual torus as $\mathbb{T}_{r,n}^* = \frac{2\pi}{r} \{-\lfloor \frac{r-1}{2} \rfloor, \dots, \lceil \frac{r-1}{2} \rceil\}^n$, so that each component of $k \in \mathbb{T}_{r,n}^*$ is between $-\pi$ and π . The reason for this identification is that the point $k = 0$ plays a special role, and we do not want to see it mirrored at the point $(2\pi, \dots, 2\pi)$. Let $k \cdot x = \sum_{j=1}^n k_j x_j$ denote the dot product of $k \in \mathbb{T}_{r,n}^*$ with $x \in \mathbb{T}_{r,n}$. The Fourier transform of $f : \mathbb{T}_{r,n} \rightarrow \mathbb{C}$ is defined by

$$(1.12) \quad \hat{f}(k) = \sum_{x \in \mathbb{T}_{r,n}} f(x) e^{ik \cdot x} \quad (k \in \mathbb{T}_{r,n}^*),$$

with the inverse Fourier transform given by

$$(1.13) \quad f(x) = \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^*} \hat{f}(k) e^{-ik \cdot x}.$$

The *convolution* of functions f, g on $\mathbb{T}_{r,n}$ is defined by

$$(1.14) \quad (f * g)(x) = \sum_{y \in \mathbb{T}_{r,n}} f(y) g(x - y),$$

and the Fourier transform of a convolution is the product of the Fourier transforms:

$$(1.15) \quad \widehat{f * g} = \hat{f} \hat{g}.$$

1.6. *The triangle diagram in Fourier form.* It is convenient to use translation invariance to regard the two-point function or triangle diagram as a function of a single variable, for example, $\tau_p(x, y) = \tau_p(y - x)$. With this identification,

$$(1.16) \quad \hat{\tau}_p(k) = \sum_{x \in \mathbb{T}_{r,n}} \tau_p(0, x) e^{ik \cdot x},$$

where 0 denotes the origin of $\mathbb{T}_{r,n}$. It is shown in [3] that $\hat{\tau}_p(k) \geq 0$ for all $k \in \mathbb{T}_{r,n}^*$. The expected cluster size and two-point function are related by

$$(1.17) \quad \chi(p) = \mathbb{E}_p |C(0)| = \sum_{x \in \mathbb{T}_{r,n}} \mathbb{E}_p I[x \in C(0)] = \sum_{x \in \mathbb{T}_{r,n}} \tau_p(0, x) = \hat{\tau}_p(0),$$

where $I[E]$ denotes the indicator function of the event E . In particular, writing $p_c = p_c(\mathbb{T}_{r,n})$,

$$(1.18) \quad \hat{\tau}_{p_c}(0) = \chi(p_c) = \lambda V^{1/3}.$$

Recalling (1.14), the triangle diagram (1.4) can be written as

$$(1.19) \quad \nabla_p(x, y) = (\tau_p * \tau_p * \tau_p)(y - x).$$

By (1.15) and (1.13), this implies that $\hat{\nabla}_p(k) = \hat{\tau}_p(k)^3$ and

$$(1.20) \quad \nabla_p(x, y) = \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^*} \hat{\nabla}_p(k) e^{-ik \cdot (y-x)} = \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^*} \hat{\tau}_p(k)^3 e^{-ik \cdot (y-x)},$$

where $\hat{\nabla}_p(k)$ denotes the Fourier transform of $\nabla_p(0, x)$. By (1.18), when $p = p_c$, the contribution to the right-hand side of (1.20) due to the term $k = 0$ is $V^{-1} \lambda^3 V = \lambda^3$. This shows a connection between the definition $\chi(p_c) = \lambda V^{1/3}$ and the triangle condition, which in turn is connected to mean-field behavior.

1.7. Main results.

1.7.1. The random walk triangle condition. For $x, y \in \mathbb{T}_{r,n}$, let

$$(1.21) \quad D(x, y) = D(y - x) = \frac{1}{\Omega} I[\{x, y\} \in \mathbb{B}],$$

where \mathbb{B} denotes a particular choice of edge set for the torus. Thus, $D(x)$ represents the 1-step transition probability for a random walk to step from 0 to a neighbor x . As in Section 1.4, we assume that \mathbb{B} is symmetric in the sense that $\{0, x\} \in \mathbb{B}$ if and only if $\{y, y \pm x\} \in \mathbb{B}$ for every vertex y . We make the following assumptions on D , which can alternatively be regarded as assumptions on the edge set \mathbb{B} .

ASSUMPTION 1.1. There exists $\beta > 0$ such that

$$(1.22) \quad \max_{x \in \mathbb{T}_{r,n}} D(x) \leq \beta$$

and

$$(1.23) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \leq \beta.$$

The assumption (1.22) is straightforward. As we will discuss in more detail in Section 2, the Fourier transform of the critical two-point function for random walks is $[1 - \hat{D}(k)]^{-1}$, and comparing with the right-hand side of (1.20), assumption (1.23) can be interpreted as a kind of generalized triangle condition for random walks. Note that the omitted term in (1.23), with $k = 0$, is infinite. For any D defined by (1.21), (1.22) implies that $\beta \geq \Omega^{-1} \geq V^{-1}$. We will require below that β be small. In particular, the degree of the graph must be large.

Random walks on each of the three tori listed in Section 1.4 obey Assumption 1.1 with β proportional to Ω^{-1} , as the following proposition shows. The proof of the proposition is given in Section 2.

PROPOSITION 1.2. *There is an $a > 0$ such that random walks on each of the three tori listed in Section 1.4 obey Assumption 1.1 with $\beta = a\Omega^{-1}$, where:*

1. *For the nearest-neighbor torus, a is a universal constant, independent of $r \geq 2$ and $n \geq 7$.*
2. *For the Hamming torus, a is a universal constant, independent of $r \geq 2$ and $n \geq 1$.*
3. *For the spread-out torus, $n \geq 7$ is fixed, r is sufficiently large depending on L and n , and a depends on n but not on L or r .*

1.7.2. The triangle condition and its consequences. Our main result is that if Assumption 1.1 holds with appropriately small parameters, then the percolation triangle condition holds. By Proposition 1.2, this establishes the triangle condition for the three tori listed in Section 1.4.

THEOREM 1.3 (The triangle condition). *Consider the torus $\mathbb{T}_{r,n}$ with edge set such that $\{0, x\}$ is an edge if and only if $\{y, y \pm x\}$ is an edge for any vertex y . There is an absolute constant $\beta_0 > 0$, not depending on r, n or the edge set of $\mathbb{T}_{r,n}$, such that the stronger triangle condition (1.11) holds in the form*

$$(1.24) \quad \nabla_p(x, y) \leq \delta_{x,y} + 13\beta + 10 \frac{\chi(p)^3}{V},$$

whenever $\lambda^3 \leq \beta_0$, $p \leq p_c$ and Assumption 1.1 holds with $\beta \leq \beta_0$.

This establishes (1.11) for our three tori, since β is proportional to Ω^{-1} in Proposition 1.2. In particular, the cases covered include the following:

- the n -cube $\mathbb{T}_{2,n}$ for $n \rightarrow \infty$,
- the complete graph (Hamming torus with $n = 1$ and $r \rightarrow \infty$),
- nearest-neighbor percolation on $\mathbb{T}_{r,n}$ with $n \geq 7$ large and $r^n \rightarrow \infty$ in any fashion, including n fixed large and $r \rightarrow \infty$, r fixed and $n \rightarrow \infty$, or $r, n \rightarrow \infty$ simultaneously,

- periodic approximations to range- L percolation on \mathbb{Z}^n for fixed $n \geq 7$ and fixed large L .

It follows that the various consequences of the triangle condition established in [7] hold for these three tori, provided λ and $a\Omega^{-1}$ are sufficiently small (as required by the smallness of the triangle), and $\lambda V^{1/3}$ is sufficiently large (as required by the additional condition on $\lambda V^{1/3}$ from [7], Theorems 1.2–1.4). Note that if λ is a fixed positive constant, then the last condition merely states that V is large. We now summarize these consequences in this context. To this end, it will be convenient to use the standard $O(\cdot)$ notation. All constants implicitly in these O -symbols are independent of the parameters of the model, except for an implicit dependence through the constant a from Proposition 1.2.

The asymptotic behavior of the critical value p_c is given in [7], Theorem 1.5, as follows.

THEOREM 1.4 (Critical threshold). *For the three tori,*

$$(1.25) \quad p_c = \frac{1}{\Omega} [1 + O(\Omega^{-1}) + O(\lambda^{-1}V^{-1/3})].$$

For the subcritical phase, the following results are consequences of [7], Theorems 1.2 and 1.5. A version of (1.27) valid for all $p \leq p_c$ is given in [7], Theorem 1.5; see (6.4) below. Let \mathcal{C}_{\max} denote a cluster of maximal size, and let

$$(1.26) \quad |\mathcal{C}_{\max}| = \max\{|C(x)| : x \in \mathbb{V}\}.$$

THEOREM 1.5 (Subcritical phase). *Let $p = p_c - \Omega^{-1}\varepsilon$ with $\varepsilon \geq 0$. For the three tori, the following hold:*

- (i) *If $\varepsilon\lambda V^{1/3} \rightarrow \infty$ as $V \rightarrow \infty$, then as $V \rightarrow \infty$,*

$$(1.27) \quad \chi(p) = \frac{1}{\varepsilon} [1 + O(\Omega^{-1}) + O((\varepsilon\lambda V^{1/3})^{-1})].$$

- (ii) *For all $\varepsilon \geq 0$,*

$$(1.28) \quad 10^{-4}\chi(p)^2 \leq \mathbb{E}_p(|\mathcal{C}_{\max}|) \leq 2\chi(p)^2 \log(V/\chi(p)^3),$$

$$(1.29) \quad \mathbb{P}_p(|\mathcal{C}_{\max}| \leq 2\chi(p)^2 \log(V/\chi(p)^3)) \geq 1 - \frac{\sqrt{e}}{[2\log(V/\chi(p)^3)]^{3/2}},$$

and, for $\omega \geq 1$,

$$(1.30) \quad \mathbb{P}_p\left(|\mathcal{C}_{\max}| \geq \frac{\chi(p)^2}{3600\omega}\right) \geq \left(1 + \frac{36\chi(p)^3}{\omega V}\right)^{-1}.$$

Inside a scaling window of width proportional to $V^{-1/3}$, the following results are consequences of [7], Theorem 1.3.

THEOREM 1.6 (Scaling window). *Fix $\lambda > 0$ sufficiently small and $\Lambda < \infty$. For the three tori, there exist constants b_1, \dots, b_8 such that the following hold for all $p = p_c + \Omega^{-1}\varepsilon$ with $|\varepsilon| \leq \Lambda V^{-1/3}$.*

(i) *If $k \leq b_1 V^{2/3}$, then*

$$(1.31) \quad \frac{b_2}{\sqrt{k}} \leq \mathbb{P}_p(|C(0)| \geq k) \leq \frac{b_3}{\sqrt{k}}.$$

(ii)

$$(1.32) \quad b_4 V^{2/3} \leq \mathbb{E}_p[|\mathcal{C}_{\max}|] \leq b_5 V^{2/3}$$

and, if $\omega \geq 1$, then

$$(1.33) \quad \mathbb{P}_p(\omega^{-1} V^{2/3} \leq |\mathcal{C}_{\max}| \leq \omega V^{2/3}) \geq 1 - \frac{b_6}{\omega}.$$

(iii)

$$(1.34) \quad b_7 V^{1/3} \leq \chi(p) \leq b_8 V^{1/3}.$$

In the above statements, the constants b_2 and b_3 can be chosen independent of λ and Λ , the constants b_5 and b_8 depend on Λ and not on λ , and the constants b_1, b_4, b_6 and b_7 depend on both λ and Λ .

For the supercritical phase, the following results are consequences of [7], Theorem 1.4.

THEOREM 1.7 (Supercritical phase). *Let $p = p_c + \varepsilon\Omega^{-1}$ with $\varepsilon \geq 0$. For the three tori:*

(i)

$$(1.35) \quad \mathbb{E}_p(|\mathcal{C}_{\max}|) \leq 21\varepsilon V + 7V^{2/3},$$

and, for all $\omega > 0$,

$$(1.36) \quad \mathbb{P}_p(|\mathcal{C}_{\max}| \leq \omega(V^{2/3} + \varepsilon V)) \geq 1 - \frac{21}{\omega}.$$

(ii)

$$(1.37) \quad \chi(p) \leq 81(V^{1/3} + \varepsilon^2 V).$$

Theorem 1.7 provides upper bounds on the size of clusters in the supercritical phase. To see that a phase transition occurs at p_c , one wants a *lower* bound. We have not proved a lower bound at the level of generality of all three tori, but we have obtained a lower bound for the case of the n -cube $\mathbb{T}_{2,n} = \{0, 1\}^n$. This is the content of the following theorem, which is proved in [8], Theorem 1.5. The statement that E_n occurs a.a.s. means that $\lim_{n \rightarrow \infty} \mathbb{P}(E_n) = 0$, assuming that λ is fixed as $n \rightarrow \infty$.

THEOREM 1.8 (Supercritical phase for the n -cube). *There are strictly positive constants c_0, c_1, c_2 such that the following holds for $\mathbb{G} = \mathbb{T}_{2,n}$, all n -independent λ with $0 < \lambda \leq c_0$ and all $p = p_c + \varepsilon n^{-1}$ with $e^{-c_1 n^{1/3}} \leq \varepsilon \leq 1$:*

$$(1.38) \quad |\mathcal{C}_{\max}| \geq c_2 \varepsilon 2^n \quad \text{a.a.s. as } n \rightarrow \infty,$$

$$(1.39) \quad \chi(p) \geq (c_2 \varepsilon)^2 2^n \quad \text{as } n \rightarrow \infty.$$

For the special case of the n -cube, Theorem 1.4 states that if λ is chosen such that $\lambda^{-1} 2^{-n/3} = O(n^{-1})$, then $p_c(n) = \frac{1}{n} + O(\frac{1}{n^2})$. This result has been extended in [20] to show that there are rational numbers a_i ($i \geq 1$) such that, for all positive integers M , all $c, c' > 0$, and all p for which $\chi(p) \in [cn^M, c'n^{-2M} 2^n]$,

$$(1.40) \quad p = \sum_{i=1}^M a_i n^{-i} + O(n^{-M-1}),$$

where the constant in the error term depends only on c, c', M . It follows from Theorem 1.4 that $a_1 = 1$, and it is shown in [19] that $a_2 = 1$ and $a_3 = \frac{7}{2}$.

1.8. Discussion.

1.8.1. Restriction to high-dimensional tori. Our results show that the phase transition for percolation on general graphs obeying the triangle condition shares several features with the phase transition for the complete graph. This mean-field behavior is expected to apply only to graphs that are in some sense high-dimensional, and our entire approach is restricted to high-dimensional graphs. As discussed in [7], Section 3.4.2, we do not expect the definition (1.9) of the critical threshold to be correct for finite approximations to low-dimensional graphs, such as \mathbb{Z}^n for $n < 6$. Neither do we expect the triangle condition to be relevant in low dimensions.

Since every finite Abelian group is a direct product of cyclic groups, our restriction to the torus actually covers all Abelian groups, apart from the fact that we consider constant widths in all directions and make a symmetry assumption. It would be straightforward to generalize our results to tori with different widths in different directions. This leaves open the case of more general graphs and non-Abelian groups, which would require a replacement for both the $x \mapsto -x$ symmetry of the torus and the commutative law.

1.8.2. The lace expansion. The derivation of the lace expansion in [13] applies immediately to finite graphs, which need not be transitive nor regular. Our proof of convergence of the expansion uses the group structure of the torus for Fourier analysis, as well as the $x \mapsto -x$ symmetry of the torus. The proof is an adaptation of the original convergence proof of [13], but is conceptually simpler and the idea of basing the proof on Assumption 1.1 is new. In addition, we benefit from working on a finite set where Fourier integrals are simply finite sums.

1.8.3. *Bulk versus periodic boundary conditions.* A natural question for \mathbb{Z}^n is the following. For $p = p_c(\mathbb{Z}^n)$, consider the restriction of percolation configurations to a large box of side r , centered at the origin. How large is the largest cluster in the box, as $r \rightarrow \infty$? The combined results of Aizenman [1] and Hara, van der Hofstad and Slade [12] show that, for spread-out models with $n > 6$, the largest cluster has size of order r^4 , and there are order r^{d-6} clusters of this size. For the nearest-neighbor model in dimensions $n \gg 6$, the same results follow from the combined results of Aizenman [1] and Hara [11]. These results apply under the *bulk* boundary condition, in which the clusters in the box are defined to be the intersection of the box with clusters in the infinite lattice (and thus clusters in the box need not be connected within the box). In terms of the volume $V = r^n$ of the box, the largest cluster at $p_c(\mathbb{Z}^n)$ therefore has size $V^{4/n}$, for $n > 6$. Aizenman [1] raised the interesting question whether the $r^4 = V^{4/n}$ would change to $r^{2n/3} = V^{2/3}$ if the periodic boundary condition is used instead of the bulk boundary condition.

Theorem 1.6 shows that, for p within a scaling window of width proportional to $V^{-1/3}$, centered at $p_c(\mathbb{T}_{r,n})$, the largest cluster is of size $V^{2/3}$ both for the sufficiently spread-out model with $n > 6$ and the nearest-neighbor model with n sufficiently large. An affirmative answer to Aizenman's question would then follow if we could prove that $p_c(\mathbb{Z}^n)$ is within this scaling window. It would be interesting to investigate this further.

1.9. *Organization.* The remainder of this paper is organized as follows. In Section 2 we analyze random walks on a torus and verify Assumption 1.1 for the three tori listed in Section 1.4. In Section 3 we give a self-contained derivation of the lace expansion. In Section 4 we estimate the Feynman diagrams that arise in the lace expansion. The results of Sections 3 apply on an arbitrary finite graph \mathbb{G} , which need not be transitive nor regular. Parts of Section 4 also apply in this general context, but in Section 4.2 we will specialize to $\mathbb{T}_{r,n}$. In Section 5 we analyze the lace expansion on an arbitrary torus that obeys Assumption 1.1, thereby proving Theorem 1.3. Finally, in Section 6 we establish a detailed relation between the Fourier transforms of the two-point functions for percolation and random walks.

2. Proof of Proposition 1.2.

2.1. *The random walk two-point function.* Consider a random walk on $\mathbb{T}_{r,n}$ where the transition probability for a step from x to y is equal to $D(x, y)$, with D given by (1.21). We assume that the edge set of the torus is invariant under translations and $x \mapsto -x$ reflections. The *two-point function* for the random walk is defined by

$$(2.1) \quad C_\mu(0, x) = \sum_{\omega: 0 \rightarrow x} \mu^{|\omega|},$$

where $0 \leq \mu < \Omega^{-1}$, the sum is over all random walks ω from 0 to x that take any number of steps $|\omega|$, and the “zero-step” walk contributes $\delta_{0,x}$. This is well defined, because the fact that there are Ω^m nearest-neighbor random walks of length m starting from the origin implies that

$$(2.2) \quad C_\mu(0, x) \leq \sum_{x \in \mathbb{T}_{r,n}} C_\mu(0, x) = \sum_{m=0}^{\infty} \Omega^m \mu^m = \frac{1}{1 - \mu\Omega} \quad (\mu < \Omega^{-1}),$$

that is, the random walk susceptibility $\sum_{x \in \mathbb{T}_{r,n}} C_\mu(0, x)$ is finite. Probabilistically, $C_{1/\Omega}(0, x)$ represents the expected number of visits to x for an infinite random walk starting at 0. Since the torus is finite, the random walk is recurrent, and hence, $C_{1/\Omega}(0, x)$ is infinite for all x . We therefore must keep $\mu < \Omega^{-1}$ when dealing with $C_\mu(0, x)$. The value $\mu = \Omega^{-1}$ plays the role of the critical point for random walks.

Using translation invariance, we can write $C_\mu(x, y) = C_\mu(y - x)$, where $C_\mu(x) = C_\mu(0, x)$. By conditioning on the first step, we see that the two-point function obeys the convolution equation

$$(2.3) \quad C_\mu(x) = \delta_{0,x} + \mu\Omega(D * C_\mu)(x).$$

Taking the Fourier transform of (2.3) gives $\hat{C}_\mu(k) = 1 + \mu\Omega\hat{D}(k)\hat{C}_\mu(k)$ and, hence,

$$(2.4) \quad \hat{C}_\mu(k) = \frac{1}{1 - \mu\Omega\hat{D}(k)}.$$

Note that $\hat{C}_\mu(0) < \infty$ for $\mu < \Omega^{-1}$ but $\hat{C}_{1/\Omega}(0) = \infty$. Although $C_{1/\Omega}(x)$ is infinite, the formula (2.4) does not diverge for $\mu = \Omega^{-1}$ for all k for which $\hat{D}(k) \neq 1$. Apart from any such singular points (usually arising only for $k = 0$), the expression $\hat{C}_{1/\Omega}(k) = [1 - \hat{D}(k)]^{-1}$ is finite. The factor $[1 - \hat{D}(k)]^{-3}$ that appears in (1.23) is thus the same as $\hat{C}_{1/\Omega}(k)^3$. Comparing with (1.20), we see that (1.23) is closely related to a triangle condition for random walks.

2.2. *Random walk estimates.* In this section we prove Proposition 1.2, which for convenience we restate as Proposition 2.1.

PROPOSITION 2.1. *There is an $a > 0$ such that random walks on each of the three tori listed in Section 1.4 obey Assumption 1.1 with $\beta = a\Omega^{-1}$, where:*

1. *For the nearest-neighbor torus, a is a universal constant, independent of $r \geq 2$ and $n \geq 7$.*
2. *For the Hamming torus, a is a universal constant, independent of $r \geq 2$ and $n \geq 1$.*
3. *For the spread-out torus, $n \geq 7$ is fixed, r is sufficiently large depending on L and n , and a depends on n but not on L or r .*

The proof is given throughout the remainder of Section 2.2. We first note that (1.22) is trivial since the maximal value of $D(x)$ is Ω^{-1} and this is less than $\beta = a\Omega^{-1}$, provided $a \geq 1$. We verify the substantial assumption (1.23) below. As a first step, we discuss the infrared bound for the random walk models.

2.2.1. *The infrared bound.* For the Hamming torus, $D(x)$ is zero unless exactly one coordinate of x is different from zero, in which case it is equal to Ω^{-1} . If we denote the number of nonzero components of $k \in \mathbb{T}_{r,n}^* = \frac{2\pi}{r} \{-\lfloor \frac{r-1}{2} \rfloor, \dots, \lceil \frac{r-1}{2} \rceil\}^n$ by $m(k)$, we therefore have

$$(2.5) \quad \hat{D}(k) = \frac{1}{\Omega} \sum_{j=1}^n \sum_{s=1}^{r-1} e^{ik_{js}} = \frac{1}{\Omega} \sum_{j=1}^n \left(\sum_{s=0}^{r-1} e^{ik_{js}} - 1 \right)$$

$$(2.6) \quad = \frac{1}{\Omega} \sum_{j=1}^n (r\delta_{k_j,0} - 1) = 1 - \frac{r}{r-1} \frac{m(k)}{n}.$$

This gives the infrared bound

$$(2.7) \quad 1 - \hat{D}(k) = \frac{r}{r-1} \frac{m(k)}{n} \geq \frac{m(k)}{n}.$$

For $k \in \mathbb{T}_{r,n}^*$, we define

$$(2.8) \quad |k|^2 = \sum_{j=1}^n k_j^2.$$

For the nearest-neighbor torus, by the symmetry of D , we have

$$(2.9) \quad \hat{D}(k) = \sum_{x \in \mathbb{T}_{r,n}} D(x) \cos(k \cdot x) = \frac{1}{n} \sum_{j=1}^n \cos k_j.$$

Since $1 - \cos t \geq 2\pi^{-2}t^2$ for $|t| \leq \pi$, this implies the infrared bound

$$(2.10) \quad 1 - \hat{D}(k) \geq \frac{2}{\pi^2} \frac{|k|^2}{n}.$$

For the spread-out torus, we first note that $\hat{D}(k)$ does not depend on r if $r \geq 2L + 1$. Thus, we can apply bounds on $\hat{D}(k)$ with $D(x)$ regarded as the step distribution of a random walk on \mathbb{Z}^n . The latter is analyzed in [17], Appendix A, where it is shown that there is an η depending only on n such that the infrared bound

$$(2.11) \quad 1 - \hat{D}(k) \geq \eta(1 \wedge L^2|k|^2)$$

holds for all $k \in \mathbb{T}_{r,n}^*$.

2.2.2. *The random walk triangle condition (1.23).*

PROOF OF (1.23) WITH $\beta = a\Omega^{-1}$ FOR THE HAMMING TORUS. Let $m(k)$ denote the number of nonzero components of k . We fix an $\varepsilon \in (0, 1)$, and divide the sum

$$(2.12) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3}$$

according to whether $m(k) \leq \varepsilon n$ or $m(k) > \varepsilon n$. It follows from (2.7) that the contribution to the sum due to $m(k) > \varepsilon n$ is bounded by

$$(2.13) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0, m(k) > \varepsilon n} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \leq \varepsilon^{-3} \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^*} \hat{D}(k)^2 = \varepsilon^{-3} \Omega^{-1},$$

since V^{-1} times the summation in the middle term is the probability that a random walk returns to its starting point after two steps.

Note that the case $m(k) \leq \varepsilon n$ does not occur for $n = 1$ if we take $\varepsilon < 1$, so we may assume that $n \geq 2$. In this case, since $|\hat{D}(k)| \leq 1$, it follows from (2.7) that

$$(2.14) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0, m(k) \leq \varepsilon n} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \leq \frac{1}{r^n} \sum_{m=1}^{\varepsilon n} \binom{n}{m} (r-1)^m \frac{n^3}{m^3}.$$

In (2.14) the binomial coefficient counts the number of ways to choose m nonzero components from n , and the factor $(r-1)^m$ counts the number of values that each nonzero component can assume. Discarding two factors of $1/m$ and using $n[m(n-m)]^{-1} \leq n(n-1)^{-1} \leq 2$, the right-hand side of (2.14) is at most

$$(2.15) \quad \begin{aligned} & n^3 \sum_{m=1}^{\varepsilon n} \frac{1}{m} \binom{n}{m} \left(1 - \frac{1}{r}\right)^m \left(\frac{1}{r}\right)^{n-m} \\ &= \frac{n^3}{r} \sum_{m=1}^{\varepsilon n} \frac{n}{m(n-m)} \binom{n-1}{m} \left(1 - \frac{1}{r}\right)^m \left(\frac{1}{r}\right)^{n-1-m} \\ &\leq 2 \frac{n^3}{r} \mathbb{P}(X \leq \varepsilon n), \end{aligned}$$

where X is a binomial random variable with parameters $(n-1, 1-r^{-1})$. Let $p = 1-r^{-1}$. Since $\varepsilon n = (p-a)(n-1)$ with $a \geq \frac{1}{2} - 2\varepsilon$, it follows from

$$(2.16) \quad \mathbb{P}(X \leq (p-a)(n-1)) \leq e^{-(n-1)a^2/2}$$

(a consequence of the Chernoff bound, see [6], page 12) that the right-hand side of (2.15) decays exponentially in n , uniformly in $r \geq 2$, if we choose an $\varepsilon < \frac{1}{4}$. This gives the desired result. \square

PROOF OF (1.23) WITH $\beta = a\Omega^{-1}$ FOR THE NEAREST-NEIGHBOR TORUS. Since the nearest-neighbor torus is the same as the Hamming torus when $r = 2$ (in which case both are the n -cube), we may assume that $r \geq 3$. Hence, $\Omega = 2n$.

We first prove that

$$(2.17) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^*} \hat{D}(k)^{2i} \leq \frac{e^{2^i i^{2i}}}{\Omega^i}.$$

For this, we observe that the left-hand side is equal to the probability that a random walk on $\mathbb{T}_{r,n}$ that starts at the origin returns to the origin after $2i$ steps. This probability is equal to Ω^{-2i} times the number of walks that make the transition from 0 to 0 in $2i$ steps. Each such walk must take an even number of steps in each coordinate direction, implying that it will live in a subspace of dimension $\ell \leq \min\{i, n\}$. If we fix the subspace, and assume $r \geq 3$, then each step in the subspace can be chosen from 2ℓ different directions, leading to a bound of $(2\ell)^{2i}$ for the number of walks in the subspace. Since the number of subspaces of fixed dimension ℓ is given by $\binom{n}{\ell} \leq n^\ell / \ell!$, we obtain the bound

$$(2.18) \quad \sum_{\ell=1}^i \frac{1}{\ell!} n^\ell (2\ell)^{2i} \leq \Omega^i i^{2i} \sum_{\ell=1}^i \frac{1}{\ell!} 2^i \leq \Omega^i e 2^i i^{2i}$$

for the number of walks that make the transition from 0 to 0 in $2i$ steps. Multiplying by Ω^{-2i} to convert the number of walks into a probability, this gives (2.17).

By (2.17) and Hölder's inequality, for any $i' > 1$,

$$(2.19) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \leq \left(\frac{e 2^{i'} (i')^{2i'}}{\Omega^{i'}} \right)^{1/i'} \left(\frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{1}{[1 - \hat{D}(k)]^{3i'}} \right)^{i'/i},$$

where $i = i'(i' - 1)^{-1}$. We choose i' large enough that $6i < 7$. By (2.19) and the infrared bound (2.10), it suffices to show that

$$(2.20) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{n^{3i}}{|k|^{6i}}$$

is bounded uniformly in $n \geq 7$ and $r \geq 3$. Let $B(0, r) = (-\frac{\pi}{r}, \frac{\pi}{r}]^n \subset \mathbb{R}^n$. For each $k \in (-\pi, \pi]^n$, there is a unique $k_r \in \mathbb{T}_{r,n}^*$ such that $k \in k_r + B(0, r)$ and we define

$$(2.21) \quad F_r(k) = \begin{cases} \frac{n^{3i}}{|k_r|^{6i}}, & k_r \neq 0, \\ 0, & k_r = 0. \end{cases}$$

Thus, $F_r(k)$ is constant on the cubes $k_r + B(0, r)$ for $k_r \in \mathbb{T}_{r,n}^*$, the identity $V|B(0, r)| = (2\pi)^n$ holds, and

$$(2.22) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{n^{3i}}{|k|^{6i}} = \int_{[-\pi, \pi]^n} F_r(k) \frac{d^n k}{(2\pi)^n}.$$

We fix $\varepsilon \in (0, 1)$, and let Q_1 and Q_2 denote the subsets of $(-\pi, \pi]^n$ for which $|k| \geq \frac{1}{1-\varepsilon} \frac{\pi}{r} \sqrt{n}$ and $|k| \leq \frac{1}{1-\varepsilon} \frac{\pi}{r} \sqrt{n}$, respectively.

For $k \in k_r + B(0, r)$, we have $|k - k_r| \leq \frac{\pi}{r} \sqrt{n}$. For $k \in Q_1$, it follows that $|k_r| \geq \varepsilon|k|$ and, hence, the contribution to the integral (2.22) due to $k \in Q_1$ is at most

$$(2.23) \quad \varepsilon^{-6i} \int_{Q_1} \frac{n^{3i}}{|k|^{6i}} \frac{d^n k}{(2\pi)^n} \leq \varepsilon^{-6i} \int_{(-\pi, \pi]^n} \frac{n^{3i}}{|k|^{6i}} \frac{d^n k}{(2\pi)^n}.$$

The integral on the right-hand side is bounded uniformly in $n \geq 7$, by the following argument. For $A > 0$ and $m > 0$,

$$(2.24) \quad \frac{1}{A^m} = \frac{1}{\Gamma(m)} \int_0^\infty t^{m-1} e^{-tA} dt.$$

Applying this with $A = |k|^2/n$ and $m = 3i$ gives

$$(2.25) \quad \int_{[-\pi, \pi]^n} \frac{n^{3i}}{|k|^{6i}} \frac{d^n k}{(2\pi)^n} = \frac{1}{\Gamma(3i)} \int_0^\infty dt t^{3i-1} \left(\int_{-\pi}^\pi \frac{d\theta}{2\pi} (e^{-t\theta^2})^{1/n} \right)^n.$$

The right-hand side is nonincreasing in n , since $\|f\|_p \leq \|f\|_q$ for $0 < p \leq q \leq \infty$ on a probability space.

For $k \in Q_2$, we use the fact that $|k_r| \geq \frac{2\pi}{r}$ for all nonzero k_r to obtain

$$(2.26) \quad \begin{aligned} \int_{Q_2} F_r(k) \frac{d^n k}{(2\pi)^n} &\leq \frac{n^{3i} r^{6i}}{(2\pi)^{6i}} \int_{Q_2} \frac{d^n k}{(2\pi)^n} \\ &= \frac{n^{3i} r^{6i}}{(2\pi)^{6i}} \frac{1}{(2\pi)^n} v_n \left(\frac{1}{1-\varepsilon} \frac{\pi}{r} \sqrt{n} \right)^n, \end{aligned}$$

where v_n denotes the volume of the unit ball in n dimensions. Since

$$(2.27) \quad v_n = \frac{\pi^{n/2}}{\Gamma(n/2 + 1)} \leq \left(\frac{2\pi e}{n} \right)^{n/2}$$

[using $\Gamma(a + 1) \geq a^a e^{-a}$], this gives

$$(2.28) \quad \int_{Q_2} F_r(k) \frac{d^n k}{(2\pi)^n} \leq \frac{n^{3i} r^{6i}}{(2\pi)^{6i}} \left(\frac{e\pi}{(1-\varepsilon)^2 2r^2} \right)^{n/2}.$$

We fix ε so that

$$(2.29) \quad \frac{e\pi}{(1-\varepsilon)^2} < 9.$$

The right-hand side of (2.28) is then bounded uniformly in $n \geq 7$ and $r \geq 3$. \square

PROOF OF (1.23) WITH $\beta = a\Omega^{-1}$ FOR THE SPREAD-OUT TORUS. Now $n \geq 7$ is fixed, L is fixed and large, and $r \rightarrow \infty$. We first use the dominated convergence theorem to show that

$$(2.30) \quad \lim_{r \rightarrow \infty} \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} = \int_{[-\pi, \pi]^n} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \frac{d^n k}{(2\pi)^n}.$$

It follows that the expression under the limit on the left-hand side is bounded above by twice the integral on the right-hand side, for r sufficiently large depending on L, n .

Recalling the definition of k_r above (2.21), we define

$$(2.31) \quad G_r(k) = \begin{cases} \hat{D}(k_r)^2 [1 - \hat{D}(k_r)]^{-3}, & k_r \neq 0, \\ 0, & k_r = 0, \end{cases}$$

so that

$$(2.32) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} = \int_{[-\pi, \pi]^n} G_r(k) \frac{d^n k}{(2\pi)^n}.$$

The function $G_r(k)$ converges pointwise to $\hat{D}(k)^2 [1 - \hat{D}(k)]^{-3}$ for $k \neq 0$ and to 0 for $k = 0$. Also, by the infrared bound (2.11),

$$(2.33) \quad G_r(k) \leq \frac{1}{\eta^3 L^3 |k_r|^6}$$

for every nonzero $k_r \in \mathbb{T}_{r,n}^*$. For each nonzero $k_r \in \mathbb{T}_{r,n}^*$ and $k \in k_r + B(0, r)$, we have $\|k_r\|_\infty \geq \frac{2\pi}{r}$ and $\|k - k_r\|_\infty \leq \frac{\pi}{r}$, so that $\|k_r\|_\infty \geq \frac{2}{3}\|k\|_\infty$. This implies that $|k_r|^2 \geq \|k_r\|_\infty^2 \geq \frac{4}{9}\|k\|_\infty^2 \geq \frac{4}{9n}|k|^2$. Therefore, for every $k \in (-\pi, \pi]^n$,

$$(2.34) \quad G_r(k) \leq \left(\frac{9}{4}\right)^3 \frac{n^3}{\eta^3 L^3 |k|^6},$$

which is integrable when $n \geq 7$. Therefore, by dominated convergence, (2.30) holds, and it suffices to bound the integral on the right-hand side of (2.30) by an L -independent multiple of Ω^{-1} .

We bound the integral on the right-hand side of (2.30) by considering separately the regions where $|k|^2 > L^{-2}$ and $|k|^2 \leq L^{-2}$. For the contribution to the integral on the right-hand side of (2.30) due to $|k|^2 > L^{-2}$, we use (2.11) and argue as in (2.13) to obtain

$$(2.35) \quad \int_{k \in [-\pi, \pi]^n : |k|^2 > L^{-2}} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \frac{d^n k}{(2\pi)^n} \leq \eta^{-3} \int_{[-\pi, \pi]^n} \hat{D}(k)^2 \frac{d^n k}{(2\pi)^n} = \eta^{-3} \Omega^{-1}.$$

For the contribution to the integral in (2.30) due to $|k|^2 \leq L^{-2}$, we use (2.11) and $\hat{D}(k)^2 \leq 1$ to obtain

$$(2.36) \quad \int_{|k|^2 \leq L^{-2}} \frac{\hat{D}(k)^2}{[1 - \hat{D}(k)]^3} \frac{d^n k}{(2\pi)^n} \leq \frac{1}{\eta^3 L^6} \int_{|k|^2 \leq L^{-2}} \frac{1}{|k|^6} \frac{d^n k}{(2\pi)^n} = C_{n,\eta} L^{-n}.$$

Summing the two contributions yields (1.23) with $\beta = a\Omega^{-1}$. \square

2.2.3. *A consequence of Assumption 1.1.* Finally, we note for future reference that (1.23) implies that

$$(2.37) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{1}{[1 - \hat{D}(k)]^3} \leq 1 + 6\beta.$$

To see this, we use the identity

$$(2.38) \quad \frac{1}{[1 - \hat{D}]^3} = 1 + 3\hat{D} + \frac{3\hat{D}^2}{1 - \hat{D}} + \frac{2\hat{D}^2}{[1 - \hat{D}]^2} + \frac{\hat{D}^2}{[1 - \hat{D}]^3}.$$

The sum of the last three terms on the right-hand side is at most $6\hat{D}^2[1 - \hat{D}]^{-3}$, and their normalized sum over k is thus at most 6β , by (1.23). Since the normalized sum over *all* $k \in \mathbb{T}_{r,n}^*$ of $3\hat{D}(k)$ is $3D(0) = 0$, its sum over nonzero k is $-3V^{-1} < 0$. This proves (2.37).

3. The lace expansion. We begin in Section 3.1 with a brief overview of the lace expansion, and then give a self-contained and detailed derivation of the expansion in Section 3.2.

The term ‘‘lace’’ was used by Brydges and Spencer [9] for a certain graphical construction that arose in the expansion they invented to study the self-avoiding walk. Although the lace expansion for percolation evolved from the expansion for the self-avoiding walk, this graphical construction does not occur for percolation, and so the term ‘‘lace’’ expansion is a misnomer in the percolation context. However, the name has stuck for historical reasons.

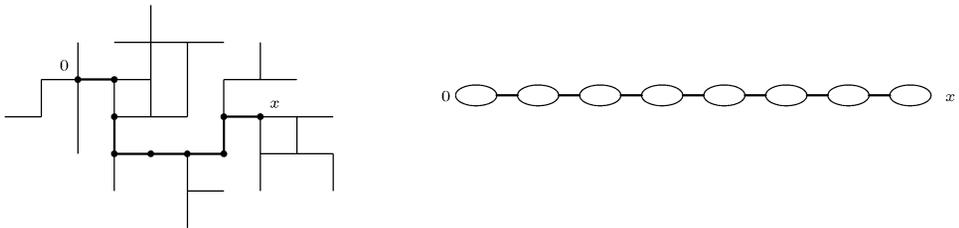


FIG. 1. A percolation cluster with a string of 8 sausages joining 0 to x , and a schematic representation of the string of sausages. The 7 pivotal bonds are shown in bold.

3.1. *Overview of the lace expansion.* In this section we give a brief introduction to the lace expansion, with an indication of how it is used to prove the triangle condition of Theorem 1.3. Since the discussion will involve the Fourier transform, we restrict attention here to percolation on the nearest-neighbor torus $\mathbb{T}_{r,n}$, with $r \geq 2$ and n large. Each vertex has degree $\Omega = 2n$ for $r \geq 3$ and $\Omega = n$ for $r = 2$. However, in Section 3.2 the expansion will be derived on an arbitrary finite graph \mathbb{G} .

Given a percolation cluster containing 0 and x , we call any bond whose removal would disconnect 0 from x a *pivotal* bond. The connected components that remain after removing all pivotal bonds are called *sausages*. Since they are separated by at least one pivotal bond by definition, no two sausages can have a common vertex. Thus, the sausages are constrained to be mutually avoiding. However, this is a weak constraint, since sausage intersections require a cycle, and cycles are unlikely. In fact, for p asymptotically proportional to Ω^{-1} , and, hence, for $p = p_c$, the probability that the origin is in a cycle of length 4 is of order $\Omega^2\Omega^{-4} = \Omega^{-2}$, and larger cycles are more unlikely. The fact that cycles are unlikely also means that sausages tend to be trees. This makes it reasonable to attempt to apply an inclusion–exclusion analysis, where the connection from 0 to x is treated as a random walk path, with correction terms taking into account cycles in sausages and the avoidance constraint between sausages. With this in mind, it makes sense to attempt to relate $\tau_p(0, x)$ to the two-point function for random walks.

The lace expansion of Hara and Slade [13] makes this procedure precise. It produces a convolution equation of the form

$$(3.1) \quad \tau_p(0, x) = \delta_{0,x} + p\Omega(D * \tau_p)(0, x) + p\Omega(\Pi_p * D * \tau_p)(0, x) + \Pi_p(0, x)$$

for the two-point function, valid for $p \leq p_c(\mathbb{T}_{r,n})$. The expansion gives explicit but complicated formulas for the function $\Pi_p : \mathbb{T}_{r,n} \times \mathbb{T}_{r,n} \rightarrow \mathbb{R}$. It will turn out that if Assumption 1.1 holds with $\beta = O(\Omega^{-1})$, then $\hat{\Pi}_p(k) = O(\Omega^{-1})$ uniformly in $p \leq p_c(\mathbb{G})$ and $k \in \mathbb{T}_{r,n}^*$. Putting $\Pi_p \equiv 0$ in (3.1) gives (2.3), and in this sense the percolation two-point function can be regarded as a small perturbation of the random walk two-point function.

Using (1.15), (3.1) can be solved to give

$$(3.2) \quad \hat{\tau}_p(k) = \frac{1 + \hat{\Pi}_p(k)}{1 - p\Omega\hat{D}(k)[1 + \hat{\Pi}_p(k)]}.$$

We will show that, under Assumption 1.1, $\hat{\Pi}_p(k)$ can be well approximated by $\hat{\Pi}_p(0)$. Since $\hat{\Pi}_p(k)$ is also small compared to 1, (3.2) suggests that the approximation

$$(3.3) \quad \hat{\tau}_p(k) \approx \frac{1}{1 - p\Omega[1 + \hat{\Pi}_p(0)]\hat{D}(k)}$$

is reasonable (where “ \approx ” denotes an uncontrolled approximation). Comparing with (2.4), this suggests that

$$(3.4) \quad \hat{\tau}_p(k) \approx \hat{C}_{\mu_p}(k) \quad \text{with} \quad \mu_p \Omega = p \Omega [1 + \hat{\Pi}_p(0)].$$

We will make this approximation precise in (6.5). Since $\hat{D}(0) = 1$, if we set $k = 0$ in (3.2) and solve for $p \Omega$, then we obtain

$$(3.5) \quad p \Omega = \frac{1}{1 + \hat{\Pi}_p(0)} - \hat{\tau}_p(0)^{-1}.$$

For $p = p_c = p_c(\mathbb{T}_{r,n})$, (3.5) states that

$$(3.6) \quad p_c \Omega = \frac{1}{1 + \hat{\Pi}_{p_c}(0)} - \lambda^{-1} V^{-1/3},$$

and, hence, $\mu_{p_c} \Omega \approx 1 - \lambda^{-1} V^{-1/3}$. This should be compared with the critical value $\mu \Omega = 1$ for the random walk.

For the triangle condition, we analyze the Fourier representation of $\nabla_p(0, x)$ given in (1.20). Extraction of the $k = 0$ term in (1.20) gives

$$(3.7) \quad \nabla_p(0, x) = \frac{\chi(p)^3}{V} + \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \hat{\tau}_p(k)^3 e^{-ik \cdot x}.$$

The second term can be estimated using (3.4) and Assumption 1.1, leading to a proof of Theorem 1.3. Details are given in Section 5.

Equation (3.6) is an implicit equation for the critical threshold. Using $\hat{\Pi}_p(k) = O(\Omega^{-1})$, (3.6) gives $p_c = \Omega^{-1} + O(\Omega^{-2})$ if $\lambda^{-1} V^{-1/3} \leq O(\Omega^{-2})$. This is the first term in an asymptotic expansion. Further terms will follow from an asymptotic expansion of $\hat{\Pi}_{p_c}(0)$ in powers of Ω^{-1} . Calculations of this sort were carried out in [15] for percolation on \mathbb{Z}^n , and have been extended in [19, 20], as was discussed in Section 1.8.

3.2. Derivation of the lace expansion. In this section we derive a version of the lace expansion (3.1) that contains a remainder term. Throughout this section, we let \mathbb{G} denote an arbitrary finite graph, which need not be transitive nor regular. We use the method of [13], which applies directly in this general setting, and we follow the presentation of [22]. We assume, for simplicity, that \mathbb{G} is finite, but with minor modifications the analysis also applies when \mathbb{G} is infinite, provided there is almost surely no infinite cluster.

Fix $p \in [0, 1]$. We define

$$(3.8) \quad \begin{aligned} J(x, y) &= p I[\{x, y\} \in \mathbb{B}] \\ &= p \Omega D(x, y) \quad \text{if } \mathbb{G} \text{ is regular,} \end{aligned}$$

with D given by (1.21). We write $\tau(x, y) = \tau_p(x, y)$ for brevity, and generally drop subscripts indicating dependence on p . For each $M = 0, 1, 2, \dots$, the expansion takes the form

$$(3.9) \quad \begin{aligned} \tau(x, y) &= \delta_{x,y} + (J * \tau)(x, y) + (\Pi_M * J * \tau)(x, y) \\ &\quad + \Pi_M(x, y) + R_M(x, y), \end{aligned}$$

where the “ $*$ ” product denotes matrix multiplication (this reduces to convolution when $\mathbb{G} = \mathbb{T}_{r,n}$). The function $\Pi_M: \mathbb{V} \times \mathbb{V} \rightarrow \mathbb{R}$ is the key quantity in the expansion, and $R_M(x, y)$ is a remainder term. The dependence of Π_M on M is given by

$$(3.10) \quad \Pi_M(x, y) = \sum_{N=0}^M (-1)^N \Pi^{(N)}(x, y),$$

with $\Pi^{(N)}(x, y)$ independent of M . The alternating sign in (3.10) arises via repeated inclusion–exclusion. In Section 6 we will prove that, for $\mathbb{G} = \mathbb{T}_{r,n}$, $p \leq p_c(\mathbb{T}_{r,n})$, and assuming Assumption 1.1 with $\lambda^3 \vee \beta$ sufficiently small,

$$(3.11) \quad \lim_{M \rightarrow \infty} \sum_y |R_M(x, y)| = 0.$$

This leads to (3.1) with $\Pi = \Pi_\infty$. Convergence properties of (3.10) when $M = \infty$ will also be established in Section 6. The remainder of this section gives the proof of (3.9).

We need the following definitions.

DEFINITION 3.1. (a) Given a bond configuration, and $A \subset \mathbb{V}$, we say x and y are *connected in A* , and write $\{x \leftrightarrow y \text{ in } A\}$, if $x = y \in A$ or if there is an occupied path from x to y having all its endpoints in A . We define a restricted two-point function by

$$(3.12) \quad \tau^A(x, y) = \mathbb{P}(x \leftrightarrow y \text{ in } \mathbb{V} \setminus A).$$

(b) Given a bond configuration, and $A \subset \mathbb{V}$, we say x and y are *connected through A* , if $x \leftrightarrow y$ and every occupied path connecting x to y has at least one bond with an endpoint in A , or if $x = y \in A$. This event is written as $x \overset{A}{\leftrightarrow} y$.

(c) Given a bond configuration, and a bond b , we define $\tilde{C}^b(x)$ to be the set of sites connected to x in the new configuration obtained by setting b to be vacant.

(d) Given a bond configuration, we say that x is *doubly connected to y* , and we write $x \Leftrightarrow y$, if $x = y$ or if there are at least two bond-disjoint paths from x to y consisting of occupied bonds.

(e) Given a bond configuration, a bond $\{u, v\}$ (occupied or not) is called *pivotal* for the connection from x to y if (i) either $x \leftrightarrow u$ and $y \leftrightarrow v$, or $x \leftrightarrow v$ and $y \leftrightarrow u$, and (ii) $y \notin \tilde{C}^{\{u,v\}}(x)$. Bonds are not usually regarded as directed. However, it will be convenient at times to regard a bond $\{u, v\}$ as directed from u to v , and we will emphasize this point of view with the notation (u, v) . A directed bond (u, v) is pivotal for the connection from x to y if $x \leftrightarrow u$, $v \leftrightarrow y$ and $y \notin \tilde{C}^{(u,v)}(x)$. We denote by $P_{(x,y)}$ the set of directed pivotal bonds for the connection from x to y .

To begin the expansion, we define

$$(3.13) \quad \Pi^{(0)}(x, y) = \mathbb{P}(x \leftrightarrow y) - \delta_{x,y}$$

and distinguish configurations with $x \leftrightarrow y$, according to whether or not there is a double connection, to obtain

$$(3.14) \quad \tau(x, y) = \delta_{x,y} + \Pi^{(0)}(x, y) + \mathbb{P}(x \leftrightarrow y \text{ and } x \not\leftrightarrow y).$$

If x is connected to y , but not doubly, then $P_{(x,y)}$ is nonempty. There is therefore a unique element $(u, v) \in P_{(x,y)}$ (the *first* pivotal bond) such that $x \leftrightarrow u$, and we can write

$$(3.15) \quad \begin{aligned} &\mathbb{P}(x \leftrightarrow y \text{ and } x \not\leftrightarrow y) \\ &= \sum_{(u,v)} \mathbb{P}(x \leftrightarrow u \text{ and } (u, v) \text{ is occupied and pivotal for } x \leftrightarrow y). \end{aligned}$$

Now comes the essential part of the expansion. Ideally, we would like to factor the probability on the right-hand side of (3.15) as

$$(3.16) \quad \begin{aligned} &\mathbb{P}(x \leftrightarrow u) \mathbb{P}((u, v) \text{ is occupied}) \mathbb{P}(v \leftrightarrow y) \\ &= (\delta_{x,u} + \Pi^{(0)}(x, u)) J(u, v) \tau(v, y). \end{aligned}$$

The expression (3.16) is the same as (3.9) with $\Pi_M = \Pi^{(0)}$ and $R_M = 0$. However, (3.15) does not factor in this way because the cluster $\tilde{C}^{(u,v)}(u)$ is constrained not to intersect the cluster $\tilde{C}^{(u,v)}(v)$, since (u, v) is pivotal. What we can do is approximate the probability on the right-hand side of (3.15) by (3.16), and then attempt to deal with the error term.

For this, we will use the next lemma, which gives an identity for the probability on the right-hand side of (3.15). In fact, we will also need a more general identity, involving the following generalizations of the event appearing on the right-hand side of (3.15). Let $x, u, v, y \in \mathbb{V}$, and $A \subset \mathbb{V}$ be nonempty. Then we define the events

$$(3.17) \quad E'(v, y; A) = \{v \overset{A}{\leftrightarrow} y\} \cap \{\bar{A}(u', v') \in P_{(v,y)} \text{ such that } v \overset{A}{\leftrightarrow} u'\}$$

and

$$(3.18) \quad \begin{aligned} &E(x, u, v, y; A) \\ &= E'(x, u; A) \cap \{(u, v) \text{ is occupied and pivotal for } x \leftrightarrow y\}. \end{aligned}$$

Note that $\{x \leftrightarrow y\} = E'(x, y; \mathbb{V})$, while $E(x, u, v, y; \mathbb{V})$ is the event appearing on the right-hand side of (3.15). A version of Lemma 3.2, with $E'(x, u; A)$ replaced by $\{0 \leftrightarrow u\}$ on both sides of (3.19), appeared in [3].

LEMMA 3.2. *Let \mathbb{G} be a finite graph, $p \in [0, 1]$, $u \in \mathbb{V}$, and let $A \subset \mathbb{V}$ be nonempty. Then*

$$(3.19) \quad \mathbb{E}(I[E(x, u, v, y; A)]) = p\mathbb{E}(I[E'(x, u; A)]\tau^{\tilde{C}^{(u,v)}(x)}(v, y)).$$

PROOF. The event appearing in the left-hand side of (3.19) is depicted in Figure 2. We first observe that the event $E'(x, u; A) \cap \{(u, v) \in P_{(x,y)}\}$ is independent of the occupation status of the bond (u, v) . This is true by definition for $\{(u, v) \in P_{(x,y)}\}$, and when (u, v) is pivotal, the occurrence or not of $E'(x, u; A)$ cannot be affected by $\{u, v\}$, since in this case $E'(x, u; A)$ is determined by the occupied paths from x to u and no such path uses the bond $\{u, v\}$. Therefore, the left-hand side of the identity in the statement of the lemma is equal to

$$(3.20) \quad p\mathbb{E}(I[E'(x, u; A) \cap \{(u, v) \in P_{(x,y)}\}]).$$

By conditioning on $\tilde{C}^{(u,v)}(x)$, (3.20) is equal to

$$(3.21) \quad p \sum_{S: S \ni x} \mathbb{E}(I[E'(x, u; A) \cap \{(u, v) \in P_{(x,y)}\} \cap \{\tilde{C}^{(u,v)}(x) = S\}]),$$

where the sum is over all finite connected sets of vertices S containing x .

In (3.21) we make the replacement, for S such that $u \in S$,

$$(3.22) \quad \{(u, v) \in P_{(x,y)}\} \cap \{\tilde{C}^{(u,v)}(x) = S\} = \{v \leftrightarrow y \text{ in } \mathbb{V} \setminus S\} \cap \{\tilde{C}^{(u,v)}(x) = S\}.$$

The event $\{v \leftrightarrow y \text{ in } \mathbb{V} \setminus S\}$ depends only on the occupation status of bonds which do not have an endpoint in S . On the other hand, given that $\{v \leftrightarrow y \text{ in } \mathbb{V} \setminus S\} \cap \{\tilde{C}^{(u,v)}(x) = S\}$ occurs, the event $E'(x, u; A)$ is determined by the occupation status of bonds which have an endpoint in $S = \tilde{C}^{(u,v)}(x)$. Similarly, the event $\{\tilde{C}^{(u,v)}(x) = S\}$ depends on bonds which have one or both endpoints

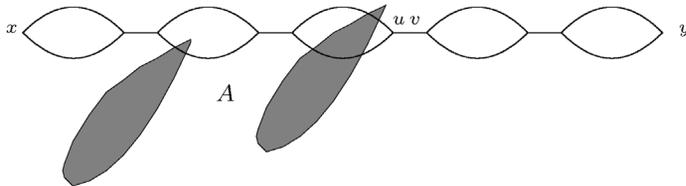


FIG. 2. *The event $E(x, u, v, y; A)$ of Lemma 3.2. The shaded regions represent the vertices in A . There is no restriction on intersections between A and $\tilde{C}^{(u,v)}(y)$.*

in S . Hence, given S , the event $E'(x, u; A) \cap \{\tilde{C}^{\{u,v\}}(x) = S\}$ is independent of the event that $\{v \leftrightarrow y \text{ in } \mathbb{V} \setminus S\}$, and, therefore, (3.21) is equal to

$$(3.23) \quad p \sum_{S: S \ni x} \mathbb{E}(I[E'(x, u; a) \cap \{\tilde{C}^{\{u,v\}}(x) = S\}]) \tau_p^S(v, y).$$

Bringing the restricted two-point function inside the expectation, replacing the superscript S by $\tilde{C}^{\{u,v\}}(x)$, and performing the sum over S , gives the desired result. □

It follows from (3.15) and Lemma 3.2 that

$$(3.24) \quad \mathbb{P}(x \leftrightarrow y \text{ and } x \not\leftrightarrow y) = \sum_{(u,v)} J(u, v) \mathbb{E}(I[x \leftrightarrow u] \tau^{\tilde{C}^{\{u,v\}}(x)}(v, y)),$$

where we write $\tilde{C}^{(u,v)}(x)$ in place of $\tilde{C}^{\{u,v\}}(x)$ to emphasize the directed nature of the bond (u, v) . On the right-hand side, $\tau^{\tilde{C}^{(u,v)}(x)}(v, y)$ is the restricted two-point function *given* the cluster $\tilde{C}^{(u,v)}(x)$ of the outer expectation, so that in the expectation defining $\tau^{\tilde{C}^{(u,v)}(x)}(v, y)$, $\tilde{C}^{(u,v)}(x)$ should be regarded as a *fixed* set. We stress this delicate point here, as it is crucial also in the rest of the expansion. The inner expectation on the right-hand side effectively introduces a second percolation model on a second graph, which depends on the original percolation model via the set $\tilde{C}^{(u,v)}(x)$.

We write

$$(3.25) \quad \begin{aligned} \tau^{\tilde{C}^{(u,v)}(x)}(v, y) &= \tau(v, y) - (\tau(v, y) - \tau^{\tilde{C}^{(u,v)}(x)}(v, y)) \\ &= \tau(v, y) - \mathbb{P}(v \overset{\tilde{C}^{(u,v)}(x)}{\longleftrightarrow} y), \end{aligned}$$

insert this into (3.24), and use (3.14) and (3.13) to obtain

$$(3.26) \quad \begin{aligned} \tau(x, y) &= \delta_{x,y} + \Pi^{(0)}(x, y) \\ &+ \sum_{(u,v)} (\delta_{x,u} + \Pi^{(0)}(x, u)) J(u, v) \tau(v, y) \\ &- \sum_{(u,v)} J(u, v) \mathbb{E}(I[x \leftrightarrow u] \mathbb{P}(v \overset{\tilde{C}^{(u,v)}(x)}{\longleftrightarrow} y)). \end{aligned}$$

With $R_0(x, y)$ equal to the last term on the right-hand side of (3.26) (including the minus sign), this proves (3.9) for $M = 0$.

To continue the expansion, we would like to rewrite the final term of (3.26) in terms of a product with the two-point function. A configuration contributing to the expectation in the final term of (3.26) is illustrated schematically in Figure 3, in which the bonds drawn with heavy lines should be regarded as living on a different graph than the bonds drawn with lighter lines, as explained previously. Our goal is to extract a factor $\tau(v', y)$, where v' is shown in Figure 3.

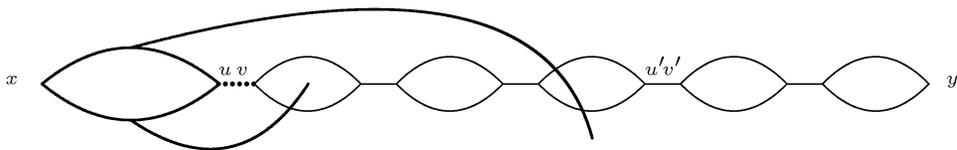


FIG. 3. A possible configuration appearing in the second stage of the expansion.

Given a configuration in which $v \overset{A}{\longleftrightarrow} y$, the *cutting bond* (u', v') is defined to be the first pivotal bond for $v \leftrightarrow y$ such that $v \overset{A}{\longleftrightarrow} u'$. It is possible that no such bond exists, as, for example, would be the case in Figure 3 if only the leftmost four sausages were included in the figure (using the terminology of Section 3.1), with y in the location currently occupied by u' . Recall the definitions of $E'(v, y; A)$ and $E(x, u, v, y; A)$ in (3.17) and (3.18). By partitioning $\{v \overset{A}{\longleftrightarrow} y\}$ according to the location of the cutting bond (or the lack of a cutting bond), we obtain the partition

$$(3.27) \quad \{v \overset{A}{\longleftrightarrow} y\} = E'(v, y; A) \dot{\bigcup}_{(u', v')} E(v, u', v', y; A),$$

which implies that

$$(3.28) \quad \mathbb{P}(v \overset{A}{\longleftrightarrow} y) = \mathbb{P}(E'(v, y; A)) + \sum_{(u', v')} \mathbb{P}(E(v, u', v', y; A)).$$

Using Lemma 3.2, this gives

$$(3.29) \quad \begin{aligned} \mathbb{P}(v \overset{A}{\longleftrightarrow} y) &= \mathbb{P}(E'(v, y; A)) \\ &+ \sum_{(u', v')} J(u', v') \mathbb{E}(I[E'(v, u'; A)] \tau^{\tilde{C}^{(u', v')}(v)}(v', y)). \end{aligned}$$

Inserting the identity (3.25) into (3.29), we obtain

$$(3.30) \quad \begin{aligned} \mathbb{P}(v \overset{A}{\longleftrightarrow} y) &= \mathbb{P}(E'(v, y; A)) \\ &+ \sum_{(u', v')} J(u', v') \mathbb{P}(E'(v, u'; A)) \tau(v', y) \\ &- \sum_{(u', v')} J(u', v') \mathbb{E}_1(I[E'(v, u'; A)] \mathbb{P}_2(v' \overset{\tilde{C}_1^{(u', v')}(v)}{\longleftrightarrow} y)). \end{aligned}$$

In the last term on the right-hand side, we have introduced subscripts for \tilde{C} and the expectations to indicate to which expectation \tilde{C} belongs.

Let

$$(3.31) \quad \Pi^{(1)}(x, y) = \sum_{(u, v)} J(u, v) \mathbb{E}_0(I[x \leftrightarrow u] \mathbb{P}_1(E'(v, y; \tilde{C}_0^{(u, v)}(x)))).$$

Inserting (3.30) into (3.26), and using (3.31), we have

$$\begin{aligned}
 \tau(x, y) &= \delta_{x,y} + \Pi^{(0)}(x, y) - \Pi^{(1)}(x, y) \\
 &+ \sum_{(u,v)} (\delta_{x,u} + \Pi^{(0)}(x, u) - \Pi^{(1)}(x, u)) J(u, v) \tau(v, y) \\
 (3.32) \quad &+ \sum_{(u,v)} J(u, v) \sum_{(u',v')} J(u', v') \\
 &\times \mathbb{E}_0(I[x \Leftrightarrow u] \mathbb{E}_1(I[E'(v, u'); \tilde{C}_0^{(u,v)}(x)]) \mathbb{P}_2(v' \overset{\tilde{C}_1^{(u',v')}(v)}{\longleftrightarrow} y))).
 \end{aligned}$$

This proves (3.9) for $M = 1$, with $R_1(x, y)$ given by the last two lines of (3.32).

We now repeat this procedure recursively, rewriting $\mathbb{P}_2(v' \overset{\tilde{C}_1^{(u',v')}(v)}{\longleftrightarrow} y)$ using (3.30), and so on. This leads to (3.9), with $\Pi^{(0)}$ and $\Pi^{(1)}$ given by (3.13) and (3.31) and, for $N \geq 2$,

$$\begin{aligned}
 \Pi^{(N)}(x, y) &= \sum_{(u_0,v_0)} \cdots \sum_{(u_{N-1},v_{N-1})} \left[\prod_{i=0}^{N-1} J(u_i, v_i) \right] \mathbb{E}_0 I[x \Leftrightarrow u_0] \\
 (3.33) \quad &\times \mathbb{E}_1 I[E'(v_0, u_1; \tilde{C}_0)] \cdots \mathbb{E}_{N-1} I[E'(v_{N-2}, u_{N-1}; \tilde{C}_{N-2})] \\
 &\times \mathbb{E}_N I[E'(v_{N-1}, y; \tilde{C}_{N-1})],
 \end{aligned}$$

$$\begin{aligned}
 R_M(x, y) &= (-1)^{M+1} \sum_{(u_0,v_0)} \cdots \sum_{(u_M,v_M)} \left[\prod_{i=0}^M J(u_i, v_i) \right] \mathbb{E}_0 I[x \Leftrightarrow u_0] \\
 (3.34) \quad &\times \mathbb{E}_1 I[E'(v_0, u_1; \tilde{C}_0)] \cdots \mathbb{E}_{M-1} I[E'(v_{M-2}, u_{M-1}; \tilde{C}_{M-2})] \\
 &\times \mathbb{E}_M [I[E'(v_{M-1}, u_M; \tilde{C}_{M-1})] \mathbb{P}_{M+1}(v_M \overset{\tilde{C}_M}{\longleftrightarrow} y)],
 \end{aligned}$$

where we have used the abbreviation $\tilde{C}_j = \tilde{C}_j^{(u_j,v_j)}(v_{j-1})$, with $v_{-1} = x$.

Since

$$(3.35) \quad \mathbb{P}_{M+1}(v_M \overset{\tilde{C}_M}{\longleftrightarrow} y) \leq \tau_p(v_M, y),$$

it follows from (3.33)–(3.34) that

$$(3.36) \quad |R_M(x, y)| \leq \sum_{u_M, v_M \in \mathbb{V}} \Pi^{(M)}(x, u_M) J(u_M, v_M) \tau_p(v_M, y).$$

4. Diagrammatic estimates for the lace expansion. In this section we prove bounds on $\Pi^{(N)}$. These bounds are summarized in Lemma 4.1. We refer to the methods of this section as diagrammatic estimates, as we use Feynman diagrams to provide a convenient representation for upper bounds on $\Pi^{(N)}$.

4.1. *The diagrams.* In this section we show how $\Pi^{(N)}$ of (3.33) can be bounded in terms of Feynman diagrams. Our approach here is essentially identical to what is done in [13], Section 2.2, apart from some notational differences, and we omit some details in the following. The results of this section apply to any graph $\mathbb{G} = (\mathbb{V}, \mathbb{B})$, finite or infinite, which need not be transitive nor regular. We do, however, assume that \mathbb{G} is connected, so that $\tau_p(x, y) > 0$ for all $x, y \in \mathbb{V}$.

Given increasing events E, F , we use the standard notation $E \circ F$ to denote the event that E and F occur disjointly. Roughly speaking, $E \circ F$ is the set of bond configurations for which there exist two disjoint sets of occupied bonds such that the first set guarantees the occurrence of E and the second guarantees the occurrence of F . The BK inequality asserts that $\mathbb{P}(E \circ F) \leq \mathbb{P}(E)\mathbb{P}(F)$, for increasing events E and F . (See [10], Section 2.3, for a proof and for a precise definition of $E \circ F$.)

Let $\mathbb{P}^{(N)}$ denote the product measure on $N + 1$ copies of percolation on \mathbb{G} . By Fubini's theorem and (3.33),

$$\begin{aligned}
 \Pi^{(N)}(x, y) &= \sum_{(u_0, v_0)} \cdots \sum_{(u_{N-1}, v_{N-1})} \left[\prod_{i=0}^{N-1} J(u_i, v_i) \right] \\
 (4.1) \quad &\times \mathbb{P}^{(N)} \left(\{x \leftrightarrow u_0\}_0 \cap \left(\bigcap_{i=1}^{N-1} E'(v_{i-1}, u_i; \tilde{C}_{i-1})_i \right) \right. \\
 &\quad \left. \cap E'(v_{N-1}, y; \tilde{C}_{N-1})_N \right),
 \end{aligned}$$

where, for an event F , we write F_i to denote that F occurs on graph i . To estimate $\Pi^{(N)}(x, y)$, it is convenient to define the events (for $N \geq 1$)

$$(4.2) \quad F_0(x, u_0, w_0, z_1) = \{x \leftrightarrow u_0\} \circ \{x \leftrightarrow w_0\} \circ \{w_0 \leftrightarrow u_0\} \circ \{w_0 \leftrightarrow z_1\},$$

$$\begin{aligned}
 (4.3) \quad &F'(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1}) \\
 &= \{v_{i-1} \leftrightarrow t_i\} \circ \{t_i \leftrightarrow z_i\} \circ \{t_i \leftrightarrow w_i\} \\
 &\quad \circ \{z_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow z_{i+1}\},
 \end{aligned}$$

$$\begin{aligned}
 (4.4) \quad &F''(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1}) \\
 &= \{v_{i-1} \leftrightarrow w_i\} \circ \{w_i \leftrightarrow t_i\} \circ \{t_i \leftrightarrow z_i\} \\
 &\quad \circ \{t_i \leftrightarrow u_i\} \circ \{z_i \leftrightarrow u_i\} \circ \{w_i \leftrightarrow z_{i+1}\},
 \end{aligned}$$

$$\begin{aligned}
 (4.5) \quad &F(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1}) \\
 &= F'(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1}) \\
 &\quad \cup F''(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1}),
 \end{aligned}$$

$$(4.6) \quad F_N(v_{N-1}, t_N, z_N, y) = \{v_{N-1} \leftrightarrow t_N\} \circ \{t_N \leftrightarrow z_N\} \circ \{t_N \leftrightarrow y\} \circ \{z_N \leftrightarrow y\}.$$

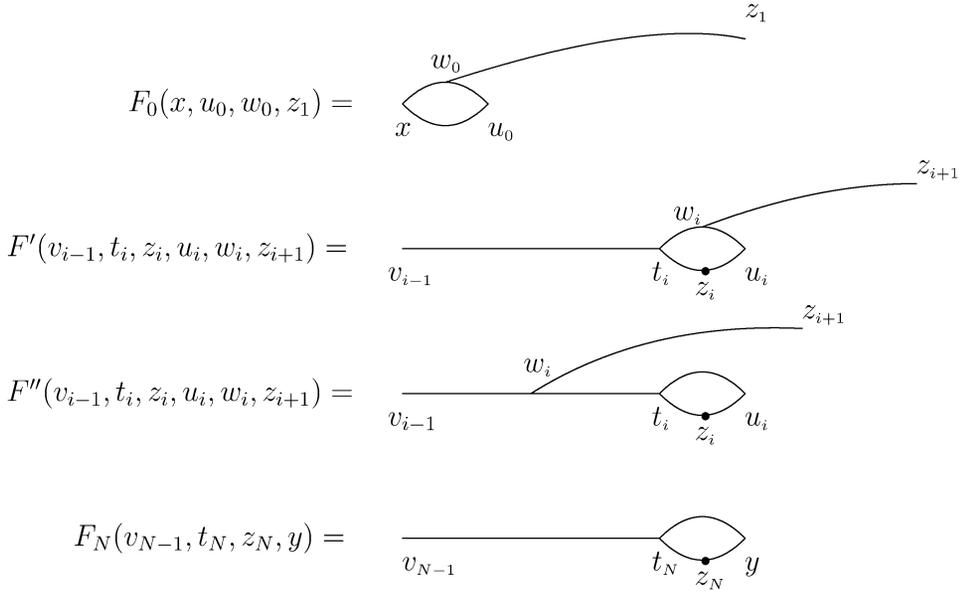


FIG. 4. Diagrammatic representations of the events $F_0(x, u_0, w_0, z_1)$, $F'(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})$, $F''(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})$, $F_N(v_{N-1}, t_N, z_N, y)$. Lines indicate disjoint connections.

The events F_0, F', F'', F_N are depicted in Figure 4. Note that

$$(4.7) \quad F_N(v, t, z, y) = F_0(y, z, t, v).$$

By the definition of E' in (3.17),

$$(4.8) \quad E'(v_{N-1}, y; \tilde{C}_{N-1})_N \subset \bigcup_{z_N \in \tilde{C}_{N-1}} \bigcup_{t_N \in \mathbb{V}} F_N(v_{N-1}, t_N, z_N, y)_N.$$

Indeed, viewing the connection from v_{N-1} to y as a string of sausages beginning at v_{N-1} and ending at y , for the event E' to occur, there must be a vertex $z_N \in \tilde{C}_{N-1}$ that lies on the last sausage, on a path from v_{N-1} to y . (In fact, both “sides” of the sausage must contain a vertex in \tilde{C}_{N-1} , but we do not need or use this.) This leads to (4.8), with t_N representing the other endpoint of the sausage that terminates at y .

Assume, for the moment, that $N \geq 2$. The condition in (4.8) that $z_N \in \tilde{C}_{N-1}$ is a condition on the graph $N - 1$ that must be satisfied in conjunction with the event $E'(v_{N-2}, u_{N-1}; \tilde{C}_{N-2})_{N-1}$. It is not difficult to see that, for $i \in \{1, \dots, N - 1\}$,

$$(4.9) \quad \begin{aligned} & E'(v_{i-1}, u_i; \tilde{C}_{i-1})_i \cap \{z_{i+1} \in \tilde{C}_i\} \\ & \subset \bigcup_{z_i \in \tilde{C}_{i-1}} \bigcup_{t_i, w_i \in \mathbb{V}} F(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})_i. \end{aligned}$$

See Figure 5 for a depiction of the inclusions in (4.8) and (4.9). Further details are given in [13], Lemma 2.5 or [22], Lemma 5.5.8.

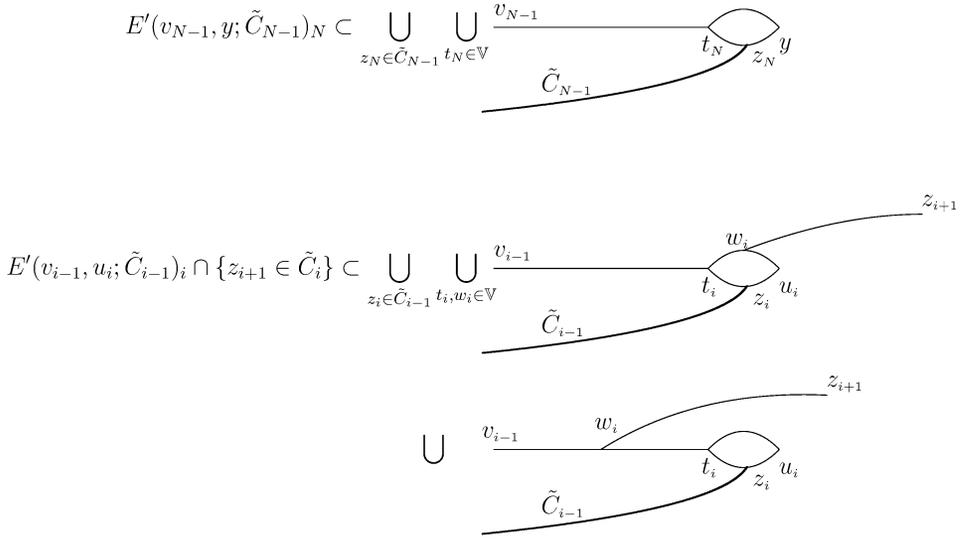


FIG. 5. Diagrammatic representations of the inclusions in (4.8) and (4.9).

With an appropriate treatment for $\{x \leftrightarrow u_0\}_0 \cap \{z_1 \in \tilde{C}_0\}$, (4.8) and (4.9) lead to

$$\begin{aligned}
 & \{x \leftrightarrow u_0\}_0 \cap \left(\bigcap_{i=1}^{N-1} E'(v_{i-1}, u_i; \tilde{C}_{i-1})_i \right) \cap E'(v_{N-1}, y; \tilde{C}_{N-1})_N \\
 (4.10) \quad & \subset \bigcup_{\vec{t}, \vec{w}, \vec{z}} \left(F_0(x, u_0, w_0, z_1)_0 \cap \left(\bigcap_{i=1}^{N-1} F(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})_i \right) \right. \\
 & \qquad \qquad \qquad \left. \cap F_N(v_{N-1}, t_N, z_N, y)_N \right),
 \end{aligned}$$

where $\vec{t} = (t_1, \dots, t_N)$, $\vec{w} = (w_0, \dots, w_{N-1})$ and $\vec{z} = (z_1, \dots, z_N)$. Therefore,

$$\begin{aligned}
 (4.11) \quad \Pi^{(N)}(x, y) & \leq \sum \left[\prod_{i=0}^{N-1} J(u_i, v_i) \right] \mathbb{P}_p(F_0(x, u_0, w_0, z_1)) \\
 & \quad \times \prod_{i=1}^{N-1} \mathbb{P}_p(F(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})) \\
 & \quad \times \mathbb{P}_p(F_N(v_{N-1}, t_N, z_N, y)),
 \end{aligned}$$

where the summation is over $z_1, \dots, z_N, t_1, \dots, t_N, w_0, \dots, w_{N-1}, u_0, \dots, u_{N-1}, v_0, \dots, v_{N-1}$. The probability in (4.11) factors because the events F_0, \dots, F_N are events on different percolation models. Each probability in (4.11) can be estimated using the BK inequality. The result is that each of the connections $\{a \leftrightarrow b\}$ present in the events F_0, F and F_N is replaced by a two-point function $\tau_p(a, b)$. This results in a large sum of two-point functions.

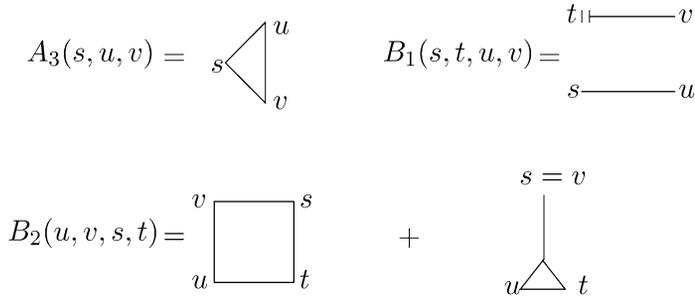


FIG. 6. Diagrammatic representations of $A_3(s, u, v)$, $B_1(s, t, u, v)$ and $B_2(u, v, s, t)$.

To organize a large sum of this form, we let

$$(4.12) \quad \begin{aligned} \tilde{\tau}_p(x, y) &= (J * \tau_p)(x, y) \\ &= p\Omega(D * \tau_p)(x, y) \quad \text{if } \mathbb{G} \text{ is regular,} \end{aligned}$$

and define

$$(4.13) \quad A_3(s, u, v) = \tau_p(s, v)\tau_p(s, u)\tau_p(u, v),$$

$$(4.14) \quad B_1(s, t, u, v) = \tilde{\tau}_p(t, v)\tau_p(s, u),$$

$$(4.15) \quad \begin{aligned} B_2(u, v, s, t) &= \tau_p(u, v)\tau_p(u, t)\tau_p(v, s)\tau_p(s, t) \\ &\quad + \sum_{a \in \mathbb{V}} \tau_p(s, a)\tau_p(a, u)\tau_p(a, t)\delta_{v,s}\tau_p(u, t). \end{aligned}$$

The two terms in B_2 arise from the two events F' and F'' in (4.5). We will write them as $B_2^{(1)}$ and $B_2^{(2)}$, respectively. The above quantities are represented diagrammatically in Figure 6. In the diagrams a line joining a and b represents $\tau_p(a, b)$. In addition, small bars are used to distinguish a line that represents $\tilde{\tau}_p$, as in B_1 .

Application of the BK inequality yields

$$(4.16) \quad \begin{aligned} &\mathbb{P}_p(F_0(x, u_0, w_0, z_1)) \\ &\leq A_3(x, u_0, w_0)\tau_p(w_0, z_1), \end{aligned}$$

$$(4.17) \quad \begin{aligned} &\sum_{v_{N-1}} J(u_{N-1}, v_{N-1})\mathbb{P}_p(F_N(v_{N-1}, t_N, z_N, y)) \\ &\leq \frac{B_1(w_{N-1}, u_{N-1}, z_N, t_N)}{\tau_p(w_{N-1}, z_N)} A_3(y, t_N, z_N). \end{aligned}$$

For F' and F'' , application of the BK inequality yields

$$\begin{aligned}
 (4.18) \quad & \sum_{v_{i-1}} J(u_{i-1}, v_{i-1}) \mathbb{P}_p(F'(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})) \\
 & \leq \frac{B_1(w_{i-1}, u_{i-1}, z_i, t_i)}{\tau_p(w_{i-1}, z_i)} B_2^{(1)}(z_i, t_i, w_i, u_i) \tau_p(w_i, z_{i+1}),
 \end{aligned}$$

$$\begin{aligned}
 (4.19) \quad & \sum_{v_{i-1}, t_i} J(u_{i-1}, v_{i-1}) \mathbb{P}_p(F''(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})) \\
 & \leq \frac{B_1(w_{i-1}, u_{i-1}, z_i, w_i)}{\tau_p(w_{i-1}, z_i)} B_2^{(2)}(z_i, w_i, w_i, u_i) \tau_p(w_i, z_{i+1}).
 \end{aligned}$$

Since the second and the third arguments of $B_2^{(2)}$ are equal by virtue of the Kronecker delta in (4.15), we can combine (4.18)–(4.19) to obtain

$$\begin{aligned}
 (4.20) \quad & \sum_{v_{i-1}, t_i} J(u_{i-1}, v_{i-1}) \mathbb{P}_p(F(v_{i-1}, t_i, z_i, u_i, w_i, z_{i+1})) \\
 & \leq \sum_{t_i} \frac{B_1(w_{i-1}, u_{i-1}, z_i, t_i)}{\tau_p(w_{i-1}, z_i)} B_2(z_i, t_i, w_i, u_i) \tau_p(w_i, z_{i+1}).
 \end{aligned}$$

Upon substitution of the bounds on the probabilities in (4.16), (4.17) and (4.20) into (4.11), the ratios of two-point functions form a telescoping product that disappears. After relabelling the summation indices, (4.11) becomes

$$\begin{aligned}
 (4.21) \quad & \Pi^{(N)}(x, y) \leq \sum_{\vec{s}, \vec{t}, \vec{u}, \vec{v}} A_3(x, s_1, t_1) \prod_{i=1}^{N-1} [B_1(s_i, t_i, u_i, v_i) B_2(u_i, v_i, s_{i+1}, t_{i+1})] \\
 & \quad \times B_1(s_N, t_N, u_N, v_N) A_3(u_N, v_N, y).
 \end{aligned}$$

The bound (4.21) is valid for $N \geq 1$, and the summation is over all $s_1, \dots, s_N, t_1, \dots, t_N, u_1, \dots, u_N, v_1, \dots, v_N$. For $N = 1, 2$, the right-hand side is represented diagrammatically in Figure 7. In the diagrams, unlabelled vertices are summed over \mathbb{V} .

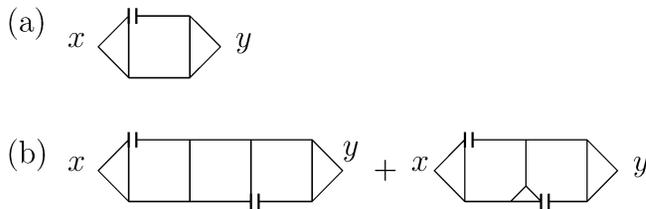


FIG. 7. The diagrams bounding (a) $\Pi^{(1)}(x, y)$ and (b) $\Pi^{(2)}(x, y)$.

4.2. *The diagrammatic bounds.* We now specialize to the case $\mathbb{G} = \mathbb{T}_{r,n}$, making use of the additive structure and the $x \mapsto -x$ symmetry of the torus. We will write $\tau_p(y - x)$ in place of $\tau_p(x, y)$, $p\Omega D(y - x)$ in place of $J(x, y)$, and $\Pi^{(N)}(y - x)$ in place of $\Pi^{(N)}(x, y)$.

The upper bounds we prove are in terms of various quantities related to the triangle diagram. Let

$$\begin{aligned}
 T_p(x) &= \sum_{y,z,u \in \mathbb{V}} \tau_p(y)\tau_p(z - y)p\Omega D(u)\tau_p(x + z - u) \\
 (4.22) \qquad &= (\tau_p * \tau_p * \tilde{\tau}_p)(x),
 \end{aligned}$$

$$(4.23) \qquad T_p = \max_{x \in \mathbb{V}} T_p(x),$$

$$\begin{aligned}
 T'_p &= \max_{x \in \mathbb{V}} \sum_{y,z \in \mathbb{V}} \tau_p(y)\tau_p(z - y)\tau_p(x - z) \\
 (4.24) \qquad &= \max_{x \in \mathbb{V}} (\tau_p * \tau_p * \tau_p)(x),
 \end{aligned}$$

and, for $k \in \mathbb{T}_{r,n}^*$, let

$$(4.25) \qquad W_p(y; k) = \sum_{x \in \mathbb{V}} [1 - \cos(k \cdot x)] \tilde{\tau}_p(x)\tau_p(x + y),$$

$$(4.26) \qquad W_p(k) = \max_{y \in \mathbb{V}} W_p(y; k).$$

Recall that $B_2^{(2)}$ denotes the second term of (4.15). For $k \in \mathbb{T}_{r,n}^*$, we also define

$$\begin{aligned}
 H_p(a_1, a_2; k) &= \sum_{u,v,s,t} [1 - \cos(k \cdot (t - u))] B_1(0, a_1, u, s) \\
 (4.27) \qquad &\qquad \qquad \times B_2^{(2)}(u, s, s, t) B_1(s, t, v, v + a_2)
 \end{aligned}$$

and

$$(4.28) \qquad H_p(k) = \max_{a_1, a_2 \in \mathbb{V}} H_p(a_1, a_2; k).$$

The remainder of this section is devoted to the proof of the following proposition.

PROPOSITION 4.1. *For $N = 0$,*

$$(4.29) \qquad \sum_{x \in \mathbb{V}} \Pi^{(0)}(x) \leq T_p,$$

$$(4.30) \qquad \sum_{x \in \mathbb{V}} [1 - \cos(k \cdot x)] \Pi^{(0)}(x) \leq W_p(0; k).$$

For $N \geq 1$,

$$(4.31) \quad \sum_{x \in \mathbb{V}} \Pi^{(N)}(x) \leq T_p'(2T_p T_p')^N,$$

$$(4.32) \quad \sum_{x \in \mathbb{V}} [1 - \cos(k \cdot x)] \Pi^{(N)}(x) \leq (4N + 3)[T_p' W_p(k)(2T_p + [1 + p\Omega]NT_p')(2T_p T_p')^{N-1} + (N - 1)(T_p^2 W_p(k) + H_p(k))(T_p')^2 (2T_p T_p')^{N-2}],$$

and, for $N = 1$, (4.32) can also be replaced by

$$(4.33) \quad \sum_{x \in \mathbb{V}} [1 - \cos(k \cdot x)] \Pi^{(1)}(x) \leq W_p(0; k) + 31T_p T_p' W_p(k).$$

4.2.1. *Proof of (4.29)–(4.30).* By (3.13) and the BK inequality,

$$(4.34) \quad \Pi^{(0)}(x) = \mathbb{P}(0 \leftrightarrow x) - \delta_{0,x} \leq \tau_p(x)^2 - \delta_{0,x}.$$

For $x \neq 0$, the event $\{0 \leftrightarrow x\}$ is the union over neighbors y of the origin of $\{\{0, y\} \text{ occupied}\} \circ \{y \leftrightarrow x\}$. Thus, by the BK inequality,

$$(4.35) \quad \tau_p(x) \leq p\Omega(D * \tau_p)(x) = \tilde{\tau}_p(x) \quad (x \neq 0).$$

Therefore,

$$(4.36) \quad \sum_{x \in \mathbb{V}} \Pi^{(0)}(x) \leq \sum_{x \in \mathbb{V}} \tau_p(x) \tilde{\tau}_p(x) \leq T_p(0).$$

Similarly,

$$(4.37) \quad \sum_{x \in \mathbb{V}} [1 - \cos(k \cdot x)] \Pi^{(0)}(x) \leq W_p(0; k).$$

This proves (4.29)–(4.30).

4.2.2. *Proof of (4.31).* For $N \geq 1$, let

$$(4.38) \quad \begin{aligned} &\Psi^{(N)}(s_{N+1}, t_{N+1}) \\ &= \sum_{\vec{s}, \vec{t}, \vec{u}, \vec{v}} A_3(0, s_1, t_1) \prod_{i=1}^N [B_1(s_i, t_i, u_i, v_i) B_2(u_i, v_i, s_{i+1}, t_{i+1})]. \end{aligned}$$

For convenience, we define $\Psi^{(0)}(x, y) = A_3(0, x, y)$, so that

$$(4.39) \quad \begin{aligned} \Psi^{(N)}(x, y) &= \sum_{u_N, v_N, s_N, t_N} \Psi^{(N-1)}(s_N, t_N) \\ &\quad \times B_1(s_N, t_N, u_N, v_N) B_2(u_N, v_N, x, y) \quad (N \geq 1). \end{aligned}$$

Since

$$(4.40) \quad \sum_x A_3(u_N, v_N, x) \leq \sum_{x,y} B_2(u_N, v_N, x, y),$$

it follows from (4.21) that

$$(4.41) \quad \sum_x \Pi^{(N)}(x) \leq \sum_{x,y} \Psi^{(N)}(x, y),$$

and bounds on $\Pi^{(N)}$ can be obtained from bounds on $\Psi^{(N)}$. We prove bounds on $\Psi^{(N)}$ and, hence, on $\Pi^{(N)}$, by induction on N .

The induction hypothesis is that

$$(4.42) \quad \sum_{x,y} \Psi^{(N)}(x, y) \leq T'_p(2T_p T'_p)^N.$$

For $N = 0$, (4.42) is true since

$$(4.43) \quad \sum_{x,y} A_3(0, x, y) \leq T'_p.$$

If we assume (4.42) is valid for $N - 1$, then by (4.39),

$$(4.44) \quad \begin{aligned} \sum_{x,y} \Psi^{(N)}(x, y) &\leq \left(\sum_{s_N, t_N} \Psi^{(N-1)}(s_N, t_N) \right) \\ &\times \left(\max_{s_N, t_N} \sum_{u_N, v_N, x, y} B_1(s_N, t_N, u_N, v_N) B_2(u_N, v_N, x, y) \right), \end{aligned}$$

and (4.42) then follows once we prove that

$$(4.45) \quad \max_{s,t} \sum_{u,v,x,y} B_1(s, t, u, v) B_2(u, v, x, y) \leq 2T_p T'_p.$$

It remains to prove (4.45). There are two terms, due to the two terms in (4.15), and we bound each term separately. The first term is bounded as

$$(4.46) \quad \begin{aligned} &\max_{s,t} \sum_{u,v,x,y} \tilde{\tau}_p(v-t) \tau_p(u-s) \tau_p(y-u) \tau_p(x-v) \tau_p(v-u) \tau_p(x-y) \\ &= \max_{s,t} \sum_{u,v} \tilde{\tau}_p(v-t) \tau_p(u-s) \tau_p(v-u) \\ &\times \left(\sum_{x,y} \tau_p(y-u) \tau_p(x-v) \tau_p(x-y) \right) \\ &\leq T'_p \max_{s,t} \sum_{u,v} \tilde{\tau}_p(v-t) \tau_p(u-s) \tau_p(v-u) \\ &= T_p T'_p. \end{aligned}$$

The second term is bounded similarly, making use of translation invariance, by

$$\begin{aligned}
 & \max_{s,t} \sum_{u,v,x,y,a} \tilde{\tau}_p(v-t)\tau_p(u-s) \\
 & \quad \times \delta_{v,x}\tau_p(y-u)\tau_p(x-a)\tau_p(u-a)\tau_p(y-a) \\
 & = \max_{s,t} \sum_{a,y,u} ((\tilde{\tau}_p * \tau)(a-t)\tau_p(u-s)) \\
 (4.47) \quad & \quad \times (\tau_p(y-u)\tau_p(u-a)\tau_p(y-a)) \\
 & = \max_{s,t} \sum_{y',a'} T_p(a'+s-t)\tau_p(y')\tau_p(a')\tau_p(y'-a') \\
 & \leq \left(\max_{a',s,t} T_p(a'+s-t) \right) \left(\sum_{y',a'} \tau_p(y')\tau_p(a')\tau_p(y'-a') \right) \\
 & \leq T_p T'_p,
 \end{aligned}$$

where $a' = a - u$, $y' = y - u$. This completes the proof of (4.45) and, hence, of (4.31).

4.2.3. *Proof of (4.32).* Next, we estimate $\sum_x [1 - \cos(k \cdot x)] \Pi^{(N)}(x)$. In a term in (4.21), there is a sequence of $2N + 1$ two-point functions along the “top” of the diagram, such that the sum of the displacements of these two-point functions is exactly equal to x . For example, in Figure 7(a) there are three displacements along the top of the diagram, and in Figure 7(b) there are five in the first diagram and four in the second. We regard the second diagram as also having five displacements, with the understanding that the third is constrained to vanish. With a similar general convention, each of the 2^{N-1} diagrams bounding $\Pi^{(N)}$ has $2N + 1$ displacements along the top of the diagram. We denote these displacements by d_1, \dots, d_{2N+1} , so that $x = \sum_{j=1}^{2N+1} d_j$. We will argue as follows to distribute the factor $1 - \cos(k \cdot x)$ among the displacements d_j .

Let $t = \sum_{j=1}^J t_j$. Taking the real part of the telescoping sum

$$(4.48) \quad 1 - e^{it} = \sum_{j=1}^J [1 - e^{it_j}] e^{i \sum_{m=1}^{j-1} t_m}$$

leads to the bound

$$(4.49) \quad 1 - \cos t \leq \sum_{j=1}^J [1 - \cos t_j] + \sum_{j=1}^J \sin t_j \sin \left(\sum_{m=1}^{j-1} t_m \right).$$

It follows from the identity $\sin(x + y) = \sin x \cos y + \cos x \sin y$ that $|\sin(x + y)| \leq |\sin x| + |\sin y|$. Applying this recursively gives

$$(4.50) \quad 1 - \cos t \leq \sum_{j=1}^J [1 - \cos t_j] + \sum_{j=1}^J \sum_{m=1}^{j-1} |\sin t_j| |\sin t_m|.$$

In the last term we use $|ab| \leq (a^2 + b^2)/2$, and then $1 - \cos^2 a \leq 2[1 - \cos a]$, to obtain

$$\begin{aligned}
 1 - \cos t &\leq \sum_{j=1}^J [1 - \cos t_j] + \frac{1}{2} \sum_{j=1}^J \sum_{m=1}^{j-1} [\sin^2 t_j + \sin^2 t_m] \\
 &\leq \sum_{j=1}^J [1 - \cos t_j] + J \sum_{j=1}^J \sin^2 t_j \\
 (4.51) \quad &= \sum_{j=1}^J [1 - \cos t_j] + J \sum_{j=1}^J [1 - \cos^2 t_j] \\
 &\leq (2J + 1) \sum_{j=1}^J [1 - \cos t_j].
 \end{aligned}$$

We apply (4.51) with $t = k \cdot x = \sum_{j=1}^{2N+1} k \cdot d_j$ to obtain a sum of $2N + 1$ diagrams like the ones for $\Pi^{(N)}(x)$, except now in the j th term, the j th line in the top of the diagram represents $[1 - \cos(k \cdot d_j)]\tau_p(d_j)$ rather than $\tau_p(d_j)$.

We distinguish three cases: (a) the displacement d_j is in a line of A_3 , (b) the displacement d_j is in a line of B_1 , (c) the displacement d_j is in a line of B_2 .

Case (a). The displacement is in a line of A_3 . We consider the case where the weight $[1 - \cos(k \cdot d_j)]$ falls on the last of the factors A_3 in (4.21). This contribution is equal to

$$\begin{aligned}
 (4.52) \quad &\sum_{u,v} \Psi^{(N-1)}(u, v) \sum_{w,x,y} B_1(u, v, w, y) \tau_p(y - w) \\
 &\quad \times [1 - \cos(k \cdot (x - y))] \tau_p(x - y) \tau_p(x - w).
 \end{aligned}$$

Applying (4.35) to $\tau_p(x - y)$, we have

$$\begin{aligned}
 (4.53) \quad &\max_{u,v} \sum_{w,x,y} B_1(u, v, w, y) \tau_p(y - w) \\
 &\quad \times [1 - \cos(k \cdot (x - y))] \tau_p(x - y) \tau_p(x - w) \\
 &\leq T_p W_p(k).
 \end{aligned}$$

It then follows from (4.42) that (4.52) is bounded above by $T'_p(2T_p T'_p)^{N-1} T_p \times W_p(k)$. By symmetry, the same bound applies when the weight falls into the first factor of A_3 , that is, when we have a factor $[1 - \cos(k \cdot d_1)]$. Thus, case (a) leads to an upper bound

$$(4.54) \quad 2T'_p(2T_p T'_p)^{N-1} T_p W_p(k).$$

Case (b). *The displacement is in a line of B_1 .* Suppose that the factor $[1 - \cos(k \cdot dj)]$ falls on the i th factor B_1 in (4.21). Depending on i , it falls either on $\tilde{\tau}_p$ or on τ_p in (4.14). We write the right-hand side of (4.21) with the extra factor as

$$(4.55) \quad \sum_x \sum_{s,t,u,v} \Psi^{(i-1)}(s, t) \tilde{B}_1(s, t, u, v) \bar{\Psi}^{(N-i)}(u - x, v - x).$$

In (4.55), either

$$(4.56) \quad \tilde{B}_1(s, t, u, v) = [1 - \cos(k \cdot (u - s))] \tilde{\tau}_p(u - s) \tau_p(v - t)$$

or

$$(4.57) \quad \tilde{B}_1(s, t, u, v) = \tilde{\tau}_p(u - s) [1 - \cos(k \cdot (v - t))] \tau_p(v - t),$$

and $\bar{\Psi}^{(N-i)}$ denotes a small variant of $\Psi^{(N-i)}$, defined inductively by $\bar{\Psi}^{(0)} = \Psi^{(0)}$ and $\bar{\Psi}^{(i)}(x, y) = \sum_{s,t,u,v} B_2(x, y, s, t) B_1(s, t, u, v) \bar{\Psi}^{(i-1)}(u, v)$. It can be verified that $\bar{\Psi}^{(N-i)}$ also obeys (4.42).

For (4.56), we let $a_1 = t - s$, $a_2 = v - u$ and $x' = u - x$. With this notation, the contribution to (4.55) due to (4.56) is bounded above by

$$(4.58) \quad \begin{aligned} & \left(\sum_{s,a_1} \Psi^{(i-1)}(s, s + a_1) \right) \left(\sum_{x',a_2} \bar{\Psi}^{(N-i)}(x', x' + a_2) \right) \\ & \times \left(\max_{s,a_1,a_2} \sum_u \tilde{B}_1(s, s + a_1, u, u + a_2) \right) \\ & = \left(\sum_{s,t} \Psi^{(i-1)}(s, t) \right) \left(\sum_{x,y} \bar{\Psi}^{(N-i)}(x, y) \right) W_p(k) \\ & \leq T'_p (2T_p T'_p)^{i-1} T'_p (2T_p T'_p)^{N-i} W_p(k) \\ & = T'_p (2T_p T'_p)^{N-1} T'_p W_p(k), \end{aligned}$$

where we used (4.42). For (4.57), we use (4.35) for $\tau_p(v - t)$, write $\tilde{\tau}_p(u - s) = \sum_y p\Omega D(y) \tau_p(u - s - y)$, estimate the sum over y with a supremum and use $\sum_y p\Omega D(y) = p\Omega$. Since there are N choices of factors B_1 , case (b) leads to an overall upper bound

$$(4.59) \quad N[1 + p\Omega] T'_p (2T_p T'_p)^{N-1} T'_p W_p(k).$$

Case (c). *The displacement is in a line of B_2 .* It is sufficient to estimate

$$(4.60) \quad \begin{aligned} & \max_{\substack{a,b,u,v \\ s,t,w,y,x}} \sum \Psi^{(i-1)}(a, b) \bar{\Psi}^{(N-i-1)}(w - x, y - x) [1 - \cos(k \cdot d)] \\ & \times B_1(a, b, u, v) B_2(u, v, s, t) B_1(s, t, w, y), \end{aligned}$$

where the maximum is over the choices $d = s - v$ or $d = t - u$. We consider separately the contributions due to $B_2^{(1)}$ and $B_2^{(2)}$ of (4.15), beginning with $B_2^{(2)}$.

Recall the definition of $H(a_1, a_2; k)$ in (4.27). The contribution to (4.60) due to $B_2^{(2)}$ can be rewritten, using $x' = w - x, a_2 = y - w, a_1 = b - a$, as

$$\begin{aligned}
 & \sum_{a, a_1, a_2, x'} \Psi^{(i-1)}(a, a + a_1) \bar{\Psi}^{(N-i-1)}(x', x' + a_2) H(a_1, a_2; k) \\
 (4.61) \quad & \leq H_p(k) \left(\sum_{x, y} \Psi^{(i-1)}(x, y) \right) \left(\sum_{x, y} \bar{\Psi}^{(N-i-1)}(x, y) \right) \\
 & \leq H_p(k) (T'_p)^2 (2T_p T'_p)^{N-2}.
 \end{aligned}$$

Since there are $N - 1$ factors B_2 to choose, this contribution to case (c) contributes at most

$$(4.62) \quad (N - 1) H_p(k) (T'_p)^2 (2T_p T'_p)^{N-2}.$$

It is not difficult to check that the contribution to case (c) due to $B_2^{(1)}$ is at most

$$(4.63) \quad (N - 1) (T_p^2 W_p(k)) (T'_p)^2 (2T_p T'_p)^{N-2}.$$

The desired estimate (4.32) then follows from (4.51), (4.54), (4.59) and (4.62)–(4.63).

4.2.4. *Proof of (4.33).* Recall from (4.21) that

$$(4.64) \quad \Pi_p^{(1)}(x) \leq \sum_{s, t, u, v \in \mathbb{V}} A_3(0, s, t) B_1(s, t, u, v) A_3(u, v, x).$$

We define $A'_3(u, v, x)$ by

$$(4.65) \quad A'_3(u, v, x) = A_3(u, v, x) - \delta_{u, x} \delta_{v, x}.$$

Then we have

$$\begin{aligned}
 & \sum_{x \in \mathbb{V}} [1 - \cos(k \cdot x)] \Pi_p^{(1)}(x) \\
 (4.66) \quad & \leq \sum_{x \in \mathbb{V}} [1 - \cos(k \cdot x)] B_1(0, 0, x, x) \\
 & \quad + \sum_{x, s, t, u, v \in \mathbb{V}} [1 - \cos(k \cdot x)] A'_3(0, s, t) B_1(s, t, u, v) A_3(u, v, x) \\
 & \quad + \sum_{x, u, v \in \mathbb{V}} [1 - \cos(k \cdot x)] B_1(0, 0, u, v) A'_3(u, v, x).
 \end{aligned}$$

The first term equals $W_p(0; k)$. The second and third terms are bounded above by $7 \cdot 3T_p T'_p W_p(k)$ and $5 \cdot 2T_p W_p(k) \leq 10T_p T'_p W_p(k)$, respectively, using (4.51) (with $J = 3$ and $J = 2$) and the methods of Section 4.2.3.

This completes the proof of Proposition 4.1. \square

5. Analysis of the lace expansion. In this section we use the lace expansion to prove the triangle condition of Theorem 1.3. The analysis is similar in spirit to the analysis of [13], but it has been simplified and reorganized, and it differs significantly in detail from the presentation of [13]. Specific improvements include the following: (i) We have reduced the number of functions in the bootstrap argument from five to three (cf. [13], Proposition 4.3), and in the bootstrap we work directly with the Fourier transform of the two-point function rather than with the triangle and related diagrams. (ii) We work with $1 - \cos(k \cdot x)$ directly, rather than expanding the cosine to second order. (iii) Our treatment of $H_p(k)$ in Lemma 5.7 below is simpler than the corresponding treatment of [13], Section 4.4.3(e).

We work in this section on an arbitrary torus $\mathbb{T}_{r,n}$ with $r \geq 2$, assuming that Assumption 1.1 is satisfied. As usual, we write the degree of the torus as Ω , and we abbreviate $p_c(\mathbb{T}_{r,n})$ to p_c .

Our analysis actually uses a slightly weaker assumption than the one stated in Assumption 1.1. Instead of (1.23), we will assume in the proof that

$$(5.1) \quad \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^* : k \neq 0} \frac{\hat{D}(k)^2}{[1 - \mu \hat{D}(k)]^3} \leq \beta$$

holds uniformly in $\mu \in [0, 1 - \frac{1}{2}\lambda^{-1}V^{-1/3}]$. Equation (5.1) is strictly weaker than (1.23), but not in a significant way. The analogue of (2.37) with μ inserted in the denominator follows from (5.1) in the same way that (2.37) follows from (1.23).

5.1. *The bootstrap argument.* Taking the Fourier transform of (3.9) and solving for $\hat{\tau}_p(k)$ gives

$$(5.2) \quad \hat{\tau}_p(k) = \frac{1 + \hat{\Pi}_M(k) + \hat{R}_M(k)}{1 - p\Omega \hat{D}(k)[1 + \hat{\Pi}_M(k)]},$$

for all $k \in \mathbb{T}_{r,n}^*$ and all $M = 0, 1, 2, \dots$. Recall from (2.4) that $\hat{C}_\mu(k) = [1 - \mu\Omega \hat{D}(k)]^{-1}$. As explained in Section 3.1, we would like to compare $\hat{\tau}_p(k)$ with $\hat{C}_\mu(k)$, with $\mu\Omega$ equal to $p\Omega[1 + \hat{\Pi}_M(0)]$. We know that $\hat{\tau}_p(0) = \chi(p) > 0$, but we do not yet know that $1 + \hat{\Pi}_M(0) + \hat{R}_M(0)$ is positive and, thus, we cannot yet be sure that the denominator of (5.2) is positive when $k = 0$. We therefore do not yet know that our choice of μ is less than Ω^{-1} . To safeguard against the possibility that $p\Omega[1 + \hat{\Pi}_M(0)] \geq 1$ or $p\Omega[1 + \hat{\Pi}_M(0)] < 0$, we define $\mu_p^{(M)}$ by

$$(5.3) \quad \mu_p^{(M)}\Omega = \min\{1 - \frac{1}{2}\lambda^{-1}V^{-1/3}, p\Omega[1 + \hat{\Pi}_M(0)]^+\},$$

where $x^+ = \max\{x, 0\}$. Later we will see that, in fact, $\mu_p^{(M)}\Omega = p\Omega[1 + \hat{\Pi}_M(0)]$. We will prove that, for all M sufficiently large (depending on p), $\lambda^3 \vee \beta$ sufficiently

small, and for all $p \leq p_c$,

$$(5.4) \quad \max_{k \in \mathbb{T}_{r,n}^*} \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)} \leq 3.$$

In fact, we will prove that the right-hand side of (5.4) can be replaced by $1 + c(\lambda^3 \vee \beta)$, where c is a universal constant. Inequality (5.4) is the key ingredient in the proof of Theorem 1.3.

The proof of (5.4) is based on the following elementary lemma. The lemma states that, under an appropriate continuity assumption, if an inequality implies a stronger inequality, then, in fact, the stronger inequality must hold. This kind of bootstrap argument has been applied repeatedly in analyses of the lace expansion, and goes back to [25] in this context.

LEMMA 5.1 (The bootstrap). *Let f be a continuous function on the interval $[p_1, p_2]$, and assume that $f(p_1) \leq 3$. Suppose for each $p \in (p_1, p_2)$, that if $f(p) \leq 4$, then, in fact, $f(p) \leq 3$. Then $f(p) \leq 3$ for all $p \in [p_1, p_2]$.*

PROOF. By hypothesis, $f(p)$ cannot be strictly between 3 and 4 for any $p \in [p_1, p_2]$. Since $f(p_1) \leq 3$, it follows by continuity that $f(p) \leq 3$ for all $p \in [p_1, p_2]$. \square

We will apply Lemma 5.1 with $p_1 = 0$, $p_2 = p_c$, and

$$(5.5) \quad f(p) = \max\{f_1(p), f_2(p), f_3(p)\},$$

where

$$(5.6) \quad f_1(p) = p\Omega, \quad f_2(p) = \max_{k \in \mathbb{T}_{r,n}^*} \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)},$$

$$(5.7) \quad f_3(p) = \max_{\substack{k,l \in \mathbb{T}_{r,n}^* \\ k \neq 0}} \frac{\hat{C}_{1/\Omega}(k)}{8} \left| \hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l-k) + \hat{\tau}_p(l+k)) \right| \\ \times (\hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l) + \hat{C}_{\mu_p^{(M)}}(l)\hat{C}_{\mu_p^{(M)}}(l+k) \\ + \hat{C}_{\mu_p^{(M)}}(l-k)\hat{C}_{\mu_p^{(M)}}(l+k))^{-1}.$$

As we will see below in Section 5.2.2, the expression $\hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l-k) + \hat{\tau}_p(l+k))$ can be interpreted as $-\frac{1}{2}$ times a discrete Laplacian of $\hat{\tau}_p$. In addition, this expression is also the Fourier transform of $[1 - \cos(k \cdot x)]\tau_p(x)$, a quantity which appears implicitly in Proposition 4.1 and in the following bounds on Π , which play an essential role in completing the bootstrap argument. The proof of

Proposition 5.2 is deferred to Section 5.3.

PROPOSITION 5.2. *Let $M = 0, 1, 2, \dots$, and assume that Assumption 1.1 holds. If $f(p)$ of (5.5) obeys $f(p) \leq K$, then there are positive constants c'_K and $\beta_0 = \beta_0(K)$ such that, for $\lambda^3 \vee \beta \leq \beta_0$,*

$$(5.8) \quad \sum_{x \in \mathbb{T}_{r,n}} |\Pi_M(x)| \leq c'_K(\lambda^3 \vee \beta),$$

$$(5.9) \quad \sum_{x \in \mathbb{T}_{r,n}} [1 - \cos(k \cdot x)] |\Pi_M(x)| \leq c'_K(\lambda^3 \vee \beta)[1 - \hat{D}(k)],$$

and for M sufficiently large (depending on K and V),

$$(5.10) \quad \sum_{x \in \mathbb{T}_{r,n}} |R_M(x)| \leq (\lambda^3 \vee \beta),$$

$$(5.11) \quad \sum_{x \in \mathbb{T}_{r,n}} [1 - \cos(k \cdot x)] |R_M(x)| \leq (\lambda^3 \vee \beta)[1 - \hat{D}(k)].$$

5.2. *The bootstrap argument completed.* We now show that f of (5.5) obeys the assumptions of Lemma 5.1, with $p_1 = 0$ and $p_2 = p_c$.

To see that $f(0) \leq 3$, we note that $\hat{\tau}_0(k) = 1$, $\mu_0^{(M)} = 0$ and, hence, $\hat{C}_{\mu_0^{(M)}}(k) = 1$, so that $f_2(0) = 1$. Since $f_1(0) = f_3(0) = 0$, we have $f(0) = 1 < 3$.

Next, we verify the continuity of f . Continuity of f_1 is clear. For f_2 , since $\mathbb{T}_{r,n}$ is finite, it follows that $\hat{\tau}_p(k)$ is a polynomial in p and, hence, is continuous. Similarly, $\hat{\Pi}_M(0)$ is a polynomial in p . Therefore, $\mu_p^{(M)}$ is continuous in p and, hence, $\hat{C}_{\mu_p^{(M)}}(k)$ also is, since $\hat{C}_\mu(k)$ is continuous in μ . The numerator and denominator in the definition of f_2 are therefore both continuous. There is no division by zero, since the denominator is positive when $\mu_p^{(M)} < 1$, by (2.4). The maximum over k is a maximum over a finite set, so f_2 is continuous. Similarly, f_3 is continuous, and thus, f is continuous.

The remaining hypothesis of Lemma 5.1 is the substantial one, and requires the detailed information about Π_M and R_M provided by Proposition 5.2. We fix $p < p_c$ and prove that $f(p) \leq 4$ implies $f(p) \leq 3$. By the assumption that $f(p) \leq 4$, the hypotheses of Proposition 5.2 are satisfied with $K = 4$. Therefore, assuming that M is sufficiently large and that $\lambda^3 \vee \beta$ is sufficiently small, the bounds (5.8)–(5.11) hold, with c'_K replaced by c'_4 .

Let

$$(5.12) \quad \lambda_p^{(M)} \Omega = p \Omega [1 + \hat{\Pi}_M(0)].$$

We now show that $\lambda_p^{(M)} \Omega \in [0, 1 - \frac{1}{2} \lambda^{-1} V^{-1/3}]$, and, hence, $\mu_p^{(M)} = \lambda_p^{(M)}$. By (5.2) with $k = 0$,

$$(5.13) \quad \chi(p) [1 - \lambda_p^{(M)} \Omega] = 1 + \hat{\Pi}_M(0) + \hat{R}_M(0).$$

Therefore,

$$(5.14) \quad \begin{aligned} 1 - \lambda_p^{(M)}\Omega &\geq \chi^{-1}(p)[1 - |\hat{\Pi}_M(0)| - |\hat{R}_M(0)|] \\ &\geq \chi^{-1}(p)[1 - (c'_4 + 1)(\lambda^3 \vee \beta)]. \end{aligned}$$

Since $\chi(p) \leq \chi(p_c) = \lambda V^{1/3}$, for $\lambda^3 \vee \beta$ sufficiently small, it follows that

$$(5.15) \quad \lambda_p^{(M)}\Omega \leq 1 - \frac{1}{2}\lambda^{-1}V^{-1/3}.$$

In addition, when λ and β are sufficiently small,

$$(5.16) \quad \lambda_p^{(M)}\Omega = p\Omega[1 + \hat{\Pi}_M(0)] \geq p\Omega[1 - c'_4(\lambda^3 \vee \beta)] \geq 0.$$

This proves that $\mu_p^{(M)}\Omega = \lambda_p^{(M)}\Omega = p\Omega[1 + \hat{\Pi}_M(0)]$.

5.2.1. *The improved bounds on $f_1(p)$ and $f_2(p)$.* First, we improve the bound on $f_1(p)$. We have already shown in (5.15) that $\mu_p^{(M)}\Omega \leq 1$. Therefore, by (5.8),

$$(5.17) \quad f_1(p) = p\Omega = \frac{\mu_p^{(M)}\Omega}{1 + \hat{\Pi}_M(0)} \leq \frac{1}{1 - c'_4(\lambda^3 \vee \beta)}.$$

The right-hand side is less than 3, if λ and β are small enough.

To improve the bound on $f_2(p)$, we write (5.2) as $\hat{\tau} = \hat{N}/\hat{F}$, with

$$(5.18) \quad \hat{N}(k) = 1 + \hat{\Pi}_M(k) + \hat{R}_M(k), \quad \hat{F}(k) = 1 - p\Omega\hat{D}(k)[1 + \hat{\Pi}_M(k)].$$

This yields

$$(5.19) \quad \begin{aligned} \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)} &= \hat{N}(k) + \hat{\tau}_p(k)[1 - \mu_p^{(M)}\Omega\hat{D}(k) - \hat{F}(k)] \\ &= [1 + \hat{\Pi}_M(k) + \hat{R}_M(k)] + \hat{\tau}_p(k)p\Omega\hat{D}(k)[\hat{\Pi}_M(k) - \hat{\Pi}_M(0)]. \end{aligned}$$

By Proposition 5.2, and by our assumptions that $\hat{\tau}_p(k) \leq 4\hat{C}_{\mu_p^{(M)}}(k)$ and $p\Omega \leq 4$, it follows from (5.19) that

$$(5.20) \quad \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)} \leq 1 + (c'_4 + 1 + 4^2c'_4\hat{C}_{\mu_p^{(M)}}(k)[1 - \hat{D}(k)])(\lambda^3 \vee \beta).$$

Since

$$(5.21) \quad 0 \leq \hat{C}_{\mu_p^{(M)}}(k)[1 - \hat{D}(k)] = 1 + \frac{\mu_p^{(M)}\Omega - 1}{1 - \mu_p^{(M)}\Omega\hat{D}(k)}\hat{D}(k) \leq 2,$$

it follows from (5.20) that

$$(5.22) \quad f_2(p) = \max_{k \in \mathbb{T}_{r,n}^*} \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p^{(M)}}(k)} \leq 1 + (c'_4 + 1 + 32c'_4)(\lambda^3 \vee \beta).$$

This is less than 3, if $\lambda^3 \vee \beta$ is small enough.

5.2.2. *Preliminaries for $f_3(p)$.* Improving the bound on f_3 is more involved, and we first develop some useful preliminaries.

The expression $\hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l - k) + \hat{\tau}_p(l + k))$ in (5.7) is closely related to a discrete second derivative of $\hat{\tau}_p(l)$. In fact, given a function \hat{f} on $\mathbb{T}_{r,n}^*$ and $k, l \in \mathbb{T}_{r,n}^*$, let

$$(5.23) \quad \partial_k^+ \hat{f}(l) = \hat{f}(l + k) - \hat{f}(l),$$

$$(5.24) \quad \partial_k^- \hat{f}(l) = \hat{f}(l) - \hat{f}(l - k),$$

and $\Delta_k \hat{f}(l) = \partial_k^- \partial_k^+ \hat{f}(l)$. Then

$$(5.25) \quad -\frac{1}{2} \Delta_k \hat{f}(l) = \hat{f}(l) - \frac{1}{2}(\hat{f}(l + k) + \hat{f}(l - k)).$$

In particular, $-\frac{1}{2} \Delta_k \hat{\tau}_p(l)$ appears in the numerator of $f_3(p)$.

The following will be useful in computations involving Δ_k . Let g be a symmetric function on the torus, meaning $g(x) = g(-x)$. Then the Fourier transform of g is actually the cosine series $\hat{g}(l) = \sum_x g(x) \cos(l \cdot x)$. We define

$$(5.26) \quad \hat{g}^{\cos}(l, k) = \sum_x g(x) \cos(l \cdot x) \cos(k \cdot x) = \frac{1}{2}[\hat{g}(l - k) + \hat{g}(l + k)],$$

$$(5.27) \quad \hat{g}^{\sin}(l, k) = \sum_x g(x) \sin(l \cdot x) \sin(k \cdot x) = \frac{1}{2}[\hat{g}(l - k) - \hat{g}(l + k)].$$

Then

$$(5.28) \quad -\frac{1}{2} \Delta_k \hat{g}(l) = \hat{g}(l) - \hat{g}^{\cos}(l, k).$$

With this observation, the following lemma can be seen as a kind of chain rule for the discrete differentiation of \hat{G} .

LEMMA 5.3. *Suppose that $g(x) = g(-x)$, and let $\hat{G}(k) = [1 - \hat{g}(k)]^{-1}$. For all $k, l \in \mathbb{T}^*$,*

$$(5.29) \quad \begin{aligned} -\frac{1}{2} \Delta_k G(l) &= \frac{1}{2}[\hat{G}(l - k) + \hat{G}(l + k)]\hat{G}(l)[\hat{g}(l) - \hat{g}^{\cos}(l, k)] \\ &\quad - \hat{G}(l - k)\hat{G}(l)\hat{G}(l + k)\hat{g}^{\sin}(l, k)^2. \end{aligned}$$

PROOF. Let $\hat{g}_{\pm} = \hat{g}(l \pm k)$ and write $\hat{g} = \hat{g}(l)$. Direct computation using (5.25) gives

$$(5.30) \quad \begin{aligned} -\frac{1}{2} \Delta_k G(l) &= \frac{1}{2} \hat{G}(l)\hat{G}(l + k)\hat{G}(l - k) \\ &\quad \times [[2\hat{g} - \hat{g}_+ - \hat{g}_-] + [2\hat{g}_+\hat{g}_- - \hat{g}\hat{g}_- - \hat{g}\hat{g}_+]] \\ &= \hat{G}(l)\hat{G}(l + k)\hat{G}(l - k) \\ &\quad \times [[\hat{g}(l) - \hat{g}^{\cos}(l, k)] + [\hat{g}_+\hat{g}_- - \hat{g}(l)\hat{g}^{\cos}(l, k)]], \end{aligned}$$

using (5.26) in the last step. By definition, and using the identity $\cos(u + v) = \cos u \cos v - \sin u \sin v$,

$$\begin{aligned} \hat{g}_- \hat{g}_+ &= \sum_{x,y} g(x)g(y) \cos((l+k) \cdot x) \cos((l-k) \cdot y) \\ (5.31) \quad &= \hat{g}^{\cos}(l, k)^2 - \hat{g}^{\sin}(l, k)^2. \end{aligned}$$

Substitution in (5.30) gives (5.29). \square

Assume that $g(x) = g(-x)$. Then

$$(5.32) \quad \frac{1}{2} |\Delta_k \hat{g}(l)| = |\hat{g}(l) - \hat{g}^{\cos}(l, k)| \leq \sum_x [1 - \cos(k \cdot x)] |g(x)|.$$

Also, by the Cauchy–Schwarz inequality and the elementary estimate $1 - \cos^2 t \leq 2[1 - \cos t]$,

$$\begin{aligned} \hat{g}^{\sin}(k, l)^2 &\leq \sum_x \sin^2(k \cdot x) |g(x)| \sum_y \sin^2(l \cdot y) |g(y)| \\ (5.33) \quad &= \sum_x [1 - \cos^2(k \cdot x)] |g(x)| \sum_y [1 - \cos^2(l \cdot y)] |g(y)| \\ &\leq 4 \sum_x [1 - \cos(k \cdot x)] |g(x)| \sum_y [1 - \cos(l \cdot y)] |g(y)|. \end{aligned}$$

In addition,

$$\begin{aligned} |\partial_k^\pm \hat{g}(l)| &\leq \sum_x |\operatorname{Re}\{e^{il \cdot x} [e^{\pm ik \cdot x} - 1]\} g(x)| \\ (5.34) \quad &\leq \sum_x [1 - \cos(k \cdot x)] + |\sin(k \cdot x)| |\sin(l \cdot x)| |g(x)| \\ &\leq \sum_x [1 - \cos(k \cdot x)] |g(x)| \\ &\quad + \left\{ 4 \sum_x [1 - \cos(k \cdot x)] |g(x)| \sum_y [1 - \cos(l \cdot y)] |g(y)| \right\}^{1/2}, \end{aligned}$$

using the same technique as in (5.33) for the third inequality.

The definition on ∂_k^\pm leads to the quotient and product rules

$$(5.35) \quad \partial_k^+ \frac{\hat{b}(l)}{\hat{d}(l)} = \frac{\partial_k^+ \hat{b}(l)}{\hat{d}(l)} - \frac{\hat{b}(l+k) \partial_k^+ \hat{d}(l)}{\hat{d}(l) \hat{d}(l+k)},$$

$$(5.36) \quad \partial_k^- \frac{\hat{b}(l)}{\hat{d}(l)} = \frac{\partial_k^- \hat{b}(l)}{\hat{d}(l)} - \frac{\hat{b}(l-k) \partial_k^- \hat{d}(l)}{\hat{d}(l) \hat{d}(l-k)},$$

$$(5.37) \quad \partial_k^+ [\hat{f}(l) \hat{h}(l)] = \partial_k^+ \hat{f}(l) \hat{h}(l+k) + \hat{f}(l) \partial_k^+ \hat{h}(l),$$

$$(5.38) \quad \partial_k^- [\hat{f}(l) \hat{h}(l)] = \partial_k^- \hat{f}(l) \hat{h}(l) + \hat{f}(l-k) \partial_k^- \hat{h}(l).$$

This gives

$$\begin{aligned}
 (5.39) \quad -\frac{1}{2}\Delta_k \frac{\hat{b}(l)}{\hat{d}(l)} &= -\frac{1}{2}\partial_k^- \left\{ \frac{\partial_k^+ \hat{b}(l)}{\hat{d}(l)} - \frac{\hat{b}(l+k) \partial_k^+ \hat{d}(l)}{\hat{d}(l)\hat{d}(l+k)} \right\} \\
 &= \frac{-(1/2)\Delta_k \hat{b}(l)}{\hat{d}(l)} + \frac{1}{2} \frac{\partial_k^+ \hat{b}(l-k) \partial_k^- \hat{d}(l)}{\hat{d}(l)\hat{d}(l-k)} + \frac{1}{2} \frac{\partial_k^- \hat{b}(l+k) \partial_k^+ \hat{d}(l)}{\hat{d}(l)\hat{d}(l+k)} \\
 &\quad + \frac{1}{2} \frac{\hat{b}(l)\Delta_k \hat{d}(l)}{\hat{d}(l)\hat{d}(l+k)} - \frac{1}{2} \frac{\hat{b}(l) \partial_k^+ \hat{d}(l-k) \partial_k^- [\hat{d}(l)\hat{d}(l+k)]}{\hat{d}(l-k)\hat{d}(l)^2\hat{d}(l+k)}.
 \end{aligned}$$

5.2.3. *The improved bound on $f_3(p)$.* We now improve the bound on $f_3(p)$. We will write

$$(5.40) \quad A = 1 + \text{const}(\lambda^3 \vee \beta),$$

where the constant is universal and may change from line to line.

First, we recall the definitions of \hat{N} and \hat{F} in (5.18) and write $\hat{\tau}_p(l)$ as

$$(5.41) \quad \hat{\tau}_p(l) = \frac{\hat{N}(l)}{\hat{F}(l)} = \frac{1}{1 - \hat{g}(l)},$$

with

$$\begin{aligned}
 (5.42) \quad \hat{g}(l) &= 1 - \frac{\hat{F}(l)}{\hat{N}(l)} \\
 &= 1 - \frac{1}{\hat{N}(l)} \{1 - \mu_p^{(M)} \Omega \hat{D}(l) + p \Omega \hat{D}(l) [\hat{\Pi}_M(0) - \hat{\Pi}_M(l)]\}.
 \end{aligned}$$

By Proposition 5.2,

$$(5.43) \quad |\hat{N}(l) - 1| \leq (c'_4 + 1)(\lambda^3 \vee \beta).$$

In particular, $\hat{N}(l) > 0$. Since $\hat{\tau}_p(l) \geq 0$ (as proved in [3]), it follows that $\hat{F}(l) \geq 0$. Proposition 5.2, (5.17) and (5.21) then imply that

$$\begin{aligned}
 (5.44) \quad 0 \leq \hat{F}(l) &\leq [1 - \mu_p^{(M)} \Omega \hat{D}(l)] + Ac'_4(\lambda^3 \vee \beta)[1 - \hat{D}(l)] \\
 &\leq [1 + 2Ac'_4(\lambda^3 \vee \beta)][1 - \mu_p^{(M)} \Omega \hat{D}(l)].
 \end{aligned}$$

By (5.22), (5.29) implies that

$$\begin{aligned}
 (5.45) \quad \hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l+k) + \hat{\tau}_p(l-k)) &= -\frac{1}{2}\Delta_k \hat{\tau}_p(l) \\
 &\leq A \frac{1}{2} (\hat{C}_{\mu_p^{(M)}}(l-k) + \hat{C}_{\mu_p^{(M)}}(l+k)) \hat{C}_{\mu_p^{(M)}}(l) |\hat{g}(l) - \hat{g}^{\cos}(l, k)| \\
 &\quad + A \hat{C}_{\mu_p^{(M)}}(l-k) \hat{C}_{\mu_p^{(M)}}(l) \hat{C}_{\mu_p^{(M)}}(l+k) \hat{g}^{\sin}(l, k)^2.
 \end{aligned}$$

We will prove that

$$(5.46) \quad |\hat{g}^{\sin}(l, k)|^2 \leq 8A[1 - \hat{D}(k)] \frac{1}{\hat{C}_{\mu_p}^{(M)}(l)},$$

$$(5.47) \quad |\hat{g}(l) - \hat{g}^{\cos}(l, k)| \leq A[1 - \hat{D}(k)].$$

These inequalities imply that the right-hand side of (5.45) is bounded above by

$$(5.48) \quad A[1 - \hat{D}(k)] \left[\frac{1}{2} (\hat{C}_{\mu_p}^{(M)}(l - k) + \hat{C}_{\mu_p}^{(M)}(l + k)) \hat{C}_{\mu_p}^{(M)}(l) + 8\hat{C}_{\mu_p}^{(M)}(l - k) \hat{C}_{\mu_p}^{(M)}(l + k) \right].$$

Recalling that $\hat{C}_{1/\Omega}(k) = [1 - \hat{D}(k)]^{-1}$, this gives

$$(5.49) \quad f_3(p) \leq 1 + \text{const}(\lambda^3 \vee \beta),$$

so that, in particular, $f_3(p) \leq 3$.

To prove (5.46), we use (5.27) and (5.42) to see that

$$(5.50) \quad |\hat{g}^{\sin}(l, k)| \leq \left| \frac{\hat{F}^{\sin}(l, k)}{\hat{N}(l - k)} \right| + \left| \frac{\hat{F}(l + k) \hat{N}^{\sin}(l, k)}{\hat{N}(l - k) \hat{N}(l + k)} \right|.$$

By (5.43), the denominators are as close as desired to 1. To deal with the first term on the right-hand side of (5.50), we use (5.18) and (5.27) to obtain

$$(5.51) \quad \hat{F}^{\sin}(l, k) = -p\Omega [\hat{D}^{\sin}(l, k) [1 + \hat{\Pi}_M(l - k)] + \hat{D}(l + k) \hat{\Pi}_M^{\sin}(l, k)].$$

By (5.33),

$$(5.52) \quad |\hat{D}^{\sin}(k, l)| \leq \{4[1 - \hat{D}(k)][1 - \hat{D}(l)]\}^{1/2},$$

and $\hat{\Pi}_M^{\sin}(l, k)$ can be bounded similarly using Proposition 5.2. By (5.17) and Proposition 5.2, the first term on the right-hand side of (5.50) is at most

$$(5.53) \quad A\{4[1 - \hat{D}(k)][1 - \hat{D}(l)]\}^{1/2}.$$

The second term on the right-hand side of (5.50) can be bounded using the same method, noting from (5.18) that the factor $\hat{F}(l + k)$ is at most $1 + 2 \cdot 1 \cdot (1 + 1) = 5$. In addition, the factor $1 - \hat{D}(l)$ can be bounded above by $2\hat{C}_{\mu_p}^{(M)}(l)^{-1}$, by (5.21).

Therefore, as required,

$$(5.54) \quad \hat{g}^{\sin}(k, l)^2 \leq 8A[1 - \hat{D}(k)] \hat{C}_{\mu_p}^{(M)}(l)^{-1}.$$

Finally, we estimate $\hat{g}(l) - \hat{g}^{\cos}(l, k) = -\frac{1}{2}\Delta_k \hat{g}(l)$ and prove (5.47). By (5.42) and (5.39),

$$(5.55) \quad -\frac{1}{2}\Delta_k \hat{g}(l) = \frac{(1/2)\Delta_k \hat{F}(l)}{\hat{N}(l)} - \frac{1}{2} \frac{\partial_k^+ \hat{F}(l - k)}{\hat{N}(l)\hat{N}(l - k)} \partial_k^- \hat{N}(l) - \frac{1}{2} \frac{\partial_k^- \hat{F}(l)}{\hat{N}(l)\hat{N}(l + k)} \partial_k^+ \hat{N}(l) - \frac{1}{2} \frac{\hat{F}(l)\Delta_k \hat{N}(l)}{\hat{N}(l)\hat{N}(l + k)} + \frac{1}{2} \frac{\hat{F}(l) \partial_k^+ \hat{N}(l - k) \partial_k^- [\hat{N}(l)\hat{N}(l + k)]}{\hat{N}(l - k)\hat{N}(l)^2 \hat{N}(l + k)}.$$

The denominators are all as close to 1 as desired, by (5.43), and we need to estimate the numerators. The first term on the right-hand side of (5.55) is the main term. By (5.37)–(5.38), its numerator obeys

$$\begin{aligned}
 (5.56) \quad \left| \frac{1}{2} \Delta_k \hat{F}(l) \right| &\leq p\Omega \left| \frac{1}{2} \Delta_k \hat{D}(l) \right| [1 + \hat{\Pi}_M(l+k)] \\
 &\quad + \frac{1}{2} p\Omega |\partial_k^+ \hat{D}(l-k) \partial_k^- \hat{\Pi}_M(l+k)| \\
 &\quad + \frac{1}{2} p\Omega |\partial_k^- \hat{D}(l) \partial_k^+ \hat{\Pi}_M(l)| \\
 &\quad + p\Omega |\hat{D}(l-k) [\frac{1}{2} \Delta_k \hat{\Pi}_M(l)]|.
 \end{aligned}$$

We bound the factors $p\Omega$ by A . The factor $|\frac{1}{2} \Delta_k \hat{D}(l)|$ is bounded above by $1 - \hat{D}(k)$, by (5.32). The last term on the right-hand side of (5.56) is bounded by a small multiple of $1 - \hat{D}(k)$, by (5.32) and Proposition 5.2. For the cross terms, we use (5.34) to obtain

$$\begin{aligned}
 (5.57) \quad |\partial_k^\pm D(l)| &\leq [1 - \hat{D}(k)] + 2[1 - \hat{D}(k)]^{1/2} [1 - \hat{D}(l)]^{1/2} \\
 &\leq [1 - \hat{D}(k)] + 2^{3/2} [1 - \hat{D}(k)]^{1/2}.
 \end{aligned}$$

Applying Proposition 5.2, similar estimates apply to $\partial_k^\pm \hat{\Pi}_M$ and $\partial_k^\pm \hat{R}_M$, but with an extra constant multiple of $\lambda^3 \vee \beta$. The two cross terms in (5.56) are therefore bounded by a small multiple of $1 - \hat{D}(k)$. We have shown that the first term on the right-hand side of (5.55) is bounded above by $A[1 - \hat{D}(k)]$.

It is sufficient to show that the remaining terms in (5.55) are at most $[1 - \hat{D}(k)]$ times a multiple of $\lambda^3 \vee \beta$. The fourth term on the right-hand side of (5.55) obeys this bound, using (5.44) to bound $\hat{F}(l)$ by a constant, and (5.32) and Proposition 5.2 to bound $\Delta_k \hat{N}(l) = \Delta_k \hat{\Pi}_M(l) + \Delta_k \hat{R}_M(l)$ by $[1 - \hat{D}(k)]$ times a multiple of $\lambda^3 \vee \beta$.

The remaining three terms in (5.55) each contain a product of a derivative of \hat{F} with a derivative of \hat{N} , or a product of two derivatives of \hat{N} [using (5.38) for the last term]. Other factors of \hat{F} or \hat{N} are bounded by harmless constants. The above arguments imply that $\partial_k^\pm \hat{N}(l)$ is bounded by $\{[1 - \hat{D}(k)] + 2^{3/2} [1 - \hat{D}(k)]^{1/2}\}$ times a multiple of $\lambda^3 \vee \beta$, as in (5.57), but with a small factor. By the definition of \hat{F} in (5.18) and by the product rule (5.37), we have

$$(5.58) \quad \partial_k^+ \hat{F}(l) = -p\Omega \partial_k^+ \hat{D}(l) [1 + \hat{\Pi}_M(l+k)] - p\Omega \hat{D}(l) \partial_k^+ \hat{\Pi}_M(l),$$

which is bounded by a multiple of the right-hand side of (5.57) (with no small factor). The same bound is obeyed by $\partial_k^- \hat{F}(l)$. Although the derivative of \hat{F} does not produce a small factor, it is accompanied by a derivative of \hat{N} which does provide the desired factor $\lambda^3 \vee \beta$. Thus, each of the remaining three terms in (5.55) is at most $[1 - \hat{D}(k)]$ times a multiple of $\lambda^3 \vee \beta$.

This completes the proof that (5.55) is bounded above by $A[1 - \hat{D}(k)]$. Therefore, we have proved (5.49). In particular, we have obtained the improved bound $f_3(p) \leq 3$.

Throughout Section 5.2, we have relied on Proposition 5.2. We now prove this proposition.

5.3. *Proof of Proposition 5.2.* In this section we prove Proposition 5.2. The main ingredient is the following lemma.

LEMMA 5.4 (Bounds on the lace expansion). *Let $N = 0, 1, 2, \dots$, and assume that Assumption 1.1 holds. For each $K > 0$, there is a constant \bar{c}_K such that if $f(p)$ of (5.5) obeys $f(p) \leq K$, then*

$$(5.59) \quad \sum_{x \in \mathbb{T}_{r,n}} \Pi^{(N)}(x) \leq [\bar{c}_K(\lambda^3 \vee \beta)]^{N \vee 1}$$

and

$$(5.60) \quad \sum_{x \in \mathbb{T}_{r,n}} [1 - \cos(k \cdot x)] \Pi^{(N)}(x) \leq [1 - \hat{D}(k)] [\bar{c}_K(\lambda^3 \vee \beta)]^{(N-1) \vee 1}.$$

Before proving Lemma 5.4, we show that it implies Proposition 5.2.

PROOF OF PROPOSITION 5.2. The bounds (5.8)–(5.9) are immediate consequences of Lemma 5.4. The constant c'_K can be taken to be equal to $4\bar{c}_K$, where the factor 4 comes from summing the geometric series.

For the remainder term $R_M(x)$, we conclude from (3.36) that

$$(5.61) \quad |R_M(x)| \leq K \sum_{u,v} \Pi^{(M)}(u) D(v-u) \tau_p(x-v)$$

and, hence, (5.10) is bounded above by $K \hat{\Pi}^{(M)}(0) \chi(p) \leq K \lambda V^{1/3} \hat{\Pi}^{(M)}(0)$. This can be made less than $\lambda^3 \vee \beta$ by taking M sufficiently large, by Lemma 5.4. For (5.11), we apply (4.51) with $J = 3$ to obtain

$$(5.62) \quad \begin{aligned} & \sum_{x \in \mathbb{T}_{r,n}} [1 - \cos(k \cdot x)] |R_M(x)| \\ & \leq 7K [1 - \hat{D}(k)] \hat{\Pi}_p^{(M)}(0) \chi(p) \\ & \quad + 7K [\hat{\Pi}_p^{(M)}(0) - \hat{\Pi}_p^{(M)}(k)] \chi(p) \\ & \quad + 7K \hat{\Pi}_p^{(M)}(0) [\hat{\tau}_p(0) - \hat{\tau}_p(k)]. \end{aligned}$$

By Lemma 5.4, we can choose M large enough that $7K \hat{\Pi}_p^{(M)}(0) \chi(p) \leq \frac{1}{3}(\lambda^3 \vee \beta)$. The second term can be treated similarly. For the last term, we apply the bound $f_3(p) \leq K$ for $l = 0$ and use (5.3) to see that

$$(5.63) \quad \begin{aligned} |\hat{\tau}_p(0) - \hat{\tau}_p(k)| &= \frac{1}{2} |\Delta_k \hat{\tau}_p(0)| \\ &\leq 24K [1 - \hat{D}(k)] [1 - \mu_p^{(M)}]^{-2} \\ &\leq 24K [1 - \hat{D}(k)] 4\lambda^2 V^{2/3}. \end{aligned}$$

Finally, we again take M large and appeal to Lemma 5.4. \square

Lemma 5.4 will follow from Proposition 4.1 combined with the following three lemmas. For these three lemmas, we recall the quantities defined in (4.22)–(4.28) and also define

$$(5.64) \quad T_p^{(2)} = \frac{1}{V} \sum_{k \in \mathbb{T}_{r,n}^*} \hat{D}(k)^2 \hat{\tau}_p(k)^3.$$

LEMMA 5.5. *Fix $p \in (0, p_c)$, assume that $f(p)$ of (5.5) obeys $f(p) \leq K$, and assume that Assumption 1.1 holds. There is a constant c_K , independent of p , such that*

$$(5.65) \quad \begin{aligned} T_p^{(2)} &\leq c_K(\lambda^3 \vee \beta), \\ T_p &\leq c_K(\lambda^3 \vee \beta), \quad T'_p \leq 1 + c_K(\lambda^3 \vee \beta). \end{aligned}$$

The bound on $T_p^{(2)}$ also applies if $\hat{\tau}_p(k)^3$ is replaced by $\hat{\tau}_p(k)$ or $\hat{\tau}_p(k)^2$ in (5.64). In addition, λ^3 can be replaced by $V^{-1}\chi(p)^3$ in each of the above bounds.

PROOF. We begin with $T_p^{(2)}$. We extract the term due to $k = 0$ in (5.64) and use $f_2(p) \leq K$ to obtain

$$(5.66) \quad T_p^{(2)} \leq V^{-1}\chi(p)^3 + V^{-1} \sum_{k \neq 0} \hat{D}(k)^2 K^3 \hat{C}_{\mu_p^{(M)}}(k)^3.$$

The first term obeys $V^{-1}\chi(p)^3 \leq V^{-1}\chi(p_c)^3 = \lambda^3$, and the desired result follows from (5.1) and (5.5). The conclusion concerning replacement of $\hat{\tau}_p(k)^3$ by $\hat{\tau}_p(k)$ or $\hat{\tau}_p(k)^2$ can be obtained by going to x -space and using $\tau_p(x) \leq (\tau_p * \tau_p)(x) \leq (\tau_p * \tau_p * \tau_p)(x)$.

For T_p , we extract the term in (4.22) due to $y = z = 0$ and $u = x$, which is $p\Omega D(x) \leq K\beta$, using $f_1(p) \leq K$ and (1.22). This gives

$$(5.67) \quad \begin{aligned} T_p(x) &\leq K\beta + \sum_{u,y,z: (y,z-y,x+z-u) \neq (0,0,0)} \tau_p(y)\tau_p(z-y) \\ &\quad \times KD(u)\tau_p(x+z-u). \end{aligned}$$

Therefore, by (4.35),

$$(5.68) \quad T_p \leq K\beta + 3K^2 \max_x \sum_{y,z \in \mathbb{T}_{r,n}} \tau_p(y)(D * \tau_p)(z-y)(D * \tau_p)(x+z),$$

where the factor 3 comes from the 3 factors τ_p whose argument can differ from 0. In terms of the Fourier transform, this gives

$$(5.69) \quad T_p \leq K\beta + 3K^2 \max_x V^{-1} \sum_{k \in \mathbb{T}_{r,n}^*} \hat{D}(k)^2 \hat{\tau}_p(k)^3 e^{-ik \cdot x} \leq K\beta + 3K^2 T_p^{(2)}.$$

Our bound on $T_p^{(2)}$ then gives the desired estimate for T_p .

The bound on T'_p is a consequence of $T'_p \leq 1 + 3T_p$. Here the term 1 is due to the contribution to (4.24) with $y = z - y = x - z = 0$, so that $x = y = z = 0$. If at least one of $y, z - y, x - z$ is nonzero, then we can use (4.35) for the corresponding two-point function. \square

LEMMA 5.6. *Fix $p \in (0, p_c)$, assume that $f(p)$ of (5.5) obeys $f(p) \leq K$, and assume that Assumption 1.1 holds. There is a constant c_K , independent of p , such that*

$$(5.70) \quad W_p(0; k) \leq c_K[1 - \hat{D}(k)](\lambda^3 \vee \beta), \quad W_p(k) \leq c_K[1 - \hat{D}(k)].$$

PROOF. For the bound on $W_p(0; k)$, we use (4.35) to obtain

$$(5.71) \quad \begin{aligned} \tilde{\tau}_p(x) &= p\Omega D(x) + \sum_{v: v \neq x} p\Omega D(v)\tau(x - v) \\ &\leq p\Omega D(x) + [p\Omega]^2(D * D * \tau_p)(x). \end{aligned}$$

We insert (5.71) into the definition (4.25) of $W_p(0; k)$ to get

$$(5.72) \quad \begin{aligned} W_p(0; k) &\leq p\Omega \sum_x [1 - \cos(k \cdot x)]D(x)\tau_p(x) \\ &\quad + [p\Omega]^2 \sum_x [1 - \cos(k \cdot x)]\tau_p(x)(D * D * \tau_p)(x). \end{aligned}$$

We begin with the first term in (5.72), which receives no contribution from $x = 0$. Using (4.35) and (5.71) again, we obtain

$$(5.73) \quad \begin{aligned} &p\Omega \sum_{x \neq 0} [1 - \cos(k \cdot x)]D(x)\tau_p(x) \\ &\leq [p\Omega]^2 \sum_x [1 - \cos(k \cdot x)]D(x)^2 \\ &\quad + [p\Omega]^2 \sum_x [1 - \cos(k \cdot x)]D(x) \sum_{v \neq x} D(v)\tau_p(x - v) \\ &\leq [p\Omega]^2 \sum_x [1 - \cos(k \cdot x)]D(x)^2 \\ &\quad + [p\Omega]^3 \sum_x [1 - \cos(k \cdot x)]D(x)(D * D)(x) \\ &\quad + [p\Omega]^3 \sum_x [1 - \cos(k \cdot x)]D(x)(D * D * \tau_p)(x). \end{aligned}$$

The first term on the right-hand side is bounded by $K^2\beta[1 - \hat{D}(k)]$, by (1.22). The second term can be bounded similarly, using $\max_x(D * D)(x) \leq \beta$. For the last

term in (5.73), we use Parseval’s identity, together with the fact that the Fourier transform of $[1 - \cos(k \cdot x)]D(x)$ is $\hat{D}(l) - \hat{D}^{\cos}(k, l)$, to obtain

$$(5.74) \quad \begin{aligned} & \sum_x [1 - \cos(k \cdot x)]D(x)(D * D * \tau_p)(x) \\ &= \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} [\hat{D}(l) - \hat{D}^{\cos}(k, l)]\hat{D}(l)^2 \hat{\tau}_p(l). \end{aligned}$$

Applying (5.32) and the bound on $T^{(2)}$ [with $\hat{\tau}_p(k)^3$ replaced by $\hat{\tau}_p(k)$], this is bounded by

$$(5.75) \quad [1 - \hat{D}(k)] \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{D}(l)^2 \hat{\tau}_p(l) \leq c_K(\lambda^3 \vee \beta)[1 - \hat{D}(k)].$$

This completes the bound on the first term of (5.72).

For the second term in (5.72), we again use Parseval’s identity to obtain

$$(5.76) \quad \begin{aligned} & \sum_x [1 - \cos(k \cdot x)]\tau_p(x)(D * D * \tau_p)(x) \\ &= \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \left[\hat{\tau}_p(l) - \frac{1}{2}(\hat{\tau}_p(l+k) + \hat{\tau}_p(l-k)) \right] \hat{D}(l)^2 \hat{\tau}_p(l). \end{aligned}$$

By the assumed bounds on $f_2(p)$ and $f_3(p)$, this is at most

$$(5.77) \quad \begin{aligned} & 8K^2 [1 - \hat{D}(k)] \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{D}(l)^2 \hat{C}_{\mu_p^{(M)}}(l) \\ & \quad \times \left[\hat{C}_{\mu_p^{(M)}}(l-k) \hat{C}_{\mu_p^{(M)}}(l) + \hat{C}_{\mu_p^{(M)}}(l) \hat{C}_{\mu_p^{(M)}}(l+k) \right. \\ & \quad \left. + \hat{C}_{\mu_p^{(M)}}(l-k) \hat{C}_{\mu_p^{(M)}}(l+k) \right]. \end{aligned}$$

We set

$$(5.78) \quad C_{\mu,k}(x) = \cos(k \cdot x)C_\mu(x).$$

Then

$$(5.79) \quad |C_{\mu,k}(x)| \leq C_\mu(x),$$

and, recalling (5.26),

$$(5.80) \quad \hat{C}_{\mu,k}(l) = \sum_{x \in \mathbb{V}} \cos(k \cdot x) \cos(l \cdot x) C_\mu(x) = \hat{C}_\mu^{\cos}(l, k).$$

Also, by (5.31),

$$(5.81) \quad \hat{C}_\mu(l-k) \hat{C}_\mu(l+k) = \hat{C}_\mu^{\cos}(l, k)^2 - \hat{C}_\mu^{\sin}(l, k)^2 \leq \hat{C}_\mu^{\cos}(l, k)^2.$$

Therefore, using (5.79) and Parseval’s identity,

$$\begin{aligned}
 & \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{D}(l)^2 \hat{C}_{\mu_p^{(M)}}(l) \hat{C}_{\mu_p^{(M)}}(l-k) \hat{C}_{\mu_p^{(M)}}(l+k) \\
 & \leq \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{D}(l)^2 \hat{C}_{\mu_p^{(M)}}(l) \hat{C}_{\mu_p^{(M)}}^{\cos}(l, k)^2 \\
 (5.82) \quad & = (D * D * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)},k} * C_{\mu_p^{(M)},k})(0) \\
 & \leq (D * D * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}})(0).
 \end{aligned}$$

Moreover, by (5.1),

$$\begin{aligned}
 (D * D * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}})(0) &= \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{D}(l)^2 \hat{C}_{\mu_p^{(M)}}(l)^3 \\
 (5.83) \quad & \leq 8\lambda^3 + \beta,
 \end{aligned}$$

where the λ^3 arises from the $l = 0$ term together with the fact that $1 - \mu \geq \frac{1}{2}\lambda^{-1}V^{-1/3}$. This proves the desired bound on the last term in (5.77).

To bound the sum of the remaining terms in (5.77), we consider

$$(5.84) \quad \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{D}^2(l) \hat{C}_{\mu_p^{(M)}}(l)^2 [\hat{C}_{\mu_p^{(M)}}(l-k) + \hat{C}_{\mu_p^{(M)}}(l+k)].$$

Applying (5.26), (5.80), (5.79) and (5.1), (5.84) equals

$$\begin{aligned}
 & \frac{2}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{D}^2(l) \hat{C}_{\mu_p^{(M)}}(l)^2 \hat{C}_{\mu_p^{(M)}}^{\cos}(l, k) \\
 & = 2(D * D * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)},k})(0) \\
 (5.85) \quad & \leq 2(D * D * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}})(0) \\
 & \leq 2(8\lambda^3 + \beta).
 \end{aligned}$$

This completes the bound on the second term of (5.72), and thus the proof that $W_p(0; k) \leq c_K(\lambda^3 \vee \beta)[1 - \hat{D}(k)]$.

Finally, we estimate $W_p(k)$. Note that no factor $\lambda^3 \vee \beta$ appears in the desired bound. By (4.25)–(4.26),

$$(5.86) \quad W_p(k) = p\Omega \max_{y \in \mathbb{V}} \sum_{x, v \in \mathbb{V}} [1 - \cos(k \cdot x)] D(v) \tau_p(x-v) \tau_p(x+y).$$

Let

$$(5.87) \quad D_k(x) = [1 - \cos(k \cdot x)] D(x), \quad \tau_{p,k}(x) = [1 - \cos(k \cdot x)] \tau_p(x).$$

Applying (4.51) with $t = k \cdot v + k \cdot (x - v)$, we obtain

$$\begin{aligned}
 W_p(k) &\leq 5p\Omega \max_{y \in \mathbb{V}} \sum_{x, v \in \mathbb{V}} [1 - \cos(k \cdot v)] D(v) \tau_p(x - v) \tau_p(y - x) \\
 (5.88) \quad &+ 5p\Omega \max_{y \in \mathbb{V}} \sum_{x, v \in \mathbb{V}} D(v) [1 - \cos(k \cdot (x - v))] \tau_p(x - v) \tau_p(y - x) \\
 &\leq 5K \max_{y \in \mathbb{V}} (D_k * \tau_p * \tau_p)(y) + 5K \max_{y \in \mathbb{V}} (D * \tau_{p,k} * \tau_p)(y).
 \end{aligned}$$

For the first term, we have

$$\begin{aligned}
 (D_k * \tau_p * \tau_p)(y) &= \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} e^{-il \cdot y} \hat{D}_k(l) \hat{\tau}_p(l)^2 \\
 (5.89) \quad &\leq \frac{K^2}{V} \sum_{l \in \mathbb{T}_{r,n}^*} |\hat{D}_k(l)| \hat{C}_{\mu_p}^{(M)}(l)^2.
 \end{aligned}$$

It follows from (5.32) that, for all $k, l \in \mathbb{T}_{r,n}^*$,

$$(5.90) \quad |\hat{D}_k(l)| = |\hat{D}(l) - \hat{D}^{\cos}(k, l)| \leq [1 - \hat{D}(k)],$$

and, hence, by (2.37),

$$\begin{aligned}
 \max_{y \in \mathbb{V}} (D_k * \tau_p * \tau_p)(y) &\leq [1 - \hat{D}(k)] \frac{K^2}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{C}_{\mu_p}^{(M)}(l)^2 \\
 (5.91) \quad &\leq c_K (\lambda^3 \vee 1) [1 - \hat{D}(k)],
 \end{aligned}$$

where the λ^3 arises from the $l = 0$ term.

The remaining term to estimate in (5.88) is

$$(5.92) \quad \max_{y \in \mathbb{V}} (D * \tau_{p,k} * \tau_p)(y) = \max_{y \in \mathbb{V}} \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} e^{-il \cdot y} \hat{D}(l) \hat{\tau}_p(l) \hat{\tau}_{p,k}(l).$$

Since

$$(5.93) \quad \hat{\tau}_{p,k}(l) = \hat{\tau}_p(l) - \frac{1}{2} (\hat{\tau}_p(l+k) + \hat{\tau}_p(l-k)),$$

we can use the bounds on $f_2(p)$ and $f_3(p)$ to see that (5.92) is at most

$$\begin{aligned}
 8K^2 [1 - \hat{D}(k)] \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} |\hat{D}(l)| \hat{C}_{\mu_p}^{(M)}(l) \\
 \times [\hat{C}_{\mu_p}^{(M)}(l-k) \hat{C}_{\mu_p}^{(M)}(l) + \hat{C}_{\mu_p}^{(M)}(l) \hat{C}_{\mu_p}^{(M)}(l+k) \\
 + \hat{C}_{\mu_p}^{(M)}(l-k) \hat{C}_{\mu_p}^{(M)}(l+k)].
 \end{aligned}$$

The above sums can all be bounded using the methods employed for the previous term. For example, the last term can be estimated using $|\hat{D}(l)| \leq 1$, (5.81), (5.79) and (2.37), by

$$\begin{aligned}
 & \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{C}_{\mu_p^{(M)}}(l) \hat{C}_{\mu_p^{(M)}}(l-k) \hat{C}_{\mu_p^{(M)}}(l+k) \\
 & \leq \frac{1}{V} \sum_{l \in \mathbb{T}_{r,n}^*} \hat{C}_{\mu_p^{(M)}}(l) \hat{C}_{\mu_p^{(M)}}^{\cos}(l, k)^2 \\
 (5.94) \quad & = (C_{\mu_p^{(M)}} * C_{\mu_p^{(M)},k} * C_{\mu_p^{(M)},k})(0) \\
 & \leq (C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}} * C_{\mu_p^{(M)}})(0). \quad \square
 \end{aligned}$$

LEMMA 5.7. Fix $p \in (0, p_c)$, assume that $f(p)$ of (5.5) obeys $f(p) \leq K$, and assume that Assumption 1.1 holds. There is a constant c_K , independent of p , such that

$$(5.95) \quad H_p(k) \leq c_K (\lambda^3 \vee \beta) [1 - \hat{D}(k)].$$

PROOF. Recall the definition of $H_p(a_1, a_2; k)$ in (4.27). In terms of the Fourier transform, recalling (5.87),

$$\begin{aligned}
 (5.96) \quad H(a_1, a_2; k) &= \frac{1}{V^3} \sum_{l_1, l_2, l_3 \in \mathbb{T}_{r,n}^*} e^{-il_1 \cdot a_1} e^{-il_2 \cdot a_2} \hat{D}(l_1) \hat{\tau}_p(l_1)^2 \hat{D}(l_2) \\
 & \quad \times \hat{\tau}_p(l_2)^2 \hat{\tau}_{p,k}(l_3) \hat{\tau}_p(l_1 - l_2) \hat{\tau}_p(l_2 - l_3) \hat{\tau}_p(l_1 - l_3).
 \end{aligned}$$

We use $f(p) \leq K$ to replace $\hat{\tau}_p(k)$ by $K \hat{C}_{\mu_p^{(M)}}(k)$ and [recalling (5.93)] $\hat{\tau}_{p,k}(l_3)$ by

$$\begin{aligned}
 (5.97) \quad & 8K [1 - \hat{D}(k)] [\hat{C}_{\mu_p^{(M)}}(l_3 - k) \hat{C}_{\mu_p^{(M)}}(l_3) + \hat{C}_{\mu_p^{(M)}}(l_3) \hat{C}_{\mu_p^{(M)}}(l_3 + k) \\
 & \quad + \hat{C}_{\mu_p^{(M)}}(l_3 - k) \hat{C}_{\mu_p^{(M)}}(l_3 + k)].
 \end{aligned}$$

This gives an upper bound for (5.96) consisting of a sum of three terms.

The last of these terms can be bounded by

$$\begin{aligned}
 (5.98) \quad & 8K^8 [1 - \hat{D}(k)] \frac{1}{V^3} \sum_{l_1, l_2, l_3 \in \mathbb{T}_{r,n}^*} |\hat{D}(l_1)| \hat{C}_{\mu_p^{(M)}}(l_1)^2 |\hat{D}(l_2)| \hat{C}_{\mu_p^{(M)}}(l_2)^2 \\
 & \quad \times \hat{C}_{\mu_p^{(M)}}(l_3 - k) \hat{C}_{\mu_p^{(M)}}(l_3 + k) \hat{C}_{\mu_p^{(M)}}(l_1 - l_2) \\
 & \quad \times \hat{C}_{\mu_p^{(M)}}(l_2 - l_3) \hat{C}_{\mu_p^{(M)}}(l_1 - l_3).
 \end{aligned}$$

Using Hölder’s inequality with $p = 3$ and $q = 3/2$, (5.98) is bounded above by $8K^8$ times

$$\begin{aligned}
 (5.99) \quad & [1 - \hat{D}(k)] \left(\frac{1}{V^3} \sum_{l_1, l_2, l_3} |\hat{D}(l_1)|^{3/2} \hat{C}_{\mu_p^{(M)}}(l_1)^3 |\hat{D}(l_2)|^{3/2} \hat{C}_{\mu_p^{(M)}}(l_2)^3 \right. \\
 & \left. \times \hat{C}_{\mu_p^{(M)}}(l_3 + k)^{3/2} \hat{C}_{\mu_p^{(M)}}(l_1 - l_3)^{3/2} \right)^{2/3} \\
 & \times \left(\frac{1}{V^3} \sum_{l_1, l_2, l_3} \hat{C}_{\mu_p^{(M)}}(l_1 - l_2)^3 \hat{C}_{\mu_p^{(M)}}(l_2 - l_3)^3 \hat{C}_{\mu_p^{(M)}}(l_3 - k)^3 \right)^{1/3}.
 \end{aligned}$$

Let

$$(5.100) \quad S_p^{(\alpha)} = V^{-1} \sum_{l \in \mathbb{T}_{r,n}^*} |\hat{D}(l)|^\alpha \hat{C}_{\mu_p^{(M)}}(l)^3.$$

The Cauchy–Schwarz inequality implies that, for all k and l_1 ,

$$(5.101) \quad \frac{1}{V} \sum_{l_3} \hat{C}_{\mu_p^{(M)}}(l_3 + k)^{3/2} \hat{C}_{\mu_p^{(M)}}(l_1 - l_3)^{3/2} \leq S_p^{(0)}.$$

Therefore, (5.99) is bounded above by

$$(5.102) \quad [1 - \hat{D}(k)] (S_p^{(0)})^{5/3} (S_p^{(3/2)})^{4/3}.$$

To complete the proof, we note that, by Hölder’s inequality,

$$(5.103) \quad S_p^{(3/2)} \leq (S_p^{(2)})^{3/4} (S_p^{(0)})^{1/4}.$$

Thus, (5.102) is bounded above by $[1 - \hat{D}(k)] S_p^{(2)} (S_p^{(0)})^2$. The latter factor can be bounded using (2.37), and the former with (5.1). This gives a bound of the desired form, with the λ^3 arising as usual from the $l = 0$ term of $S_p^{(2)}$.

Routine bounds can be used to deal with the other two terms in a similar fashion. □

PROOF OF LEMMA 5.4. This is an immediate consequence of Proposition 4.1 and Lemmas 5.5–5.7. The bound (4.33) is used for (5.60) when $N = 1$ [as (4.32) is not sufficient]. □

5.4. The triangle condition. The hypotheses of Lemma 5.1 have all been verified, and we conclude from the lemma that $f(p) \leq 3$ for all $p \leq p_c$. Moreover, we have seen in (5.17), (5.22) and (5.49) that it follows from $f(p) \leq 4$ that, in fact,

$$(5.104) \quad f(p) \leq K_0 = 1 + \text{const}(\lambda^3 \vee \beta),$$

where the constant is universal. Therefore, (5.104) indeed holds for all $p \leq p_c$. In particular, the bounds of Proposition 5.2 and Lemmas 5.4–5.7 all hold, with K equal to the K_0 of (5.104).

PROOF OF THEOREM 1.3. By definition, $1 \leq \nabla_p(x, x) \leq T'_p$, and it was noted in the proof of Lemma 5.5 that $T'_p \leq 1 + 3T_p$. For $x \neq y$, it follows from (4.35) that

$$(5.105) \quad \nabla_p(x, y) = (\tau_p * \tau_p * \tau_p)(y - x) \leq 3T_p(y - x) \quad (x \neq y),$$

where the factor 3 arises since there are three factors τ_p that could have a nonzero argument and, hence, permit application of (4.35). Thus, it suffices to show that $3T_p \leq 10V^{-1}\chi(p)^3 + 13\beta$. By (5.66) and (5.69) with $K = K_0$ of (5.104),

$$(5.106) \quad T_p \leq K_0\beta + 3K_0^2 \frac{\chi(p)^3}{V} + 3K_0^5\beta,$$

and the desired result follows from the fact that K_0 can be taken to be as close as desired to 1 by taking $\lambda^3 \vee \beta$ sufficiently small. \square

6. Asymptotics for $\hat{\tau}_p(k)$. In this section we restrict attention to the torus $\mathbb{T}_{r,n}$ and assume that Assumption 1.1 holds with $\lambda^3 \vee \beta$ small. We will show that it is possible to extend (1.27) to an asymptotic formula for $\hat{\tau}_p(k)$ when $p \leq p_c$, for all $k \in \mathbb{T}_{r,n}^*$. This result is not used elsewhere in the paper.

The observation below (5.104) can be used in conjunction with (5.59) and (5.61) to see that $\lim_{M \rightarrow \infty} \sum_x |R_M(0, x)| = 0$, so that, by (5.2), we have

$$(6.1) \quad \hat{\tau}_p(k) = \frac{1 + \hat{\Pi}_p(k)}{1 - p\Omega\hat{D}(k)[1 + \hat{\Pi}_p(k)]},$$

where Π_p denotes $\Pi_{M=\infty}$. Similarly, the limit $M \rightarrow \infty$ can be taken in (5.19) to conclude that

$$(6.2) \quad \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p}(k)} - 1 = \hat{\Pi}_p(k) + \hat{\tau}_p(k)p\Omega\hat{D}(k)[\hat{\Pi}_p(k) - \hat{\Pi}_p(0)],$$

where $\mu_p = \mu_p^{(\infty)} = p[1 + \hat{\Pi}_p(0)]$.

THEOREM 6.1 (Asymptotics for the two-point function). *Suppose that Assumption 1.1 holds for percolation on $\mathbb{T}_{r,n}$, with $\lambda^3 \vee \beta$ sufficiently small. For $p \leq p_c$,*

$$(6.3) \quad \hat{\tau}_p(k) = (1 + O(\lambda^3 \vee \beta))\hat{C}_{m_p}(k) = \frac{1 + O(\lambda^3 \vee \beta)}{1 - m_p\Omega\hat{D}(k)},$$

where $m_p\Omega = 1 - \Omega(p_c - p) - \lambda^{-1}V^{-1/3}$ and the error term is uniform in $k \in \mathbb{T}_{r,n}^*$ and $p \leq p_c$.

PROOF. Let $\varepsilon = \Omega(p_c - p) \geq 0$. We first consider the case $k = 0$. The combination of [7], Theorem 1.2(i) and Theorem 1.3 implies that, for all $p \leq p_c$,

$$(6.4) \quad \frac{1}{\lambda^{-1}V^{-1/3} + \varepsilon} \leq \chi(p) \leq \frac{1}{\lambda^{-1}V^{-1/3} + [1 - O(\lambda^3 \vee \beta)]\varepsilon}.$$

This implies (6.3) for $k = 0$, and we therefore assume $k \neq 0$ henceforth.

Using (5.4), (5.17), Proposition 5.2 and (5.21), (6.2) leads to

$$(6.5) \quad \left| \frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p}(k)} - 1 \right| = O(\lambda^3 \vee \beta).$$

Since

$$(6.6) \quad \frac{\hat{\tau}_p(k)}{\hat{C}_{m_p}(k)} - 1 = \left(\frac{\hat{\tau}_p(k)}{\hat{C}_{\mu_p}(k)} - 1 \right) \frac{\hat{C}_{\mu_p}(k)}{\hat{C}_{m_p}(k)} + \left(\frac{\hat{C}_{\mu_p}(k)}{\hat{C}_{m_p}(k)} - 1 \right)$$

and since

$$(6.7) \quad \left| \frac{\hat{C}_{\mu_p}(k)}{\hat{C}_{m_p}(k)} - 1 \right| = \frac{|(\mu_p - m_p)\Omega\hat{D}(k)|}{1 - \mu_p\Omega\hat{D}(k)} \leq \frac{|\mu_p - m_p|\Omega}{1 - \mu_p\Omega},$$

it suffices to show that

$$(6.8) \quad \frac{|\mu_p - m_p|\Omega}{1 - \mu_p\Omega} = O(\lambda^3 \vee \beta).$$

But by definition and (6.1),

$$(6.9) \quad \mu_p\Omega = 1 - [1 + \hat{\Pi}_p(0)]\chi(p)^{-1}.$$

Also, by (6.4),

$$(6.10) \quad m_p\Omega = 1 - [1 + O(\lambda^3 \vee \beta)]\chi(p)^{-1}.$$

Therefore, as required,

$$(6.11) \quad \frac{|\mu_p - m_p|\Omega}{1 - \mu_p\Omega} = \frac{O(\lambda^3 \vee \beta)\chi(p)^{-1}}{[1 + \hat{\Pi}_p(0)]\chi(p)^{-1}} = O(\lambda^3 \vee \beta). \quad \square$$

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REFERENCES

[1] AIZENMAN, M. (1997). On the number of incipient spanning clusters. *Nuclear Phys. B [FS]* **485** 551–582.

- [2] AIZENMAN, M. and BARSKY, D. J. (1987). Sharpness of the phase transition in percolation models. *Comm. Math. Phys.* **108** 489–526.
- [3] AIZENMAN, M. and NEWMAN, C. M. (1984). Tree graph inequalities and critical behavior in percolation models. *J. Statist. Phys.* **36** 107–143.
- [4] ALON, N. and SPENCER, J. H. (2000). *The Probabilistic Method*, 2nd ed. Wiley, New York.
- [5] BARSKY, D. J. and AIZENMAN, M. (1991). Percolation critical exponents under the triangle condition. *Ann. Probab.* **19** 1520–1536.
- [6] BOLLOBÁS, B. (2001). *Random Graphs*, 2nd ed. Cambridge Univ. Press, Cambridge.
- [7] BORGS, C., CHAYES, J. T., VAN DER HOFSTAD, R., SLADE, G. and SPENCER, J. (2005). Random subgraphs of finite graphs: I. The scaling window under the triangle condition. *Random Structures Algorithms* **27** 137–184.
- [8] BORGS, C., CHAYES, J. T., VAN DER HOFSTAD, R., SLADE, G. and SPENCER, J. (2005). Random subgraphs of finite graphs: III. The phase transition for the n -cube. *Combinatorica*. To appear.
- [9] BRYDGES, D. C. and SPENCER, T. (1985). Self-avoiding walk in 5 or more dimensions. *Comm. Math. Phys.* **97** 125–148.
- [10] GRIMMETT, G. (1999). *Percolation*, 2nd ed. Springer, Berlin.
- [11] HARA, T. (2005). Decay of correlations in nearest-neighbour self-avoiding walk, percolation, lattice trees and animals. Preprint.
- [12] HARA, T., VAN DER HOFSTAD, R. and SLADE, G. (2003). Critical two-point functions and the lace expansion for spread-out high-dimensional percolation and related models. *Ann. Probab.* **31** 349–408.
- [13] HARA, T. and SLADE, G. (1990). Mean-field critical behaviour for percolation in high dimensions. *Comm. Math. Phys.* **128** 333–391.
- [14] HARA, T. and SLADE, G. (1994). Mean-field behaviour and the lace expansion. In *Probability and Phase Transition* (G. R. Grimmett, ed.) 87–122. Kluwer, Dordrecht.
- [15] HARA, T. and SLADE, G. (1995). The self-avoiding-walk and percolation critical points in high dimensions. *Combin. Probab. Comput.* **4** 197–215.
- [16] HARA, T. and SLADE, G. (2000). The scaling limit of the incipient infinite cluster in high-dimensional percolation. II. Integrated super-Brownian excursion. *J. Math. Phys.* **41** 1244–1293.
- [17] VAN DER HOFSTAD, R. and SLADE, G. (2002). A generalised inductive approach to the lace expansion. *Probab. Theory Related Fields* **122** 389–430.
- [18] VAN DER HOFSTAD, R. and SLADE, G. (2003). The lace expansion on a tree with application to networks of self-avoiding walks. *Adv. in Appl. Math.* **30** 471–528.
- [19] VAN DER HOFSTAD, R. and SLADE, G. (2005). Expansion in n^{-1} for percolation critical values on the n -cube and \mathbb{Z}^n : The first three terms. *Combin. Probab. Comput.* To appear.
- [20] VAN DER HOFSTAD, R. and SLADE, G. (2005). Asymptotic expansions in n^{-1} for percolation critical values on the n -cube and \mathbb{Z}^n . *Random Structures Algorithms*. To appear.
- [21] JANSON, S., ŁUCZAK, T. and RUCIŃSKI, A. (2000). *Random Graphs*. Wiley, New York.
- [22] MADRAS, N. and SLADE, G. (1993). *The Self-Avoiding Walk*. Birkhäuser, Boston.
- [23] MENSHIKOV, M. V. (1986). Coincidence of critical points in percolation problems. *Soviet Math. Dokl.* **33** 856–859.
- [24] NGUYEN, B. G. (1987). Gap exponents for percolation processes with triangle condition. *J. Statist. Phys.* **49** 235–243.
- [25] SLADE, G. (1987). The diffusion of self-avoiding random walk in high dimensions. *Comm. Math. Phys.* **110** 661–683.

- [26] SLADE, G. (1999). Lattice trees, percolation and super-Brownian motion. In *Perplexing Problems in Probability: Festschrift in Honor of Harry Kesten* (M. Bramson and R. Durrett, eds.) 35–51. Birkhäuser, Basel.

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