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LONG-TIME TAILS IN THE PARABOLIC ANDERSON MODEL WITH BOUNDED POTENTIAL

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Abstract: We consider the parabolic Anderson problem $\partial_t u = \kappa\Delta u + \xi u$ on $(0,\infty)\times\mathbb{Z}^d$ with random i.i.d. potential $\xi = (\xi(z))_{z\in\mathbb{Z}^d}$ and the initial condition $u(0,\cdot)\equiv 1$. Our main assumption is that $\operatorname{esssup}\xi(0) = 0$. In dependence of the thickness of the distribution $\operatorname{Prob}(\xi(0)\in\cdot)$ close to its essential supremum, we identify both the asymptotics of the moments of $u(t,0)$ and the almost-sure asymptotics of $u(t,0)$ as $t\to\infty$ in terms of variational problems. As a by-product, we establish Lifshitz tails for the random Schrödinger operator $-\kappa\Delta-\xi$ at the bottom of its spectrum. In our class of ξ-distributions, the Lifshitz exponent ranges from $\frac{d}{2}$ to ∞; the power law is typically accompanied by lower-order corrections.

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1. Introduction and statement of results

1.1 Model and assumptions

In this paper we study the so-called parabolic Anderson model, which is the Euclidean-time version of the Schrödinger equation with a random potential and a homogeneous initial condition. More precisely, consider the initial problem

$$\begin{aligned} \partial_t\, u(t,z) &= \kappa\Delta^{\mathrm{d}} u(t,z) + \xi(z)u(t,z), && (t,z)\in(0,\infty)\times\mathbb{Z}^d,\\ u(0,z) &= 1, && z\in\mathbb{Z}^d, \end{aligned} \tag{1.1}$$

where ∂_t is the time derivative, $u\colon [0,\infty)\times\mathbb{Z}^d\to[0,\infty)$ is a function, $\kappa>0$ is a diffusion constant, Δ^{d} is the discrete Laplacian $(\Delta^{\mathrm{d}} f)(z)=(2d)^{-1}\sum_{y\sim z}(f(y)-f(z))$, and $\xi=(\xi(z))_{z\in\mathbb{Z}^d}$ is a random i.i.d. potential. Let $\langle\,\cdot\,\rangle$ be the expectation with respect to ξ and let $\operatorname{Prob}(\cdot)$ be the corresponding probability measure. Our main subject of interest is the large-t behavior of the p-th moment $\langle u(t,0)^p\rangle$ for all $p>0$ and the almost-sure asymptotics of $u(t,0)$ as $t\to\infty$.

It is clear that these asymptotics are determined by the upper tails of the random variable $\xi(0)$. Our principal assumption is that the potential distribution is bounded from above. As then follows by applying a criterion derived in Gärtner and Molchanov [GM90], there is a unique non-negative solution to (1.1) for almost all ξ. Moreover, since $\xi(\cdot)\to\xi(\cdot)+a$ is compensated by $u(t,\cdot)\to e^{at}u(t,\cdot)$ in (1.1), we assume without loss of generality that $\xi(0)$ is a non-degenerate random variable with

$$\operatorname{esssup}\xi(0)=0. \tag{1.2}$$

Thus $\operatorname{Prob}(\xi(0)\le 0)=1$, and $\operatorname{Prob}(\xi(0)\ge-\varepsilon)>0$ for every $\varepsilon>0$. *A priori*, $\operatorname{Prob}(\xi(0)=-\infty)$ may be non-vanishing, but some restrictions to its size have to be imposed in order to have an interesting almost-sure asymptotics (see Theorem 1.5).

The main representative of the distributions we shall study is

$$\operatorname{Prob}\bigl(\xi(0)>-x\bigr)\approx\exp\Bigl\{-\text{const.}\, x^{-\frac{\gamma}{1-\gamma}}\Bigr\},\qquad x\downarrow 0, \tag{1.3}$$

where $\gamma\in[0,1)$ is a parameter. For $\gamma=0$ this class of distributions includes the discrete version of the so-called "Wiener sausage problem", which has extensively been studied (see, e.g., Donsker and Varadhan [DV79], Antal [A95], and Sznitman [S98]).

Our assumptions on the thickness of $\operatorname{Prob}(\xi(0)\in\cdot)$ at 0 will be described in terms of scaling properties of the cumulant generating function

$$H(\ell)=\log\langle e^{\ell\xi(0)}\rangle,\qquad \ell\ge 0. \tag{1.4}$$

Note that H is convex and, by (1.2), decreasing and strictly negative on $(0,\infty)$.

Scaling Assumption. *We assume that there is a non-decreasing function $t\mapsto\alpha_t\in(0,\infty)$ and a function $\widetilde H\colon[0,\infty)\to(-\infty,0]$, $\widetilde H\not\equiv 0$, such that*

$$\lim_{t\to\infty}\frac{\alpha_t^{d+2}}{t}H\left(\frac{t}{\alpha_t^d}\,y\right)=\widetilde H(y),\qquad y\ge 0, \tag{1.5}$$

uniformly on compact sets in $(0,\infty)$.

Note that finiteness and non-triviality of $\widetilde H$ necessitate that $t/\alpha_t^d\to\infty$ and $\alpha_t=O(t^{1/(d+2)})$. In the asymptotic sense, (1.5) and non-triviality of $\widetilde H$ determine the pair $(\alpha_t,\widetilde H)$ uniquely up to a constant multiple resp. scaling. Indeed, if $(\widehat{\alpha}_t, \widehat{H})$ is another pair satisfying the Scaling Assumption then, necessarily, $\widehat{\alpha}_t/\alpha_t \to c \neq 0, \infty$ and $\widehat{H}(\cdot) = c^{d+2}\widetilde{H}(\cdot/c^d)$. Moreover, if $t \mapsto \widehat{\alpha}_t$ is a positive function with $\widehat{\alpha}_t/\alpha_t \to 0$, then the limit in (1.5) equals $\widehat{H} \equiv 0$. Similarly, if $\widehat{\alpha}_t/\alpha_t \to \infty$, then $\widehat{H} \equiv -\infty$. These assertions follow directly from convexity of H (see also Subsection 3.2). For bounds on the growth of $t \mapsto \alpha_t$, see Proposition 1.1.

1.2 Main results

1.2.1 Power-law asymptotic scaling. Remarkably, our Scaling Assumption constrains the form of possible $\widetilde{H}$ to a two-parameter family and forces the scale function α_t to be regularly varying. The following claim is proved in Subsection 3.2.

Proposition 1.1 *Suppose* (1.2) *and the Scaling Assumption hold. Then*

$$\widetilde{H}(y) = \widetilde{H}(1)y^\gamma, \qquad y > 0, \tag{1.6}$$

for some $\gamma \in [0,1]$. *Moreover,*

$$\lim_{t\to\infty} \frac{\alpha_{pt}}{\alpha_t} = p^\nu \quad \text{for all } p > 0, \text{ and} \quad \lim_{t\to\infty} \frac{\log \alpha_t}{\log t} = \nu, \tag{1.7}$$

where

$$\nu = \frac{1-\gamma}{d+2-d\gamma} \in \big(0, \tfrac{1}{d+2}\big]. \tag{1.8}$$

This leads us to the following concept:

Definition. *Given a* $\gamma \in [0,1]$, *we say that* H *is* in the γ-class, *if there is a function* $t \mapsto \alpha_t$ *such that* (H, α_t) *satisfies the Scaling Assumption and the limiting* $\widetilde{H}$ *is homogeneous with exponent* γ, *as in* (1.6).

As is seen from (1.3), each value $\gamma \in [0,1)$ can be attained. Note that, despite the simplicity of possible $\widetilde{H}$, the richness of the class of all ξ-distributions persists in the scaling behavior of $\alpha_t = t^{\nu+o(1)}$. For instance, the case $\gamma = 0$ includes both the distributions with an atom at 0 and those with no atom but with a density ρ (w.r.t. the Lebesgue measure) having the asymptotic behavior $\rho(x) \sim (-x)^\sigma$ $(x \uparrow 0)$ for a $\sigma > -1$. It is easy to find that $\alpha_t = t^{\frac{1}{d+2}}$ [and $\widetilde{H}(1) = \log \mathrm{Prob}(\xi(0) = 0)$] in the first case while $\alpha_t = (t/\log t)^{\frac{1}{d+2}}$ in the second one. Yet thinner a tail has $\rho(x) \sim \exp(-\log^\tau |x|^{-1})$ with $\tau > 1$, for which $\alpha_t = (t/\log^\tau t)^{\frac{1}{d+2}}$.

Throughout the remainder of this paper, we restrict ourselves to the case $\gamma < 1$. The case $\gamma = 1$ is qualitatively different from that of $\gamma < 1$; for more explanation see Subsections 2.2 and 2.5.

1.2.2 Moment asymptotics. We proceed by describing the logarithmic asymptotics of the p-th moment of $u(0,t)$. First we introduce four classes of objects:

- *Function spaces*: Let

$$\mathcal{F} = \left\{ f \in C_{\mathrm{c}}(\mathbb{R}^d, [0,\infty)) \colon \|f\|_1 = 1 \right\}, \tag{1.9}$$

and for $R > 0$, let $\mathcal{F}_R$ be set of $f \in \mathcal{F}$ with support in $[-R,R]^d$. By $C^+(R)$ (resp. $C^-(R)$) we denote the set of continuous functions $[-R,R]^d \to [0,\infty)$ (resp. $[-R,R]^d \to (-\infty, 0]$). Note that functions in $\mathcal{F}_R$ vanish at the boundary of $[-R,R]^d$, while those in $C^\pm(R)$ may not.

• *Functionals*: Let $\mathcal{I}\colon \mathcal{F}\to[0,\infty]$ be the Donsker-Varadhan rate functional

$$\mathcal{I}(f)=\begin{cases}\kappa\big\|(-\Delta)^{\frac12}\sqrt{f}\big\|_2^2 & \text{if } \sqrt{f}\in\mathcal{D}\big((-\Delta)^{\frac12}\big),\\ \infty & \text{otherwise,}\end{cases}\tag{1.10}$$

where Δ is the Laplace operator on $L^2(\mathbb{R}^d)$ (defined as a self-adjoint extension of $\sum_i(\partial^2/\partial x_i^2)$ from, e.g., the Schwarz class on $\mathbb{R}^d$) and $\mathcal{D}((-\Delta)^{1/2})$ denotes the domain of its square root. Note that $\mathcal{I}(f)$ is nothing but the Dirichlet form of the Laplacian evaluated at $f^{1/2}$.

For $R>0$ we define the functional $\mathcal{H}_R\colon C^+(R)\to(-\infty,0]$ by putting

$$\mathcal{H}_R(f)=\int_{[-R,R]^d}\widetilde{H}\big(f(x)\big)\,dx.\tag{1.11}$$

Note that for H in the γ-class, $\mathcal{H}_R(f)=\widetilde{H}(1)\int f(x)^\gamma dx$, with the interpretation $\mathcal{H}_R(f)=\widetilde{H}(1)|\operatorname{supp} f|$ when $\gamma=0$. Here $|\cdot|$ denotes the Lebesgue measure.

• *Legendre transforms*: Let $\mathcal{L}_R\colon C^-(R)\to[0,\infty]$ be the Legendre transform of $\mathcal{H}_R$,

$$\mathcal{L}_R(\psi)=\sup\big\{(f,\psi)-\mathcal{H}_R(f)\colon f\in C^+(R),\,\operatorname{supp} f\subset\operatorname{supp}\psi\big\},\tag{1.12}$$

where we used the shorthand notation $(f,\psi)=\int f(x)\psi(x)\,dx$. If H is in the γ-class, we get $\mathcal{L}_R(\psi)=\text{const.}\int|\psi(x)|^{-\frac{\gamma}{1-\gamma}}\,dx$ for $\gamma\in(0,1)$ and $\mathcal{L}_R(\psi)=-\widetilde{H}(1)\,|\operatorname{supp}\psi|$ for $\gamma=0$.

For any potential $\psi\in C^-(R)$, we also need the principal (i.e., the largest) eigenvalue of the operator $\kappa\Delta+\psi$ on $L^2([-R,R]^d)$ with Dirichlet boundary conditions, expressed either as the Legendre transform of $\mathcal{I}$ or in terms of the Rayleigh-Ritz principle:

$$\begin{aligned}\lambda_R(\psi)&=\sup\big\{(f,\psi)-\mathcal{I}(f)\colon f\in\mathcal{F}_R,\,\operatorname{supp} f\subset\operatorname{supp}\psi\big\}\\&=\sup\big\{(\psi,g^2)-\kappa\|\nabla g\|_2^2\colon g\in C_{\mathrm{c}}^\infty(\operatorname{supp}\psi,\mathbb{R}),\,\|g\|_2=1\big\},\end{aligned}\tag{1.13}$$

with the interpretation $\lambda_R(0)=-\infty$.

• *Variational principles*: Here is the main quantity of this subsection:

$$\chi=\inf_{R>0}\inf\big\{\mathcal{I}(f)-\mathcal{H}_R(f)\colon f\in\mathcal{F}_R\big\}\tag{1.14}$$

$$=\inf_{R>0}\inf\big\{\mathcal{L}_R(\psi)-\lambda_R(\psi)\colon \psi\in C^-(R)\big\}.\tag{1.15}$$

where (1.15) is obtained from (1.14) by inserting (1.12) and the second line in (1.13). Note that χ depends on γ and the constant $\widetilde{H}(1)$.

The main result of this subsection is the following theorem; for the proof see Section 4.

Theorem 1.2 *Suppose* (1.2) *and our Scaling Assumption hold. Let H be in the γ-class for some $\gamma\in[0,1)$. Then $\chi\in(0,\infty)$ and, for every $p\in(0,\infty)$,*

$$\lim_{t\to\infty}\frac{\alpha_{pt}^2}{pt}\log\big\langle u(t,0)^p\big\rangle=-\chi.\tag{1.16}$$

Both (1.14) and (1.15) arise in well-known large-deviation statements: the former for an exponential functional of Brownian occupation times, the latter for the principal eigenvalue for a scaled version of the field ξ. Our proof addresses the first formula; an approach based on the second formula is heuristically explained in Subsection 2.1.1.

Formula (1.16), together with the results of Proposition 1.1, imply that

$$\lim_{t\to\infty}\frac{\alpha_t^2}{t}\log\frac{\langle u(t,0)^p\rangle^{1/p}}{\langle u(t,0)^q\rangle^{1/q}}=\chi\big(q^{-2\nu}-p^{-2\nu}\big),\qquad p,q\in(0,\infty),\tag{1.17}$$

whenever H is in the γ-class, where $\nu>0$ is as in (1.8). In particular, $\langle u(t,0)^p\rangle$ for $p>1$ decays much slower than $\langle u(t,0)\rangle^p$. This is the type of behavior typical for *intermittency* (for the definition and significance of this notion we refer to Gärtner and Molchanov [GM90] and the monograph of Carmona and Molchanov [CM94]).

1.2.3 Lifshitz tails. Based on Theorem 1.2, we can compute the asymptotics of so-called *integrated density of states* (IDS) of the operator $-\kappa\Delta^{\mathrm{d}}-\xi$ on the right-hand side of (1.1), at the bottom of its spectrum. Below we define the IDS and list some of its basic properties. For a comprehensive treatment and proofs we refer to the book by Carmona and Lacroix [CL90].

The IDS is defined as follows: Let $R>0$ and consider the operator $\mathfrak{H}_R=-\kappa\Delta^{\mathrm{d}}-\xi$ in $[-R,R]^d\cap\{x\in\mathbb{Z}^d\colon \xi(x)>-\infty\}$ with Dirichlet boundary conditions. Clearly, $\mathfrak{H}_R$ has a finite number of eigenvalues that we denote E_k, so it is meaningful to consider the quantity

$$N_R(E)=\#\{k\colon E_k\le E\},\qquad E\in\mathbb{R}.\tag{1.18}$$

The integrated density of states is then the limit

$$n(E)=\lim_{R\to\infty}\frac{N_R(E)}{(2R)^d},\tag{1.19}$$

giving $n(E)$ the interpretation as the number of energy levels below E per unit volume. The limit exists and is almost surely constant, as can be proved using e.g. subadditivity.

It is clear that $E\mapsto n(E)$ is monotone and that $n(E)=0$ for all $E<0$, provided (1.2) is assumed. In the 1960's, based on heuristic arguments, Lifshitz postulated that $n(E)$ behaves like $\exp(-\text{const.}\,E^{-\delta})$ as $E\downarrow 0$. This asymptotic form has been established rigorously in the cases treated by Donsker and Varadhan [DV79] and Sznitman [S98], with $\delta=d/2$. Here we generalize this result to our class of distributions with $\gamma<1$; however, in our cases the power-law is typically supplemented with a lower-order correction. The result can be concisely formulated in terms of the inverse function of $t\mapsto\alpha_t$:

Theorem 1.3 *Suppose* (1.2) *and the Scaling Assumption hold. Let H be in the γ-class for some $\gamma\in[0,1)$ and let α^{-1} be the inverse to the scaling function $t\mapsto\alpha_t$. Then*

$$\lim_{E\downarrow 0}\frac{\log n(E)}{E\alpha^{-1}(E^{-\frac12})}=-\frac{2\nu}{1-2\nu}\big[(1-2\nu)\chi\big]^{-\frac{1}{2\nu}}\tag{1.20}$$

where χ is as in (1.14) *and ν is defined in* (1.8).

Invoking (1.7), $E\alpha^{-1}(E^{-1/2})=E^{-1/\beta+o(1)}$ as $E\downarrow 0$, where

$$\beta=\frac{2}{d+2\frac{\gamma}{1-\gamma}}=\frac{2\nu}{1-2\nu}\in\big(0,\tfrac{2}{d}\big].\tag{1.21}$$

In particular, $1/\beta$ is the Lifshitz exponent. Theorem 1.3 is proved in Subsection 4.3.

1.2.4 Almost-sure asymptotics. The almost-sure behavior of $u(t,0)$ depends strongly on whether the origin belongs to a finite or infinite component of the set $\mathcal{C} = \{z \in \mathbb{Z}^d : \xi(z) > -\infty\}$. Indeed, if 0 is in a finite component of $\mathcal{C}$, then $u(t,0)$ decays exponentially with t. Thus, in order to get a non-trivial almost-sure behavior of $u(t,0)$ as $t \to \infty$, we need that $\mathcal{C}$ contains an infinite component $\mathcal{C}_\infty$ and that $0 \in \mathcal{C}_\infty$ occurs with a non-zero probability. In $d \geq 2$, this is guaranteed by requiring that $\mathrm{Prob}(\xi(0) > -\infty)$ exceed the percolation threshold $p_{\mathrm{c}}(d)$ for site percolation on $\mathbb{Z}^d$. In $d = 1$, $\mathcal{C}$ is percolating if and only if $\mathrm{Prob}(\xi(0) > -\infty) = 1$; sufficient "connectivity" can be ensured only under an extra condition on the *lower* tail of $\xi(0)$.

Suppose, without loss of generality, that $t \mapsto t/\alpha_t^2$ is strictly increasing (recall that $\alpha_t = t^{\nu + o(1)}$ with $\nu \leq 1/3$). Then we can define another scale function $t \mapsto b_t \in (0,\infty)$ by setting

$$\frac{b_t}{\alpha_{b_t}^2} = \log t, \qquad t > 0. \tag{1.22}$$

(In other words, b_t is the inverse function of $t \mapsto t/\alpha_t^2$ evaluated at $\log t$.) Let

$$\widetilde{\chi} = -\sup_{R>0} \sup \left\{ \lambda_R(\psi) \colon \psi \in C^-(R),\, \mathcal{L}_R(\psi) \leq d \right\}. \tag{1.23}$$

In our description of the almost sure asymptotics, the pair $(\alpha_{b_t}, \widetilde{\chi})$ will play a role analogous to the pair (α_t, χ) in Theorem 1.2. It is clear from Proposition 1.1 that

$$b_t = (\log t)^{\frac{1}{1-2\nu} + o(1)} \quad \text{and} \quad \alpha_{b_t}^2 = \big(\log t\big)^{\beta + o(1)}, \qquad t \to \infty, \tag{1.24}$$

where β is as in (1.21). It turns out that $\widetilde{\chi}$ can be computed from χ:

Proposition 1.4 *Suppose* (1.2) *and our Scaling Assumption hold. Let H be in the γ-class for some $\gamma \in [0,1)$. Then $\widetilde{\chi} \in (0,\infty)$ and*

$$\widetilde{\chi} = \chi^{\frac{1}{1-2\nu}} (1 - 2\nu) \left(\frac{2\nu}{d} \right)^{\beta}, \tag{1.25}$$

where χ and $\widetilde{\chi}$ are as in (1.14) *and* (1.23), *and $\nu = \frac{1-\gamma}{d+2-d\gamma}$.*

The proof of Proposition 1.4 is given in Subsection 3.3. In the special case $\gamma = 0$, the relation (1.25) can independently be verified by inserting the explicit expressions for χ and $\widetilde{\chi}$ derived e.g. in Sznitman [S98].

Our main result on the almost sure asymptotics reads as follows:

Theorem 1.5 *Suppose* (1.2) *and our Scaling Assumption hold. Let H be in the γ-class for some $\gamma \in [0,1)$. In $d \geq 2$, suppose that* $\mathrm{Prob}(\xi(0) > -\infty) > p_{\mathrm{c}}(d)$*; in $d = 1$, suppose* $\langle \log(-\xi(0) \vee 1) \rangle < \infty$. *Then*

$$\lim_{t \to \infty} \frac{\alpha_{b_t}^2}{t} \log u(t,0) = -\widetilde{\chi} \qquad \mathrm{Prob}(\,\cdot\,|0 \in \mathcal{C}_\infty)\text{-almost surely.} \tag{1.26}$$

The condition on the lower tail of the distribution of $\xi(0)$ in $d = 1$ is possibly not optimal. For more comments on this issue, see Subsection 2.5. Theorem 1.5 is proved in Section 5; for a heuristic derivation see Subsection 2.1.2.

2. Heuristics, literature remarks, and open problems

2.1 Heuristic derivation

The quantity $u(t,0)$ can be given a dynamical-system interpretation: Imagine a particle system on $\mathbb{Z}^d$ with particles performing independent (continuous-time) simple random walk in a landscape of "soft" traps. The power of each trap is described by the field ξ: particles at site z become trapped (killed) at rate $-\xi(z)$. For this particle system, $u(t,0)$ is the total expected number of particles located at the origin at time t, provided the initial configuration had exactly one particle at each lattice site.

It is clear from (1.2) that, by time t, the origin is not likely to be reached by any particle from regions having distance more than t from the origin. If $u_t(t,0)$ is the expected number of particles at the origin at time t under the constraint that no particle from outside the box $Q_t=[-t,t]^d\cap\mathbb{Z}^d$ has reached the origin, then this should imply that

$$u(t,0)\approx u_t(t,0). \tag{2.1}$$

The particle system in the box Q_t is driven by the operator $\kappa\Delta^{\mathrm{d}}+\xi$ on the right-hand side of (1.1) with zero boundary conditions in Q_t and the leading-order behavior of u_t should be governed by its principal (i.e., the largest) eigenvalue $\lambda_t^{\mathrm{d}}(\xi)$ in the sense that

$$u_t(t,0)\approx e^{t\lambda_t^{\mathrm{d}}(\xi)}. \tag{2.2}$$

Based on (2.2), we can give a plausible explanation of our Theorems 1.2 and 1.5.

2.1.1 Moment asymptotics. Under the expectation with respect to ξ, there is a possibility that $\langle u(t,0)\rangle$ will be dominated by a set of ξ's with exponentially small probability. But then the decisive contribution to the average particle-number at zero may come from much smaller a box than Q_t. Let $R\alpha_t$ denote the diameter of the purported box. Then we should have

$$\big\langle u_t(t,0)\big\rangle\approx\big\langle e^{t\lambda^{\mathrm{d}}_{R\alpha_t}}\big\rangle. \tag{2.3}$$

The proper choice of the scale function α_t is determined by balancing the gain in $\lambda^{\mathrm{d}}_{R\alpha_t}(\xi)$ and the loss due to taking ξ's with exponentially small probability. Introducing the scaled field

$$\bar\xi_t(x)=\alpha_t^2\xi\big(\lfloor x\alpha_t\rfloor\big), \tag{2.4}$$

the condition that these scales match for $\bar\xi_t\approx\psi\in C^-(R)$ reads

$$\log\operatorname{Prob}(\bar\xi_t\approx\psi)\asymp t\lambda^{\mathrm{d}}_{R\alpha_t}\big(\alpha_t^{-2}\psi(\cdot\,\alpha_t^{-1})\big). \tag{2.5}$$

By scaling properties of the continuous Laplace operator, the right-hand side is approximately equal to $(t/\alpha_t^2)\lambda_R(\psi)$, where $\lambda_R(\psi)$ is defined in (1.13). On the other hand, by our Scaling Assumption,

$$\log\operatorname{Prob}(\bar\xi_t\approx\psi)\approx-\frac{t}{\alpha_t^2}\mathcal{L}_R(\psi), \tag{2.6}$$

i.e., we expect $\bar\xi_t$ to satisfy a large-deviation principle with rate t/α_t^2 and rate function $\mathcal{L}_R$. Then the scales on both sides of (2.5) are identical and, comparing also the prefactors, we have

$$\big\langle e^{t\lambda^{\mathrm{d}}_{R\alpha_t}}1\{\bar\xi_t\approx\psi\}\big\rangle\approx\exp\big\{\tfrac{t}{\alpha_t^2}[\lambda_R(\psi)-\mathcal{L}_R(\psi)]\big\}. \tag{2.7}$$

Now collect (2.1), (2.3) and (2.7) and maximize over $\psi \in C^-(R)$ and over $R > 0$ to obtain formally the statement on the moment asymptotics in Theorem 1.2 for $p = 1$. Note that, by the above heuristic argument, α_t is the spatial scale of the "islands" in the potential landscape that are only relevant for the moments of $u(t,0)$.

2.1.2 Almost-sure asymptotics. Based on the intuition developed for the moment asymptotics, the decisive contribution to (2.2) should come from some quite localized region in Q_t. Suppose this region has size α_{b_t}, where b_t is some new running time scale; for instance, divide Q_t regularly into boxes of diameter $R\alpha_{b_t}$ ("microboxes") with some $R > 0$. According to (2.6) with t replaced by b_t, we have for any $\psi \in C^-(R)$ with $\mathcal{L}_R(\psi) \le d$ that

$$\operatorname{Prob}(\bar\xi_{b_t} \approx \psi) \approx \exp\bigl\{-\tfrac{b_t}{\alpha_{b_t}^2}\mathcal{L}_R(\psi)\bigr\} \ge e^{-db_t/\alpha_{b_t}^2}, \tag{2.8}$$

Suppose that b_t obeys (1.22). Then the right-hand side of (2.8) decays as fast as t^{-d}. Since there are of order t^d microboxes in Q_t, a Borel-Cantelli argument implies that for any ψ with $\mathcal{L}_R(\psi) < d$, there will be a microbox in Q_t where $\bar\xi_{b_t} \approx \psi$. As before, $t\lambda^{\mathrm{d}}_{R\alpha_{b_t}}(\psi(\cdot/\alpha_{b_t})/\alpha_{b_t}^2) \approx (t/\alpha_{b_t}^2)\lambda_R(\psi)$, and by optimizing over ψ, any value smaller than $\widetilde\chi$ can be attained by $\lambda_R(\psi)$ in some microbox in Q_t.

This suggests that $u(t,\cdot)$ in the favorable microbox decays as described by (1.26). It remains to ensure, and this is a non-trivial part of the argument, that the particles that have survived in this microbox by t can always reach the origin within a negligible portion of time t. This requires, in particular, that sites with $\xi > -\infty$ form an infinite cluster containing the origin. If the connection between 0 and the microbox can be guaranteed, $u(t,0)$ should exhibit the same leading-order decay, which is the essence of the claim in Theorem 1.5. Note that, as before, α_{b_t} is the spatial scale of the islands relevant for the random variable $u(t,0)$.

2.2 The case $\gamma = 1$

In the boundary case $\gamma = 1$ the relevant islands grow (presumably) slower than any polynomial as $t \to \infty$ (i.e., $\alpha_t = t^{o(1)}$), and $\widetilde H$ is linear. As a consequence, the asymptotic expansion of $\langle u(t,0)^p\rangle$ starts with a *deterministic* term $\exp[\alpha_{pt}^d H(pt/\alpha_{pt}^d)]$. Even though Theorem 1.2 is formally satisfied in this case, no variational problem is involved at this order and no information about the "typical" configuration of the fields is gained.

To understand which ξ dominate the moments of $u(t,0)$ and, in particular, $u(t,0)$ itself, we have to analyze the next-order term. This requires imposing an additional scaling assumption: We suppose the existence of a new scale function $\eta_t = o(t\alpha_t^{-d-2})$ such that

$$\lim_{t\to\infty}\frac{1}{\eta_t}\Bigl[H\bigl(\tfrac{t}{\alpha_t^d}y\bigr) - H\bigl(\tfrac{t}{\alpha_t^d}\bigr)y\Bigr] = \widehat H(y) \tag{2.9}$$

exists locally uniformly in $y \in (0,\infty)$. In analogy with Theorem 1.2, this should lead to the asymptotic expansion of the moments

$$\bigl\langle u(t,0)^p\bigr\rangle = \exp\Bigl[\alpha_{pt}^d H\bigl(\tfrac{pt}{\alpha_{pt}^d}\bigr) - \eta_{pt}\alpha_{pt}^d\bigl(\widehat\chi + o(1)\bigr)\Bigr], \tag{2.10}$$

where $\widehat\chi$ is defined as in Subsection 1.2.2 with $\widetilde H$ replaced by $\widehat H$. On the other hand, the almost-sure asymptotics should solely be determined by the second-order scale. Indeed, (1.12) outputs either value 0 or ∞, depending whether $\sup\psi \le \widetilde H(1)$ or not. Setting $\psi = \widetilde H(1) + \alpha_t^{d+2}\psi_\star/(t\eta_t)$

with some $\psi_\star \in C^-(R)$, (2.8) should read as $\mathrm{Prob}(\bar{\xi}_{b_t} \approx \psi) \approx \exp\{-(b_t/\alpha_{b_t}^2)\mathcal{L}_R^\star(\psi_\star)\}$, where $\mathcal{L}_R^\star$ is defined by (1.12) with $\widetilde{H}$ replaced by $\widehat{H}$. Let $b_t^\star$ solve for s in $\eta_s\alpha_s^d = \log t$. Then $u(t,0) = \exp\{-(t/\alpha_{b_t^\star}^2)[\widehat{\chi}_\star + o(1)]\}$ should hold, where and $\widehat{\chi}_\star$ is defined by (1.23) with $\widetilde{H}$ replaced by $\widehat{H}$. However, we have not made any serious attempt to carry out the details.

Surprisingly, the function $\widehat{H}$ takes a *unique* form in this case:

$$\widehat{H}(y) = \sigma y \log y, \tag{2.11}$$

where $\sigma > 0$ is a parameter. To establish this, we just need to apply a couple of observations from Proposition 1.1. For any $p \in (0,\infty)$, let $\phi_t(p) = \alpha_{pt}/\alpha_t$. Then we have

$$\begin{aligned} &H\big(\tfrac{pt}{\alpha_{pt}^d} y\big) - yH\big(\tfrac{pt}{\alpha_{pt}^d}\big) \\ &\qquad = \Big[H\big(\tfrac{t}{\alpha_t^d} py\phi_t(p)^{-d}\big) - py\phi_t(p)^{-d}H\big(\tfrac{t}{\alpha_t^d}\big)\Big] - y\Big[H\big(\tfrac{t}{\alpha_t^d} p\phi_t(p)^{-d}\big) - p\phi_t(p)^{-d}H\big(\tfrac{t}{\alpha_t^d}\big)\Big] \end{aligned} \tag{2.12}$$

By dividing both sides by η_t, interpreting pt as the time variable on the left-hand side, and recalling that $\phi_t(p) \to 1$ as $t \to \infty$ by Proposition 1.1 in this case ($\gamma = 1$), we have

$$\frac{\eta_{pt}}{\eta_t}\big(\widehat{H}(y) + o(1)\big) = \widehat{H}(py) - y\widehat{H}(p) + o(1), \tag{2.13}$$

where we also used continuity of $y \mapsto \widehat{H}(y)$. This proves that $\eta_{pt}/\eta_t \to \widehat{\phi}(p)$ satisfying $\widehat{\phi}(p) = [\widehat{H}(py) - \widehat{H}(p)y]/\widehat{H}(y)$ for any $p, y \in (0,\infty)$. Since $\widehat{\phi}(p)$ is clearly multiplicative, after some work we find out that the pair $(\widehat{\phi}(\,\cdot\,), \widehat{H}(\,\cdot\,))$ must be of the form $\widehat{\phi}(p) = p^{1+\varkappa}$ and $\widehat{H}(y) = \sigma y \frac{y^\varkappa - 1}{\varkappa}$ for some $\sigma > 0$ and $\varkappa \in [-1, 0]$ (the cases $\varkappa < -1$ violate the convexity of $\widehat{H}$; $\varkappa > 0$ is incompatible with $\eta_t = o(t/\alpha_t^{d+2})$). For $\varkappa = 0$, the formula reads as $\widehat{H}(y) = \sigma y \log y$. Observe that, by differentiability of $y \mapsto \widehat{H}(y)$, the limit in (2.9) is uniform on $[0, M]$ for any $M > 0$.

To rule out the cases with negative $\varkappa$, suppose $\varkappa < 0$ and note that, analogously to (1.7), $\widehat{\phi}(p) = p^{1+\varkappa}$ implies $\eta_t = t^{1+\varkappa+o(1)}$. Let y_t and δ_t be defined by the formulas

$$\frac{t}{\alpha_t^d}\, y_t = 1 \qquad \text{and} \qquad \eta_t\, y_t^{\delta_t} = 1. \tag{2.14}$$

Clearly, since $y_t = t^{-1+o(1)}$ by the first relation and (1.7), we have $\delta_t = 1 + \varkappa + o(1)$. Moreover, by $y_t^{\delta_t} \to 0$ we also have $H(\,\cdot\, y_t) \le H(\,\cdot\, y_t^{1-\delta_t})y_t^{\delta_t}$. Now insert (2.14) into the bracket in (2.9), divide by $y_t^{\delta_t}$ and use the preceding observation to get that

$$H(1) - \frac{\alpha_t^d}{t} H\Big(\frac{t}{\alpha_t^d}\Big) \le \frac{1}{\eta_t}\Big[H\big(\tfrac{t}{\alpha_t^d} y_t^{1-\delta_t}\big) - H\big(\tfrac{t}{\alpha_t^d}\big)y_t^{1-\delta_t}\Big] = \widehat{H}(y_t^{1-\delta_t}) + o(1), \quad t \to \infty, \tag{2.15}$$

where we used that $y_t^{1-\delta_t}/\eta_t = \alpha_t^d/t$ and invoked the uniformity of (2.9). Since $\lim_{t\to\infty} y_t^{1-\delta_t} = 0$, the right-hand side vanishes as $t \to \infty$. But this is a contradiction, because $H(1)$ is finite while $(\alpha_t^d/t)H(t/\alpha_t^d) = \alpha_t^2[\widetilde{H}(1) + o(1)] \to -\infty$, by $\lim_{t\to\infty} \alpha_t = \infty$. Consequently, only the case $\varkappa = 0$ is compatible with (1.2) and the Scaling Assumption (the cases $\varkappa < 0$ vaguely correspond to measures with the tail (1.3) but with $\operatorname{esssup} \xi(0) < 0$, which implies that $\alpha_t = \text{const.}$).

2.3 An application: Self-attractive random walks

One of our original source of motivation for this work have been self-attractive path measures as models for "squeezed polymers". Consider a polymer $S = (S_0, \ldots, S_n)$ of length n

modeled by a path of simple random walk with a transformed path measure proportional to $\exp[\beta\sum_x V(\ell_n(x))]$. Here $V\colon \mathbb{Z}\to(-\infty,0]$, and $\ell_n(x)=\#\{k\le n\colon S_k=x\}$ is the local time at x. Assuming that V is convex and $V(0)=0$, e.g., $V(\ell)=-\ell^\gamma$ with $\gamma\in[0,1)$, the interaction has an attractive effect. A large class of such functions V (i.e., the completely monotonous ones) are the cumulant generating functions of probability distributions on $[-\infty,0]$, like H in (1.4). Via the Feynman-Kac representation, this makes the study of the above path measure essentially equivalent to the study of the moments of a parabolic Anderson model. In fact, the only difference is that for polymer models the time of the walk is discrete.

We have no doubt that Theorem 1.2 extends to the discrete-time case. Hence, the endpoint S_n of the polymer fluctuates on the scale α_n as in our Scaling Assumption, which is $\alpha_n=n^\nu$ in the $V(\ell)=-\ell^\gamma$ case. Since $\gamma\mapsto\nu$ is decreasing, we are confronted with the counterintuitive feature that the squeezing effect is the more extreme the "closer" is V to the linear function. This is even more surprising if one recalls that for the boundary case $\gamma=1$, the Hamiltonian $\sum_x V(\ell_n(x))$ is deterministic, and therefore the endpoint runs on scale $n^{1/2}$. Note that, on the other hand, for $\gamma>1$, which is the self-repellent case, it is expected in dimensions $d=2$ and 3 (and known in $d=1$) that the scale of the endpoint is larger than $\frac12$. Hence, at least in low dimensions, there is an intriguing phase transition for the path scale at $\gamma=1$.

As a nice side-remark, the following model of an *annealed randomly-charged polymer* also falls into the class of models considered above. Consider an n-step simple random walk $S=(S_0,\dots,S_n)$ with weight $e^{-\beta\mathcal{I}_n(S)}$ where $\beta>0$ and

$$\mathcal{I}_n(S)=\sum_{0\le i<j\le n}\omega_i\omega_j 1\{S_i=S_j\}. \tag{2.16}$$

Here $\omega=(\omega_i)_{i\in\mathbb{N}_0}$ is an i.i.d. sequence with a symmetric distribution on $\mathbb{R}$ having variance one. Think of ω_i as an electric charge at site i of the polymer. (For continuous variants of this model and more motivation see e.g., Buffet and Pulé [BP97]).

If the charges equilibrate faster than the walk, the interaction they effectively induce on the walk is given by the expectation $E(e^{-\beta\mathcal{I}_n(S)})$ and is thus of the above type with

$$V(\ell)=-\log E\exp\bigl((\omega_0+\dots+\omega_\ell)^2\bigr), \tag{2.17}$$

where E denotes the expectation with respect to ω. By the invariance principle, we have $V(\ell)=-(1/2+o(1))\log\ell$ as $\ell\to\infty$, which means that V satisfies our Scaling Assumption with $\alpha_n=(n/\log n)^{1/(d+2)}$. Hence, we can identify the logarithmic asymptotics of the partition function $\mathbb{E}_0\otimes E(e^{-\beta\mathcal{I}_n})$ and see that the typical end-to-end distance of the annealed charged polymer runs on the scale α_n, i.e., the averaging over the charges has a self-attractive effect.

2.4 Relation to earlier work

General mathematical aspects of the problem (1.1), including the existence and uniqueness of solutions and a criterion for intermittency (see (1.17) and the comments thereafter), were first addressed by Gärtner and Molchanov [GM90]. In a subsequent paper [GM98] (see also [GM96]), the same authors focused on the case of *double-exponential* distributions

$$\mathrm{Prob}(\xi(0)>x)\sim\exp\bigl\{-e^{x/\varrho}\bigr\},\qquad x\to\infty. \tag{2.18}$$

For $0<\varrho<\infty$, it turns out that the main contribution to $\langle u(t,0)^p\rangle$ comes from islands in $\mathbb{Z}^d$ of asymptotically finite size (which corresponds to a constant α_t in our notation). When the upper tails of $\mathrm{Prob}(\xi(0) \in \cdot)$ are yet thicker (i.e., $\varrho = \infty$), e.g., when $\xi(0)$ is Gaussian, then the overwhelming contribution to $\langle u(t,0)^p \rangle$ comes from very high peaks of ξ concentrated at single sites. (In a continuous setting the scaling can still be non-trivial, see Gärtner and König [GK98], and Gärtner, König and Molchanov [GKM99].) For thinner tails than double-exponential (i.e., when $\varrho = 0$, called the *almost bounded* case in [GM98]), the relevant islands grow unboundedly as $t \to \infty$, i.e., $\alpha_t \to \infty$ in our notation. The distribution (2.18) thus constitutes a certain critical class for having a non-degenerate but still discrete spatial structure.

The opposite extreme of the tail behaviors was addressed by Donsker and Varadhan [DV79] (moment asymptotics) and by Antal [A95] (almost-sure asymptotics), see also [A94]. The distribution that these authors considered was $\xi(0) = 0$ or $-\infty$ with probability p and $1-p$, respectively. The analysis of the moments boils down to a self-interacting polymer problem (see Subsection 2.3), which is essentially the route taken by Donsker and Varadhan. In the case of a fixed field, the problem is a discrete analogue of the Brownian motion in a Poissonian potential analyzed extensively by Sznitman in the 1990's using his celebrated method of enlargement of obstacles (MEO), see Sznitman [S98].

Interpret points z with $\xi(z) = -\infty$ as a trap where the simple random walk is killed. If $\mathcal{O} = \{z \in \mathbb{Z}^d \colon \xi(z) = -\infty\}$ denotes the trap region and $T_{\mathcal{O}} = \inf\{t > 0 \colon X(t) \in \mathcal{O}\}$ the entrance time, then

$$u(t,z) = \mathbb{P}_z(T_{\mathcal{O}} > t), \tag{2.19}$$

i.e., $u(t,z)$ is the survival probability at time t for a walk started at z. In his thesis [A94], Antal derives a discrete version of the MEO and demonstrates its value in [A94] and [A95] by proving results which are (refinements of) our Theorems 1.2 and 1.5 for $\gamma = 0$ and $\alpha_t = t^{1/(d+2)}$.

2.5 Discussion and open problems

(1) *"Almost-bounded" cases.* We gave ourselves the task to fill in the gap between the two regimes considered in [GM98] and [DV79] resp. [A95], i.e., we wanted to study the general case in which the diameter α_t of the relevant islands grows to infinity. The present paper investigates the case in which the field is bounded from above and α_t diverges at least like a power of t. As already noted in Subsection 2.2, in the boundary case $\alpha_t = t^{o(1)}$ (i.e., $\gamma = 1$) another phenomenon occurs which cannot be handled in a unified manner. We believe that the $\gamma = 1$ case reflects the *whole* regime of "almost bounded" but unbounded potentials, i.e., those interpolating between our cases $\gamma < 1$ and the double exponential distribution. For these reasons, we leave its investigation to future work.

(2) *Generalized MEO.* Our proofs closely follow Gärtner and König [GK98] and Gärtner, König and Molchanov [GKM99]. The argument for the moment asymptotics essentially goes back to the seminal papers by Donsker and Varadhan [DV75] and [DV79]. However, unlike Donsker and Varadhan, we do not use folding to compactify the space, but rather a comparison technique for Dirichlet eigenvalues in large and small boxes. This technique, which is adopted from Gärtner and Molchanov [GM96], makes the proof of the almost-sure asymptotics coherent with the part on moment asymptotics and it works, at least in principle, also for correlated fields. Unfortunately, it seems to be applicable only to the leading-order term.

It would be interesting to develop an extension of the MEO for other fields in our class (in particular, those with $\gamma \neq 0$), which should allow us to go beyond the leading order term.

However, this requires the knowledge of the *shape* of the field that brings the main contribution to the moments of $u(t,0)$ resp. to $u(t,0)$ itself. While the MEO can help in controlling the "probability part" of the statements (1.16) and (1.26), an analysis of the minimizers in (1.14) and (1.23) is also needed. The latter is expected to be delicate in higher dimensions.

(3) *Correlation structure.* Another open problem concerns the asymptotic correlation structure of the random field $u(t,\cdot)$, as has been analysed by Gärtner and den Hollander [GH99] in the case of the double-exponential distribution. Also for answering this question, quite some control of the minimizers in (1.14) and (1.23) is required. Unfortunately, the compactification technique of [GH99] cannot be applied without additional work, since it seems to rely on the discreteness of the underlying space in several important places.

(4) *Lower-tail in $d=1$.* For the almost-sure asymptotics of $u(t,0)$, suitable conditions on the lower tail of the distribution of $\xi(0)$ had to be imposed in order to derive our Theorem 1.5. In $d\geq 2$, percolation of sites z with $\xi(z)>-\infty$ turned out to be sufficient, while in $d=1$ we additionally had to assume that $\log(-\xi(0)\vee 1)$ has the first moment. The reason for this is that, in $d=1$, no site in $\mathbb{Z}\setminus\{0\}$ can be reached from 0 avoiding any of the sites in-between. As a consequence, if the lower tail of $\xi(0)$ is too thick, the sites with large negative ξ may screen off the favorable regions where $u(\cdot,t)$ is governed by the variational problem (1.23) (see Subsection 2.1.1 for an informal explanation). It would be interesting to determine to what extent can our condition be still relaxed and how the almost sure asymptotics of $u(t,0)$ depends on the lower tail of $\xi(0)$ when it is robustly violated.

3. Preliminaries

In this section we first introduce some necessary notation needed in the proof of Theorems 1.2 and 1.5 and then prove Propositions 1.1 and 1.4. In the last subsection, we prove a claim on the convergence of certain approximants to the variational problem (1.14).

3.1 Feynman-Kac formula and Dirichlet eigenvalues

Our analysis is based on the link between the random-walk and random-field descriptions provided by the Feynman-Kac formula. Let $(X(s))_{s\in[0,\infty)}$ be the continuous-time simple random walk on $\mathbb{Z}^d$ with generator $\kappa\Delta^{\mathrm{d}}$. By $\mathbb{P}_z$ and $\mathbb{E}_z$ we denote the probability measure resp. the expectation with respect to the walk starting at $X(0)=z\in\mathbb{Z}^d$.

3.1.1 General initial problem. For any potential $V\colon\mathbb{Z}^d\to[-\infty,0]$, we denote by u^V the unique solution to the initial problem

$$\begin{aligned} \partial_t u(t,z) &= \kappa\Delta^{\mathrm{d}}u(t,z)+V(z)u(t,z), \qquad && (t,z)\in(0,\infty)\times\mathbb{Z}^d,\\ u(0,z) &= 1, && z\in\mathbb{Z}^d. \end{aligned} \tag{3.1}$$

The Feynman-Kac formula allows us to express u^V as

$$u^V(t,z)=\mathbb{E}_z\left[\exp\int_0^t V\big(X(s)\big)\,ds\right],\qquad z\in\mathbb{Z}^d,\ t>0. \tag{3.2}$$

Introduce the local times of the walk

$$\ell_t(z)=\int_0^t 1\{X(s)=z\}\,ds,\qquad z\in\mathbb{Z}^d,\ t>0, \tag{3.3}$$

i.e., $\ell_t(z)$ is the amount of time the random walk has spent at $z \in \mathbb{Z}^d$ by time t. Note that $\int_0^t V(X(s))\,ds = (V, \ell_t)$, where $(\cdot,\cdot)$ stands for the inner product on $\ell^2(\mathbb{Z}^d)$.

Let $R > 0$ and let $Q_R = [-R, R]^d \cap \mathbb{Z}^d$. The solution of the initial-boundary value problem

$$\begin{aligned} \partial_t u(t,z) &= \kappa\Delta^{\mathrm{d}} u(t,z) + V(z)u(t,z), && (t,z) \in (0,\infty)\times Q_R,\\ u(0,z) &= 1, && z \in Q_R,\\ u(t,z) &= 0, && t > 0,\ z \notin Q_R, \end{aligned} \tag{3.4}$$

will be denoted by $u_R^V\colon [0,\infty)\times\mathbb{Z}^d \to [0,\infty)$. Similarly to (3.2), we have the representation

$$u_R^V(t,z) = \mathbb{E}_z\Big[\exp\Big\{\int_0^t V\big(X(s)\big)\,ds\Big\}1\{\tau_R > t\}\Big], \qquad z\in\mathbb{Z}^d,\ t>0, \tag{3.5}$$

where τ_R is the first exit time from the set Q_R, i.e.,

$$\tau_R = \inf\big\{t > 0\colon X(t) \notin Q_R\big\}. \tag{3.6}$$

Alternatively,

$$u_R^V(t,z) = \mathbb{E}_z\Big[e^{(V,\ell_t)}1\big\{\operatorname{supp}(\ell_t) \subset Q_R\big\}\Big], \tag{3.7}$$

where we recalled (3.3). Note that, for $0 < r < R < \infty$,

$$u_r^V \le u_R^V \le u^V \qquad \text{in } [0,\infty)\times\mathbb{Z}^d, \tag{3.8}$$

as follows by (3.5) because $\{\tau_r > t\} \subset \{\tau_R > t\}$.

Apart from u^V, we also need the fundamental solution $p_R^V(t,\cdot,z)$ of (3.4), i.e., the solution to (3.4) with $p_R^V(0,\cdot,z) = \delta_z(\cdot)$ instead of the second line. The Feynman-Kac representation is

$$p_R^V(t,y,z) = \mathbb{E}_y\Big[e^{(V,\ell_t)}1\big\{\operatorname{supp}(\ell_t)\subset Q_R\big\}1\big\{X(t)=z\big\}\Big] \qquad y,z\in\mathbb{Z}^d. \tag{3.9}$$

Note that $\sum_{z\in Q_R} p_R^V(t,y,z) = u^V(t,y)$.

3.1.2 Eigenvalue representations. The second crucial tool for our proofs will be the principal (i.e., the largest) eigenvalue $\lambda_R^{\mathrm{d}}(V)$ of the operator $\kappa\Delta^{\mathrm{d}} + V$ in Q_R with Dirichlet boundary condition. The Rayleigh-Ritz formula reads

$$\lambda_R^{\mathrm{d}}(V) = \sup\big\{(V,g^2) - \kappa\|\nabla g\|_2^2\colon g\in\ell^2(\mathbb{Z}^d), \|g\|_2 = 1, \operatorname{supp}(g)\subset Q_R\big\}. \tag{3.10}$$

Here ∇ denotes the discrete gradient.

Let $\lambda_1 > \lambda_2 \ge \lambda_3 \ge \dots \ge \lambda_n$, $n = \#Q_R$, be the eigenvalues of the operator $\kappa\Delta^{\mathrm{d}}+V$ in $\ell^2(Q_R)$ with Dirichlet boundary condition (some of them can be $-\infty$). We also write $\lambda_R^{\mathrm{d},k}(V) = \lambda_k$ for the k-th eigenvalue to emphasize its dependence on the potential and the box Q_R. Let $(\mathrm{e}_k)_k$ be an orthonormal basis in $\ell^2(Q_R)$ consisting of corresponding eigenfunctions $\mathrm{e}_k = \mathrm{e}_R^{\mathrm{d},k}(V)$. (Conventionally, e_k vanishes outside Q_R.) Then we have the Fourier expansions

$$p_R^V(t,y,z) = \sum_k e^{t\lambda_k}\mathrm{e}_k(y)\mathrm{e}_k(z) \tag{3.11}$$

and, by summing this over all $y\in Q_R$,

$$u_R^V(t,\cdot) = \sum_k e^{t\lambda_k}(\mathrm{e}_k, 1)_R\,\mathrm{e}_k(\cdot), \tag{3.12}$$

where we used $(\cdot,\cdot)_R$ to denote the inner product in $\ell^2(Q_R)$. Here and henceforth "1" is the function taking everywhere value 1.

3.2 Power-law scaling

Proof of Proposition 1.1. Let $\widetilde H_t$ be the function given by

$$\widetilde H_t(\,\cdot\,)=\frac{\alpha_t^{d+2}}{t}H\Bigl(\frac{t}{\alpha_t^d}\,\cdot\Bigr). \tag{3.13}$$

By our Scaling Assumption, $\lim_{t\to\infty}\widetilde H_t=\widetilde H$ on $[0,\infty)$. Note that both $\widetilde H_t$ and $\widetilde H$ are convex, non-positive and not identically vanishing with value 0 at zero. Consequently, $\widetilde H_t$ and $\widetilde H$ are continuous and strictly negative in $(0,\infty)$. Moreover, by applying Jensen's inequality to the definition of H, we have that $y\mapsto \widetilde H_t(y)/y$ and $y\mapsto\widetilde H(y)/y$ are both non-decreasing functions.

Next we shall show that α_{pt}/α_t tends to a finite non-zero limit for all p. Let us pick a $y>0$ and a $p\in(0,\infty)$ and consider the identity

$$\widetilde H_t\biggl(p\Bigl(\frac{\alpha_t}{\alpha_{pt}}\Bigr)^d y\biggr)=p\Bigl(\frac{\alpha_t}{\alpha_{pt}}\Bigr)^{d+2}\widetilde H_{pt}(y), \tag{3.14}$$

which results by comparing (3.13) with the "time" parameter interpreted once as t and next time as pt. Invoking the monotonicity of $y\mapsto\widetilde H_t(y)/y$, it follows that

$$p\Bigl(\frac{\alpha_t}{\alpha_{pt}}\Bigr)^2\widetilde H_{pt}(y)\ge\widetilde H_t(py)\quad\text{whenever}\quad\alpha_t\ge\alpha_{pt}. \tag{3.15}$$

This implies that α_{pt}/α_t is bounded away from zero, because we have

$$\liminf_{t\to\infty}\Bigl(\frac{\alpha_{pt}}{\alpha_t}\Bigr)^2\ge\frac{p\widetilde H(y)}{\widetilde H(py)}\wedge 1>0, \tag{3.16}$$

where "$\wedge$" stands for minimum. Since $p\in(0,\infty)$ was arbitrary, α_{pt}/α_t is also uniformly bounded, by replacing t with t/p.

Let $\phi(p)$ be defined for each p as a subsequential limit of α_{pt}/α_t, i.e., $\phi(p)=\lim_{n\to\infty}\alpha_{pt_n}/\alpha_{t_n}$ with some (p-dependent) $t_n\to\infty$. By our previous reasoning $\phi(p)^{-1}$ is non-zero, finite and, for all $y>0$, it solves for z in the equation

$$\widetilde H\bigl(pz^dy\bigr)=pz^{d+2}\widetilde H(y). \tag{3.17}$$

Here we were allowed to pass to the limiting function $\widetilde H$ on the left-hand side of (3.14) because $\widetilde H$ is continuous and the scaling limit (1.5) is uniform on compact sets in $(0,\infty)$. But $z\mapsto\widetilde H(pz^dy)/z^d$ is non-decreasing while $z\mapsto pz^2\widetilde H(y)$ is *strictly* decreasing, so the solution to (3.17) is unique. Hence, the limit $\phi(p)=\lim_{t\to\infty}\alpha_{pt}/\alpha_t$ exists in $(0,\infty)$ for all $p\in(0,\infty)$.

It is easily seen that ϕ is multiplicative on $(0,\infty)$, i.e., $\phi(pq)=\phi(p)\phi(q)$. Since $\phi(p)\ge1$ for $p\ge1$, by the same token we also have that $p\mapsto\phi(p)$ is non-decreasing. These two properties imply that $\phi(2^n)=\phi(2)^n$ and that $\phi(2)^{\frac{n}{m}}\le\phi(p)\le\phi(2)^{\frac{n+1}{m}}$ for any $p>0$, and m, n integer such that $2^n\le p^m<2^{n+1}$. Consequently, $\phi(p)=p^\nu$ with $\nu=\log_2\phi(2)$. By plugging this back into (3.17) and setting $y=1$ we get that

$$\widetilde H\bigl(p^{1-d\nu}\bigr)=\widetilde H(1)\,p^{1-(d+2)\nu}. \tag{3.18}$$

The claims (1.6) and (1.7) are thus established by putting $\gamma(1-d\nu)=1-(d+2)\nu$, which is (1.8). Clearly, $\gamma\in[0,1]$, in order to have the correct monotonicity properties of $y\mapsto\widetilde{H}(y)$ and $y\mapsto\widetilde{H}(y)/y$.

To prove also the second statement in (1.7), we first write

$$\alpha_{2^N}=\alpha_1\prod_{m=0}^{N-1}\frac{\alpha_{2^{m+1}}}{\alpha_{2^m}} \tag{3.19}$$

which, after taking the logarithm, dividing by $\log 2^N$, and noting that $\alpha_{2^{m+1}}/\alpha_{2^m}\to\phi(2)$ as $m\to\infty$, allows us to conclude that

$$\lim_{N\to\infty}\frac{\log\alpha_{2^N}}{\log 2^N}=\log_2\phi(2)=\nu. \tag{3.20}$$

The limit for general t is then proved again by sandwiching t between 2^{N-1} and 2^N and invoking the monotonicity of $t\mapsto\alpha_t$. □

3.3 Relation between χ and $\widetilde{\chi}$

Proof of Proposition 1.4. Suppose H is in the γ-class and define ν as in Proposition 1.1. Suppose $\chi\neq 0,\infty$ (for a proof of this statement, see Proposition 3.1). The argument hinges on particular scaling properties of the functionals $\psi\mapsto\mathcal{L}_R(\psi)$ and $\psi\mapsto\lambda_R(\psi)$, which enable us to convert (1.14) into (1.23). Given $\psi\in C^-(R)$, let us for each $b\in(0,\infty)$ define $\psi_b\in C^-(bR)$ by

$$\psi_b(x)=\frac{1}{b^2}\psi\left(\frac{x}{b}\right). \tag{3.21}$$

Then we have

$$\mathcal{L}_{bR}(\psi_b)=b^{\frac{1}{\nu}-2}\mathcal{L}_R(\psi)\qquad\text{and}\qquad\lambda_{bR}(\psi_b)=b^{-2}\lambda_R(\psi), \tag{3.22}$$

where in the first relation we used that ψ_b can be converted into ψ in (1.12) by substituting $b^{2/(1-\gamma)}f(\cdot/b)$ in the place of $f(\,\cdot\,)$; the second relation is a result of a simple spatial scaling of the first line in (1.13). Note that $\frac{1}{\nu}-2\geq 1>0$.

Let $\psi^{(n)}\in C^-(R_n)$ be a minimizing sequence of the variational problem in (1.15). Suppose, without loss of generality, that $\mathcal{L}_{R_n}(\psi^{(n)})\to\bar{\mathcal{L}}$ and $\lambda_{R_n}(\psi^{(n)})\to\bar{\lambda}$. Then we have

$$\chi=\bar{\mathcal{L}}-\bar{\lambda}. \tag{3.23}$$

Now pick any $b\in(0,\infty)$ and consider instead the sequence $(\psi_b^{(n)})$. Clearly,

$$\chi\leq\lim_{n\to\infty}\left[\mathcal{L}_{bR_n}(\psi_b^{(n)})-\lambda_{bR_n}(\psi_b^{(n)})\right]=b^{\frac{1}{\nu}-2}\bar{\mathcal{L}}-b^{-2}\bar{\lambda} \tag{3.24}$$

for all b. By (3.23), the derivative of the right-hand side must vanish at $b=1$, i.e.,

$$\left(\tfrac{1}{\nu}-2\right)\bar{\mathcal{L}}+2\bar{\lambda}=0. \tag{3.25}$$

By putting (3.23) and (3.25) together, we easily compute that

$$\bar{\mathcal{L}}=2\nu\chi. \tag{3.26}$$

Note that while $b \mapsto \mathcal{L}_{bR}(\psi_b)$ is strictly increasing, $b \mapsto \lambda_{bR}(\psi_b)$ is strictly decreasing. This allows us to recast (1.15) as

$$\chi = \bar{\mathcal{L}} + \inf_{R>0} \inf \bigl\{-\lambda_R(\psi) \colon \psi \in C^-(R),\, \mathcal{L}_R(\psi) \le \bar{\mathcal{L}}\bigr\}\,. \tag{3.27}$$

Indeed, we begin by observing that "$\le$" holds in (3.27), as is verified by pulling $\bar{\mathcal{L}}$ inside the bracket, replacing it with $\mathcal{L}_R(\psi)$, and dropping the last condition. To prove the "$\ge$" part, note that the above sequence $(\psi_b^{(n)})$ for $b<1$ eventually fulfills the last condition in (3.27) because $\mathcal{L}_{bR_n}(\psi_b^{(n)}) \to b^{\frac{1}{\nu}-2}\bar{\mathcal{L}} < \bar{\mathcal{L}}$. Since $\lambda_{bR_n}(\psi_b^{(n)}) \to b^{-2}\bar{\lambda}$, the right-hand side of (3.27) is no more than $\bar{\mathcal{L}} - b^{-2}\bar{\lambda}$ for any $b<1$. Taking $b \uparrow 1$ and recalling (3.23) proves the equality in (3.27).

With (3.27) in the hand we can finally prove (1.25). By using ψ_b instead of ψ in (3.27), the condition $\mathcal{L}_R(\psi) \le \bar{\mathcal{L}}$ becomes $\mathcal{L}_R(\psi) \le b^{\frac{1}{\nu}-2}\bar{\mathcal{L}}$ and the factor b^{-2} appears in front of the infimum. Thus, setting $b^{\frac{1}{\nu}-2}\bar{\mathcal{L}} = d$, which by (3.26) requires that

$$b = \Bigl(\frac{2\nu\chi}{d}\Bigr)^{\frac{\nu}{1-2\nu}}, \tag{3.28}$$

(note that $b \ne 0, \infty$) and invoking (3.26), we recover the variational problem (1.23). Therefore,

$$\chi = \bar{\mathcal{L}} + b^{-2}\widetilde{\chi} = 2\nu\chi + \Bigl(\frac{2\nu\chi}{d}\Bigr)^{-\frac{2\nu}{1-2\nu}}\widetilde{\chi}. \tag{3.29}$$

From this, (1.25) follows by simple algebraic manipulations. The claim $\widetilde{\chi} \in (0,\infty)$ is a consequence of (1.25) and the fact that $\chi \in (0,\infty)$. $\square$

3.4 Approximation variational problems

The proof of Theorem 1.2 will require some knowledge of the properties of the variational problem (1.14). Let

$$\chi_R = \inf\bigl\{\mathcal{I}(f) - \mathcal{H}_R(f) \colon f \in \mathcal{F}_R\bigr\}, \qquad R>0. \tag{3.30}$$

In particular, we need to prove that certain approximation quantities converge to χ_R. Suppose H is in the γ-class and introduce the following quantities: In the case $\gamma \in (0,1)$, let

$$\chi_R^\star(M) = \inf\bigl\{\mathcal{I}(f) - \mathcal{H}_R(f \wedge M) \colon f \in \mathcal{F}_R\bigr\}, \qquad M>0, \tag{3.31}$$

for any $R>0$. For $\gamma = 0$ and any $R>0$, let

$$\chi_R^{\#}(\varepsilon) = \inf\bigl\{\mathcal{I}(f) - \widetilde{H}(1)|\{f>\varepsilon\}| \colon f \in \mathcal{F}_R\bigr\}, \qquad 0<\varepsilon \ll R. \tag{3.32}$$

The needed properties are summarized as follows:

Proposition 3.1 *Let H be in the γ-class and let χ be as in* (1.14). *Then*

(1) $\chi \in (0,\infty)$.

(2) For $\gamma \in (0,1)$ and any $R>0$, $\lim_{M\to\infty} \chi_R^\star(M) = \chi_R$.

(3) For $\gamma = 0$ and any $R>0$, $\lim_{\varepsilon\downarrow 0} \chi_R^{\#}(\varepsilon) = \chi_R$.

Proof of (1) and (2). Assertion (1) for $\gamma = 0$ is well-known. Assume that $\gamma \in (0,1)$ and observe that, due to the perfect scaling properties of both $f \mapsto \mathcal{I}(f)$ and $f \mapsto \mathcal{H}_R(f)$, (3.30) can alternatively be written as

$$\chi_R = \inf\big\{R^{-2}\mathcal{I}(f) - R^{d(1-\gamma)}\mathcal{H}_1(f)\colon f \in \mathcal{F}_1\big\}. \tag{3.33}$$

Let $(\lambda_1, \widehat{g})$ be the principal eigenvalue resp. an associated eigenvector of $-\Delta$ in $[-1,1]^d$ with Dirichlet boundary condition. Then $\mathcal{I}(\widehat{g}^2) = \kappa\lambda_1 \neq 0, \infty$, which means that

$$\chi_R \leq R^{-2}\kappa\lambda_1 - R^{d(1-\gamma)}\widetilde{H}(1)\int |\widehat{g}|^{2\gamma} =: \bar{\chi}_R. \tag{3.34}$$

Since $\widehat{g}$ is continuous and bounded, the integral is finite, whereby $\chi \leq \inf_{R>0}\bar{\chi}_R < \infty$.

Claim (2) and the remainder of (1) are then simple consequences of the following observation, whose justification we defer to the end of this proof:

$$\inf\big\{\mathcal{I}(f)\colon f \in \mathcal{F}_R,\ \|f\mathbb{1}_{\{f\geq M\}}\|_1 \geq \varepsilon\big\} \geq \kappa\,\frac{\varepsilon}{2}\Big(\frac{M}{8\pi_d}\Big)^{2/d}, \qquad R, \varepsilon > 0,\ M \geq 8\pi_d d^d, \tag{3.35}$$

where π_d is the volume of the unit sphere in $\mathbb{R}^d$. Indeed, to get that χ is non-vanishing, let $R > 0$ be fixed, set $\varepsilon = 1/2$ and choose M such that the infimum in (3.35) is strictly larger than $-\widetilde{H}(1)M^{\gamma-1}/2$. Let $C := -\widetilde{H}(1)M^{\gamma-1}/2$. Then for any $f \in \mathcal{F}_R$ either $\|f\mathbb{1}_{\{f\geq M\}}\|_1 \geq 1/2$, which implies $\mathcal{I}(f) \geq C$, or $\|f\mathbb{1}_{\{f\geq M\}}\|_1 < 1/2$ which implies

$$-\mathcal{H}_R(f) \geq -\widetilde{H}(1)\int f^\gamma\,\mathbb{1}_{\{f<M\}} \geq -\widetilde{H}(1)M^{\gamma-1}\int f\,\mathbb{1}_{\{f<M\}} \geq -\widetilde{H}(1)M^{\gamma-1}/2 = C. \tag{3.36}$$

Thus, in both cases, $\mathcal{I}(f) - \mathcal{H}_R(f) \geq C > 0$ independent of R. This finishes part (1).

To prove also part (2), note first that $\chi_R^\star(M) \leq \chi_R$ for all $M > 0$. Given $\varepsilon > 0$, let $M \geq 1$ be such that the infimum in (3.35) is larger than $\bar{\chi}_R$ in (3.34). Consider (3.31) restricted to $f \in \mathcal{F}_R$ with $\|f\mathbb{1}_{\{f\geq M\}}\|_1 < \varepsilon$. Since for any such f

$$\begin{aligned} -\mathcal{H}_R(f\wedge M) &\geq -\widetilde{H}(1)\int f^\gamma\,\mathbb{1}_{\{f<M\}} \geq -\mathcal{H}_R(f) + \widetilde{H}(1)\int f^\gamma\,\mathbb{1}_{\{f\geq M\}} \\ &\geq -\mathcal{H}_R(f) + \widetilde{H}(1)\int f\,\mathbb{1}_{\{f\geq M\}} \geq -\mathcal{H}_R(f) + \widetilde{H}(1)\varepsilon, \end{aligned} \tag{3.37}$$

the restricted infimum is no less than $\chi_R + \widetilde{H}(1)\varepsilon$. Therefore, $\chi_R^\star(M) \geq \bar{\chi}_R \wedge (\chi_R + \widetilde{H}(1)\varepsilon)$, which by $\varepsilon \downarrow 0$ and (3.34) proves part (2) of the claim.

It remains to prove (3.35). To that end, denote the infimum by $\Psi_R(\varepsilon, M)$ and note that

$$\Psi_R(\varepsilon, M) = R^{-2}\Psi_1(\varepsilon, MR^d). \tag{3.38}$$

Indeed, denoting $f^*(\,\cdot\,) = R^d f(\cdot R)$ for any $f \in \mathcal{F}_R$, we have $f^* \in \mathcal{F}_1$, $\mathcal{I}(f^*) = R^2\mathcal{I}(f)$, and $\|f^*\mathbb{1}_{\{f^*\geq MR^d\}}\|_1 = \|f\mathbb{1}_{\{f\geq M\}}\|_1$, whereby (3.38) immediately follows. Since $R^{-2}(MR^d)^{2/d} = M^{2/d}$, it suffices to prove (3.35) just for $R = 1$.

Recall that the operator $-\Delta$ on $[-1,1]^d$ with Dirichlet boundary condition has a compact resolvent, so its spectrum $\sigma(-\Delta)$ is a discrete set of finitely-degenerate eigenvalues. For each $k \in \mathbb{N}$, define the function

$$\varphi_k(x) = \begin{cases} \cos\big(\frac{\pi}{2}kx\big) & \text{if } k \text{ is odd}, \\ \sin\big(\frac{\pi}{2}kx\big) & \text{if } k \text{ is even}. \end{cases} \tag{3.39}$$

Then $\sigma(-\Delta) = \{\pi^2|k|_2^2/4\colon k \in \mathbb{N}^d\}$, with $|k|_2^2 = k_1^2 + \cdots + k_d^2$ and the eigenvectors given as $\omega_k = \varphi_{k_1} \otimes \cdots \otimes \varphi_{k_d}$. Note that the latter form a (Fourier) basis in $L^2([-1,1]^d)$.

Let $\varepsilon > 0$ and $M > 0$ be fixed. Let r be such that $8\pi_d r^d = M$. Note that $r \geq d$. Pick a function $f \in \mathcal{F}_1$ such that $\|f\mathbb{1}_{\{f\geq M\}}\|_1 \geq \varepsilon$ and let $g = \sqrt{f}$. Let g_1 resp. g_2 be the normalized projections of g onto the Hilbert spaces generated by (ω_k) with $|k|_2 \leq r$ resp. $|k|_2 > r$. Then $g = a_1 g_1 + a_2 g_2$ with $|a_1|^2 + |a_2|^2 = 1$. We claim that $\|g_1\|_\infty \leq \sqrt{M}/2$. Indeed, $g_1 = \sum_k c_k \omega_k$ where $(c_k) \in \ell^2(\mathbb{N}^d)$ is such that $c_k = 0$ for all $k \in \mathbb{N}^d$ with $|k|_2 > r$ and

$$\|g_1\|_\infty \leq \sum_k |c_k| \|\omega_k\|_\infty \leq \sqrt{\#\{k\colon c_k \neq 0\}} \leq \sqrt{2\pi_d r^d} = \sqrt{M}/2. \tag{3.40}$$

Here we used that $\|\omega_k\|_\infty \leq 1$, then we applied Cauchy-Schwarz inequality and noted that (c_k) is normalized to one in $\ell^2(\mathbb{N}^d)$, because $\|\omega_k\|_2 = 1$ for all $k \in \mathbb{N}^d$. The third inequality follows by the observation $\#\{k\colon c_k \neq 0\} \leq \pi_d(r+1)^d/2d \leq 2\pi_d r^d$ implied by $r \geq d$.

Let x be such that $g(x) \geq \sqrt{M}$. Then we have $\sqrt{M} \leq g(x) \leq |g_1(x)| + |a_2||g_2(x)|$. Using (3.40), we derive that $|a_2||g_2(x)| \geq \sqrt{M}/2$, whereby we have that $g(x) \leq 2|a_2||g_2(x)|$. This gives us the bound

$$\varepsilon \leq \|f\mathbb{1}_{\{f\geq M\}}\|_1 = \|g\mathbb{1}_{\{g\geq\sqrt{M}\}}\|_2^2 \leq 4|a_2|^2\|g_2\|_2^2 = 4|a_2|^2, \tag{3.41}$$

i.e., $|a_2|^2 \geq \varepsilon/4$. On the other hand,

$$\mathcal{I}(f) = \kappa\|\nabla g\|_2^2 \geq \kappa|a_2|^2\|\nabla g_2\|_2^2 \geq \kappa|a_2|^2\frac{\pi^2}{4}r^2. \tag{3.42}$$

where we used that $g_1 \perp g_2$ and that g_2 has no overlap with ω_k such that $|k|_2 \leq r$. By putting (3.41) and (3.42) together and noting that $\pi^2/16 \geq 1/2$, (3.35) for $R = 1$ follows. $\square$

Proof of (3). Let $\varepsilon \ll (2R)^d$ and consider $f \in \mathcal{F}_R$. Let $g = \sqrt{f}$ and define $g_\varepsilon = (g - \sqrt{\varepsilon})\mathbb{1}\{g \geq \sqrt{\varepsilon}\}$. By a straightforward calculation, $\|g_\varepsilon\|_2^2 \geq 1 - 2\varepsilon(2R)^d - 2\sqrt{\varepsilon(2R)^d}$. Let $f_\varepsilon = (g_\varepsilon/\|g_\varepsilon\|_2)^2$. Then $\mathcal{I}(f) \geq \|g_\varepsilon\|_2^2\mathcal{I}(f_\varepsilon)$, while $|\{f > \varepsilon\}| = |\{f_\varepsilon > 0\}|$. This implies that $\chi_R^\#(\varepsilon) \geq \chi_R(1 - O(\sqrt{\varepsilon}))$. Since $\chi_R^\#(\varepsilon) \leq \chi_R$, the proof is finished. $\square$

4. Proof of Theorems 1.2 and 1.3

We begin by deriving the logarithmic asymptotics for the moments of $u(t,0)$ as stated in Theorem 1.2. The proof is divided into two parts: we separately prove the lower bound and the upper bound. Whenever convenient, we write $\alpha(t)$ instead of α_t.

4.1 The lower bound

We translate the corresponding proof of [GK98] into the discrete setting. Let u denote the solution to (1.1), denoted by u^ξ in Section 3. Similarly, let u_R stand for u_R^ξ for any $R > 0$. Fix $p \in (0,\infty)$, $R > 0$, and consider the box $Q_{R\alpha(pt)} = [-R\alpha(pt), R\alpha(pt)]^d \cap \mathbb{Z}^d$. Note that $\#Q_{R\alpha(pt)} = e^{o(t\alpha_{pt}^{-2})}$ as $t \to \infty$. Recall that $u_{R\alpha(pt)}(t,\cdot) = 0$ outside $Q_{R\alpha(pt)}$ and that $(\cdot,\cdot)$ denotes the inner product in $\ell^2(\mathbb{Z}^d)$. Our first observation is the following.

Lemma 4.1 *As $t \to \infty$,*

$$\big\langle u(t,0)^p \big\rangle \geq e^{o(t\alpha_{pt}^{-2})}\big\langle (u_{R\alpha(pt)}(t,\cdot), 1)^p \big\rangle. \tag{4.1}$$

Proof. In the case $p \geq 1$, use the shift-invariance of $z \mapsto u(t,z)$, Jensen's inequality, and the monotonicity assertion (3.8) to obtain

$$\begin{aligned}\big\langle u(t,0)^p\big\rangle &= \Big\langle \frac{1}{\#Q_{R\alpha(pt)}} \sum_{z\in Q_{R\alpha(pt)}} u(t,z)^p \Big\rangle \\ &\geq \Big\langle \Big(\frac{1}{\#Q_{R\alpha(pt)}} \sum_{z\in Q_{R\alpha(pt)}} u(t,z)\Big)^p \Big\rangle \geq e^{o(t\alpha_{pt}^{-2})} \big\langle (u_{R\alpha(pt)}(t,\cdot),1)^p \big\rangle.\end{aligned} \tag{4.2}$$

In the case $p < 1$, instead of Jensen's inequality we apply

$$\sum_{i=1}^n x_i^p \geq \Big(\sum_{i=1}^n x_i\Big)^p, \qquad x_1,\dots,x_n \geq 0,\ n\in\mathbb{N}, \tag{4.3}$$

to deduce similarly as in (4.2) that

$$\begin{aligned}\big\langle u(t,0)^p\big\rangle &= e^{o(t\alpha_{pt}^{-2})} \Big\langle \sum_{z\in Q_{R\alpha(pt)}} u(t,z)^p \Big\rangle \\ &\geq e^{o(t\alpha_{pt}^{-2})} \Big\langle \Big(\sum_{z\in Q_{R\alpha(pt)}} u(t,z)\Big)^p \Big\rangle \geq e^{o(t\alpha_{pt}^{-2})} \big\langle (u_{R\alpha(pt)}(t,\cdot),1)^p \big\rangle. \quad \square\end{aligned} \tag{4.4}$$

The following Lemma 4.2 carries out the necessary large-deviation arguments for the case $p = 1$. Lemma 4.3 then reduces the proof of arbitrary p to the case $p = 1$. Recall the "finite-R" version χ_R of (1.14) defined in (3.30).

Lemma 4.2 *Let $R > 0$. Then for $t \to \infty$,*

$$-\chi_R + o(1) \leq \frac{\alpha_t^2}{t} \log \big\langle (u_{R\alpha(t)}(t,\cdot),1)\big\rangle \leq -\chi_{3R} + o(1), \tag{4.5}$$

$$\frac{\alpha_t^2}{t} \log \Big\langle \sum_k e^{t\lambda^{\mathrm{d},k}_{R\alpha(t)}(\xi)} \Big\rangle \leq -\chi_{3R} + o(1). \tag{4.6}$$

Lemma 4.3 *Let $R > 0$. Then for $t \to \infty$,*

$$\big\langle (u_{R\alpha(pt)}(t,\cdot),1)^p \big\rangle \geq e^{o(t\alpha_{pt}^{-2})} \big\langle (u_{R\alpha(pt)}(pt,\cdot),1)\big\rangle. \tag{4.7}$$

Lemmas 4.1, 4.2, and 4.3 make the proof of the lower bound immediate:

Proof of Theorem 1.2, lower bound. By combining (4.1), (4.7) and the left inequality in (4.5) for pt instead of t, we see that $(\alpha_{pt}^2/pt)\log\langle u(t,0)^p\rangle \geq -\chi_R + o(1)$. Since $\lim_{R\to\infty}\chi_R = \chi$, the left-hand side of (1.16), with "lim inf" instead of "lim", is bounded below by $-\chi$. By Proposition 3.1(1), χ positive, finite and non-zero. $\square$

The remainder of this subsection is devoted to the proof of the two lemmas.

Proof of Lemma 4.2. Recall the notation of Subsection 3.1. By taking the expectation over ξ (and using that ξ is an i.i.d. field) and recalling (3.7), we have for any $z \in Q_{R\alpha(t)}$ that

$$\begin{aligned}\big\langle u_{R\alpha(t)}(t,z)\big\rangle = \Big\langle \mathbb{E}_z\big[e^{(\xi,\ell_t)}1\{\tau_{R\alpha(t)}>t\}\big]\Big\rangle &= \mathbb{E}_z\Big[\prod_{y\in\mathbb{Z}^d}\big\langle e^{\ell_t(y)\xi(y)}\big\rangle 1\{\tau_{R\alpha(t)}>t\}\Big]\\ &= \mathbb{E}_z\Big[\exp\Big\{\sum_{y\in\mathbb{Z}^d} H\big(\ell_t(y)\big)\Big\}1\big\{\operatorname{supp}(\ell_t)\subset Q_{R\alpha(t)}\big\}\Big],\end{aligned}\tag{4.8}$$

Consider the scaled version $\bar\ell_t\colon \mathbb{R}^d\to[0,\infty)$ of the local times

$$\bar\ell_t(x) = \frac{\alpha_t^d}{t}\ell_t\big(\lfloor x\alpha_t\rfloor\big), \qquad x\in\mathbb{R}^d. \tag{4.9}$$

Let $\widetilde{\mathcal F}$ be the space of all non-negative Lebesgue almost everywhere continuous functions in $L^1(\mathbb{R}^d)$ with a bounded support. Clearly, $\mathcal F\subset\widetilde{\mathcal F}$ and $\bar\ell_t\in\widetilde{\mathcal F}$. Introduce the functional $\mathcal H^{(t)}\colon \widetilde{\mathcal F}\to[-\infty,0]$, assigning each $f\in\widetilde{\mathcal F}$ the value

$$\mathcal H^{(t)}(f) = \int_{\mathbb{R}^d}\widetilde H_t\big(f(x)\big)\,dx, \tag{4.10}$$

where we recalled (3.13). Substituting $\bar\ell_t$ and $\mathcal H^{(t)}$ into (4.8), we obtain

$$\big\langle (u_{R\alpha(t)}(t,\cdot),1)\big\rangle = \sum_{z\in Q_{R\alpha(t)}}\mathbb{E}_z\Big[\exp\Big\{\frac{t}{\alpha_t^2}\mathcal H^{(t)}\left(\bar\ell_t\right)\Big\}1\big\{\operatorname{supp}(\bar\ell_t)\subset[-R,R+\alpha_t^{-1}]^d\big\}\Big]. \tag{4.11}$$

Using shift-invariance and the fact that $\mathcal H^{(t)}(f)\le\mathcal H^{(t)}(f\wedge M)$ for any $M>0$, we have

$$\begin{aligned}\mathbb{E}_0\Big[\exp\Big\{\frac{t}{\alpha_t^2}\mathcal H^{(t)}\left(\bar\ell_t\right)\Big\}1\big\{\operatorname{supp}(\bar\ell_t)\subset[-R,R]^d\big\}1\{\bar\ell_t\le M\}\Big] &\le \big\langle (u_{R\alpha(t)}(t,\cdot),1)\big\rangle\\ &\le e^{o(t\alpha_t^{-2})}\,\mathbb{E}_0\Big[\exp\Big\{\frac{t}{\alpha_t^2}\mathcal H^{(t)}\left(\bar\ell_t\wedge M\right)\Big\}1\big\{\operatorname{supp}(\bar\ell_t)\subset[-3R,3R]^d\big\}\Big].\end{aligned}\tag{4.12}$$

It is well known that the family of scaled local times $(\bar\ell_t)_{t>0}$ satisfies a weak large-deviation principle on $L^1(\mathbb{R}^d)$ with rate $t\alpha_t^{-2}$ and rate function $\mathcal I$ defined in (1.10). This fact has been first derived by Donsker and Varadhan [DV79] for the discrete-time random walk; for the changes of the proof in the continuous time case we refer to Chapter 4 of the monograph by Deuschel and Stroock [DS89]. The large-deviation principle allows us to use Varadhan's integral lemma to convert both bounds in (4.12) into corresponding variational formulas. Note that, if both $\mathcal I$ and $\mathcal H$ are appropriately extended to $L^1([-R,R]^d)$, all infima (3.30), (3.31) and (3.32) can be taken over $f\in L^1([-R,R]^d)$ with the same result. In the sequel, we have to make a distinction between the cases $\gamma\in(0,1)$ and $\gamma=0$.

In the case $\gamma\in(0,1)$, our Scaling Assumption implies that, for every $M>0$, $f\mapsto\mathcal H(f)$ is continuous and $\mathcal H^{(t)}$ converges to $\mathcal H$ uniformly on the space of all measurable functions $[-R,R]^d\to[0,M]$ with L^∞ topology. Indeed, for any such function f and any $\varepsilon>0$, the integral (4.10) can be split into $\mathcal H^{(t)}(f1_{\{f>\varepsilon\}})$ and $\mathcal H^{(t)}(f1_{\{0<f\le\varepsilon\}})$. The former then converges uniformly to $\mathcal H(f1_{\{f>\varepsilon\}})$, while the latter can be bounded as

$$0\ge\mathcal H^{(t)}\big(f1_{\{0<f\le\varepsilon\}}\big)\ge\widetilde H_t(\varepsilon)\big|\{0<f\le\varepsilon\}\big|\ge(2R)^d\widetilde H_t(\varepsilon), \tag{4.13}$$

where we invoked the monotonicity of $y \mapsto \widetilde{H}_t(y)$. Taking $\varepsilon \downarrow 0$ proves that this part is negligible for $\mathcal{H}^{(t)}(f)$ and, if $t \to \infty$ is invoked before $\varepsilon \downarrow 0$, it also shows that $\mathcal{H}(f 1_{\{f>\varepsilon\}}) \to \mathcal{H}(f)$ uniformly in f as $\varepsilon \downarrow 0$. Having verified continuity, Varadhan's lemma (and $M \to \infty$) readily outputs the left inequality in (4.5), while on the right-hand side it yields a bound in terms of the quantity $\chi^\star_{3R}(M)$ defined in (3.31). By Proposition 3.1(2), $\chi^\star_{3R}(M)$ tends to χ_{3R} as $M \to \infty$, which proves the inequality on the right of (4.5).

In the case $\gamma = 0$, the lower bound goes along the same line, but we have to be more careful with (4.13), since $\lim_{\varepsilon\downarrow 0} \lim_{t\to\infty} \widetilde{H}_t(\varepsilon) \neq 0$ in this case. Let us estimate

$$\begin{aligned}\mathcal{H}^{(t)}(f) = \mathcal{H}^{(t)}\big(f 1_{\{0<f\le\varepsilon\}}\big) + \mathcal{H}^{(t)}\big(f 1_{\{f>\varepsilon\}}\big) &\ge \widetilde{H}_t(\varepsilon)\big|\{0<f\le\varepsilon\}\big| + \mathcal{H}^{(t)}\big(f 1_{\{f>\varepsilon\}}\big)\\ &\ge \mathcal{H}(f) - \big|\mathcal{H}^{(t)}(f 1_{\{f>\varepsilon\}}) - \mathcal{H}(f 1_{\{f>\varepsilon\}})\big| - (2R)^d\big|\widetilde{H}_t(\varepsilon) - \widetilde{H}(\varepsilon)\big|,\end{aligned} \quad (4.14)$$

where we invoked the explicit form of $f \mapsto \mathcal{H}(f)$. Since both absolute values on the right-hand side tend to 0 as $t \to \infty$ uniformly in $f \le M$, the lower bound in (4.5) follows again by Varadhan's lemma and limit $M \to \infty$. For the upper bound, the estimate and uniform limit $\mathcal{H}^{(t)}(f) \le \mathcal{H}^{(t)}(f 1_{\{f>\varepsilon\}}) \to \mathcal{H}(f 1_{\{f>\varepsilon\}})$ give us a bound in terms of the quantity $\chi^\#_{3R}(\varepsilon)$ defined in (3.32). By then M is irrelevant, so by invoking Proposition 3.1(3), the claim is proved by taking $\varepsilon \downarrow 0$.

It remains to prove (4.6). Recall the shorthand $\lambda_k = \lambda^{\mathrm{d},k}_{R\alpha(t)}(\xi)$. By (3.11), (3.9) and analogously to (4.8), we have

$$\Big\langle \sum_k e^{t\lambda_k} \Big\rangle = \sum_{z \in Q_{R\alpha(t)}} \big\langle p_{R\alpha(t)}(t,z,z) \big\rangle = \Big\langle \sum_{z \in Q_{R\alpha(t)}} \mathbb{E}_z\Big[e^{(\xi,\ell_t)} 1\{\tau_{R\alpha_t} > t\} 1\big\{X(t) = z\big\}\Big]\Big\rangle. \quad (4.15)$$

Noting that $1\{X(t) = z\} \le 1$, we thus have $\langle \sum_k e^{t\lambda_k} \rangle \le \langle (u_{R\alpha(t)}(t,\cdot), 1)\rangle$. With this in the hand, (4.6) directly follows by the right inequality in (4.5). □

Proof of Lemma 4.3. In the course of the proof, we use abbreviations $r = R\alpha(pt)$ and $\lambda_k = \lambda^{\mathrm{d},k}_r(\xi)$. Recall that $(\mathrm{e}_k)_k$ denotes an orthonormal basis in $\ell^2(Q_r)$ (with inner product $(\cdot,\cdot)_r$) consisting of the eigenfunctions of $\kappa\Delta^{\mathrm{d}} + \xi$ with Dirichlet boundary condition.

We first turn to the case $p \ge 1$. Use the Fourier expansion (3.12) and the inequality

$$\Big(\sum_{i=1}^n x_i\Big)^p \ge \sum_{i=1}^n x_i^p, \qquad x_1, \dots, x_n \ge 0,\, n \in \mathbb{N}, \quad (4.16)$$

to obtain

$$\big\langle (u_r(t,\cdot),1)^p \big\rangle = \Big\langle \Big(\sum_k e^{t\lambda_k} \,(\mathrm{e}_k,1)_r^2 \Big)^p \Big\rangle \ge \Big\langle \sum_k e^{pt\lambda_k}\,(\mathrm{e}_k,1)_r^{2p} \Big\rangle. \quad (4.17)$$

By Jensen's inequality for the probability measure $(l, d\xi) \mapsto \langle \sum_k e^{pt\lambda_k}\rangle^{-1} e^{pt\lambda_l}\mathrm{Prob}(d\xi)$,

$$\begin{aligned}\text{r.h.s. of (4.17)} &\ge \bigg(\frac{\langle \sum_k e^{pt\lambda_k} (\mathrm{e}_k,1)_r^2 \rangle}{\langle \sum_k e^{pt\lambda_k}\rangle} \bigg)^p \Big\langle \sum_k e^{pt\lambda_k} \Big\rangle \\ &\ge e^{o(t\alpha_{pt}^{-2})} \Big\langle \sum_k e^{pt\lambda_k}\,(\mathrm{e}_k,1)_r^2 \Big\rangle = e^{o(t\alpha_{pt}^{-2})} \big\langle (u_r(pt,\cdot),1)\big\rangle,\end{aligned} \quad (4.18)$$

where we recalled from the end of the proof of Lemma 4.2 that $\langle\sum_k e^{pt\lambda_k}\rangle \le \langle (u_r(pt,\cdot),1)\rangle = \langle\sum_k e^{pt\lambda_k}(\mathrm{e}_k,1)_r^2\rangle$, inserted $1 \ge e^{o(t\alpha_{pt}^{-2})}(\mathrm{e}_k,1)_r^2$, and applied (3.12).

In the case $p\in(0,1)$, we apply Jensen's inequality as follows:

$$\big\langle (u_r(t,\cdot),1)^p\big\rangle = (1,1)_r^p\Big\langle\Big(\sum_k e^{t\lambda_k}\frac{(\mathrm{e}_k,1)_r^2}{(1,1)_r}\Big)^p\Big\rangle \ge (1,1)_r^p\Big\langle\sum_k e^{pt\lambda_k}\frac{(\mathrm{e}_k,1)_r^2}{(1,1)_r}\Big\rangle. \tag{4.19}$$

Invoking that $(1,1)_r = e^{o(t\alpha_{pt}^{-2})}$, the proof is finished by recalling (3.12) once again. □

4.2 The upper bound

Recall that Q_R denotes the discrete box $[-R,R]^d\cap\mathbb{Z}^d$. We abbreviate $r(t)=t\log t$ for $t>0$. For $z\in\mathbb{Z}^d$ and $R>0$, we denote by $\lambda^{\mathrm{d}}_{z;R}(V)$ the principal eigenvalue of the operator $\kappa\Delta^{\mathrm{d}}+V$ with Dirichlet boundary conditions in the *shifted* box $z+Q_R$. The main ingredient in the proof of the upper bound in Theorem 1.2 is (the following) Proposition 4.4, which provides an estimate of $u(t,0)$ in terms of the maximal principal eigenvalue of $\kappa\Delta^{\mathrm{d}}+V$ in small subboxes ("microboxes") of the "macrobox" $Q_{r(t)}$.

Proposition 4.4 *Let $B_R(t)=Q_{r(t)+2\lfloor R\rfloor}$. Then there is a constant $C=C(d,\kappa)>0$ such that, for any $R,t>C$ and any potential $V\colon\mathbb{Z}^d\to[-\infty,0]$,*

$$u^V(t,0)\le e^{-t}+e^{Ct/R^2}\big(3r(t)\big)^d\exp\Big\{t\max_{z\in B_R(t)}\lambda^{\mathrm{d}}_{z;2R}(V)\Big\}. \tag{4.20}$$

By Proposition 4.4 and inequality (4.6), the upper bound in Theorem 1.2 is now easy:

Proof of Theorem 1.2, upper bound. Let $p\in(0,\infty)$. First, notice that the second term in (4.20) can be estimated in terms of a sum:

$$\exp\Big\{t\max_{z\in B_R(t)}\lambda^{\mathrm{d}}_{z;2R}(V)\Big\}\le\sum_{z\in B_R(t)}e^{t\lambda^{\mathrm{d}}_{z;2R}(V)}. \tag{4.21}$$

Thus, applying (4.20) to $u(t,0)$ (i.e., for $V=\xi$) with R replaced by $R\alpha(pt)$ for some fixed $R>0$, raising both sides to the p-th power, and using (4.21) we get

$$u(t,0)^p\le 2^p\max\Big\{e^{-pt},\,e^{Cpt/(R^2\alpha(pt)^2)}\big(3r(t)\big)^{pd}\sum_{z\in B_{R\alpha(pt)}(t)}e^{pt\lambda^{\mathrm{d}}_{z;2R\alpha(pt)}(\xi)}\Big\}. \tag{4.22}$$

Next we take the expectation w.r.t. ξ and note that, by the shift-invariance of ξ, the distribution of $\lambda^{\mathrm{d}}_{z;2R\alpha(pt)}(\xi)$ does not depend on $z\in\mathbb{Z}^d$. Take logarithm, multiply by $\alpha_{pt}^2/(pt)$ and let $t\to\infty$. Then we have that

$$\limsup_{t\to\infty}\frac{\alpha_{pt}^2}{pt}\log\big\langle u(t,0)^p\big\rangle\le\frac{C}{R^2}+\limsup_{t\to\infty}\frac{\alpha_{pt}^2}{pt}\log\big\langle\exp\{pt\lambda^{\mathrm{d}}_{2R\alpha(pt)}(\xi)\}\big\rangle, \tag{4.23}$$

where we also used that e^{-pt}, $r(t)^{pd}$, and $\#B_{R\alpha(pt)}(t)$ are all $e^{o(t\alpha_{pt}^{-2})}$ as $t\to\infty$. Since

$$\exp\big\{pt\lambda^{\mathrm{d}}_{R\alpha(pt)}(\xi)\big\}\le\sum_k\exp\big\{pt\lambda^{\mathrm{d},k}_{R\alpha(pt)}(\xi)\big\}, \tag{4.24}$$

(4.6) for pt instead of t implies that the second term on the right-hand side of (4.23) is bounded by $-\chi_{6R}$. The upper bound in Theorem 1.2 then follows by letting $R\to\infty$. □

Now we can turn to the proof of Proposition 4.4. We begin by showing that $u^V(t,0)$ is very close to the solution $u^V_{r(t)}(t,0)$ of the initial-boundary problem (3.4), whenever the size $r(t) = t\log t$ of the "macrobox" $Q_{r(t)}$ is large enough.

Lemma 4.5 *For sufficiently large $t > 0$,*

$$u^V(t,0) \le e^{-t} + u^V_{r(t)}(t,0). \tag{4.25}$$

Proof. It is immediate from (3.2) and (3.5) with $r = r(t)$ that

$$u^V(t,0) - u^V_{r(t)}(t,0) = \mathbb{E}_0\left[\exp\left\{\int_0^t V\big(X(s)\big)\,ds\right\} 1\{\tau_{r(t)} \le t\}\right]. \tag{4.26}$$

According to Lemma 2.5(a) in [GM98], we have, for every $r > 0$,

$$\mathbb{P}_0(\tau_r \le t) \le 2^{d+1}\exp\left\{-r\left(\log\frac{r}{d\kappa t} - 1\right)\right\}. \tag{4.27}$$

Using this for $r = r(t) = t\log t$ in (4.26), we see that, for sufficiently large t (depending only on d and κ), the right-hand side of (4.26) is no more than e^{-t}. $\square$

The crux of our proof of Proposition 4.4 is that the principal eigenvalue in a box Q_r of size r can be bounded by the maximal principal eigenvalue in "microboxes" $z + Q_R$ contained in Q_r, at the cost of changing the potential slightly. This will later allow us to move the t-dependence of the principal eigenvalue from the *size* of $Q_{r(t)}$ to the *number* of "microboxes". The following lemma is a discrete version of Proposition 1 of [GK98] and is based on ideas from [GM96]. However, for the sake of completeness, no familiarity with [GK98] is assumed.

Lemma 4.6 *There is a number $C > 0$ such that for every integer R, there is a function $\Phi_R\colon \mathbb{Z}^d \to [0,\infty)$ with the following properties:*
(1) Φ_R is $2R$-periodic in every component.
(2) $\|\Phi_R\|_\infty \le C/R^2$.
(3) For any potential $V\colon \mathbb{Z}^d \to [-\infty,0]$ and any $r > R$,

$$\lambda^{\mathrm{d}}_r(V - \Phi_R) \le \max_{z\in Q_{r+2R}} \lambda^{\mathrm{d}}_{z;2R}(V). \tag{4.28}$$

Proof. The idea is to construct a partition of unity

$$\sum_{k\in\mathbb{Z}^d} \eta_k^2(z) = 1, \qquad z \in \mathbb{Z}^d, \tag{4.29}$$

where $\eta_k(z) = \eta(z - 2Rk)$ with

$$\eta\colon \mathbb{Z}^d \to [0,1] \text{ such that } \eta \equiv 1 \text{ on } Q_{R/2},\ \operatorname{supp}(\eta) \subset Q_{3R/2}. \tag{4.30}$$

Then we put

$$\Phi_R(z) = \kappa \sum_{k\in\mathbb{Z}^d} \big|\nabla\eta_k(z)\big|^2, \qquad z \in \mathbb{Z}^d, \tag{4.31}$$

where ∇ is the discrete gradient. Obviously, Φ_R is $2R$-periodic in every component. The construction of η such that Φ_R satisfies (2) is given at the end of this proof.

Assuming the existence of the above partition of unity, we turn to the proof of (4.28). Recall the Rayleigh-Ritz formula (3.10), which can be shortened as $\lambda_r^{\mathrm{d}}(V) = \sup G^V(g)$, where

$$G^V(g) = \sum_{z\in\mathbb{Z}^d} \bigl(-\kappa|\nabla g(z)|^2 + V(z)g^2(z)\bigr), \tag{4.32}$$

and where the supremum is over normalized $g \in \ell^2(\mathbb{Z}^d)$ with support in Q_r. Let g be such a function, and define $g_k(z) = g(z)\eta_k(z)$ for $k, z \in \mathbb{Z}^d$. Note that, according to (4.29) and (4.30), we have $\sum_k \|g_k\|_2^2 = 1$ and $\operatorname{supp}(g_k) \subset 2kR + Q_{3R/2}$. Then we claim that

$$G^{V-\Phi_R}(g) = \sum_{k\in\mathbb{Z}^d} \|g_k\|_2^2\, G^V\Bigl(\frac{g_k}{\|g_k\|_2}\Bigr). \tag{4.33}$$

Indeed, invoking (4.29) and (4.31), it is easily seen that

$$\kappa \sum_{k\in\mathbb{Z}^d} |\nabla g_k|^2 = \kappa \sum_{k\in\mathbb{Z}^d} \bigl(g^2|\nabla\eta_k|^2 + \tfrac12\nabla\eta_k^2\cdot\nabla g^2 + \eta_k^2|\nabla g|^2\bigr) = g^2\Phi_R + \kappa|\nabla g|^2. \tag{4.34}$$

Therefore,

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \|g_k\|_2^2\, G^V\Bigl(\frac{g_k}{\|g_k\|_2}\Bigr) &= \sum_{k\in\mathbb{Z}^d} G^V(g_k) = \sum_{z\in\mathbb{Z}^d}\sum_{k\in\mathbb{Z}^d}\Bigl[-\kappa\bigl|\nabla g_k(z)\bigr|^2 + V(z)g_k^2(z)\Bigr]\\ &= \sum_{z\in\mathbb{Z}^d}\Bigl[-\kappa\bigl|\nabla g(z)\bigr|^2 + \bigl(V(z)-\Phi_R(z)\bigr)g^2(z)\Bigr] = G^{V-\Phi_R}(g),\end{aligned} \tag{4.35}$$

which is (4.33). Since the support of g_k is contained in $2kR + Q_{3R/2}$, the Rayleigh-Ritz formula yields that $G^V(g_k/\|g_k\|_2) \le \lambda^{\mathrm{d}}_{2kR;3R/2}(V) \le \lambda^{\mathrm{d}}_{2kR;2R}(V)$ whenever $\|g_k\|_2 \neq 0$ (which requires, in particular, that $2R|k| - \frac32 R \le r$). Estimating these eigenvalues by their maximum and taking into account that $\sum_{k\in\mathbb{Z}^d}\|g_k\|_2^2 = \|g\|_2^2 = 1$, we find that the right-hand side of (4.33) does not exceed the right-hand side of (4.28). By passing to the supremum over g on the left-hand side of (4.33), we arrive at the claim (4.28).

For the proof to be complete, it remains to construct the functions η and Φ_R with the properties (4.29) and (4.30) and such that $\|\Phi_R\|_\infty \le C/R^2$ for some $C > 0$. First, it is easily checked that the ansatz

$$\eta(z) = \prod_{i=1}^d \zeta(z_i), \qquad z = (z_1,\dots,z_d)\in\mathbb{Z}^d, \tag{4.36}$$

reduces the construction of η to the case $d = 1$ (with η replaced by ζ). In order to define $z \mapsto \zeta(z)$, let $\varphi\colon \mathbb{R}\to[0,1]$ be such that both $\sqrt{\varphi}$ and $\sqrt{1-\varphi}$ are smooth, $\varphi \equiv 0$ on $(-\infty,-1]$ and $\varphi\equiv 1$ on $[1,\infty)$ and $\varphi(-x) = 1-\varphi(x)$ for all $x\in\mathbb{R}$. Then we put

$$\zeta(z) = \sqrt{\varphi\bigl(\tfrac12 + \tfrac{z}{R}\bigr)\bigl[1-\varphi\bigl(-\tfrac12+\tfrac{z}{R}\bigr)\bigr]}, \qquad z\in\mathbb{Z}. \tag{4.37}$$

It is straightforward to verify that the functions $\zeta_k^2(z) = \zeta^2(z+2Rk)$ with $k\in\mathbb{Z}$ form a partition of unity on $\mathbb{R}$. Moreover, as follows by a direct computation, $\sup_{z\in\mathbb{Z}}\sum_k|\nabla\zeta_k(z)|^2 \le 4\|(\sqrt{\varphi})'\|_\infty^2 R^{-2}$, which means that (2) is satisfied with $C = 4d\|(\sqrt{\varphi})'\|_\infty^2$. This finishes the construction and also the proof. □

Proof of Proposition 4.4. Having all the prerequisites, the proof is easily completed. First,

$$\int_0^t V\big(X(s)\big)\,ds \le t\frac{C}{R^2} + \int_0^t (V-\Phi_R)\big(X(s)\big)\,ds, \qquad t>0. \tag{4.38}$$

by Lemma 4.6(2). Therefore, combining (3.2) with Lemma 4.5, we have that

$$u^V(t,0) \le e^{-t} + e^{tC/R^2} u_{r(t)}^{V-\Phi_R}(t,0) \tag{4.39}$$

whenever t is large enough. Invoking also the Fourier expansion (3.12) w.r.t. the eigenfunctions of $\kappa\Delta^{\mathrm{d}} + V - \Phi_R$ in $\ell^2(Q_{r(t)})$ and the fact that $(1,1)_{r(t)} = \#Q_{r(t)}$, we find that

$$u_{r(t)}^{V-\Phi_R}(t,0) \le \sum_{z\in Q_{r(t)}} u_{r(t)}^{V-\Phi_R}(t,z) \le \#Q_{r(t)} \exp\big\{t\lambda_{r(t)}^{\mathrm{d}}(V-\Phi_R)\big\}. \tag{4.40}$$

Now apply Lemma 4.6 for $r = r(t) = t\log t$ to finish the proof. □

4.3 Proof of Lifshitz tails

Let ν_R denote the empirical measure on the spectrum of $\mathfrak{H}_R$, i.e.,

$$\nu_R = \frac{1}{\#Q_R}\sum_k \delta_{\{-\lambda_k\}}, \tag{4.41}$$

where $\lambda_k = \lambda_R^{\mathrm{d},k}(\xi) = -E_k$ denotes the eigenvalues of $-\mathfrak{H}_R$. Note that ν_R has total mass at most 1, because the dimension of the underlying Hilbert space is bounded by $\#Q_R$. Due to (1.2), ν_R is supported on $[0,\infty)$. Moreover, $N_R(E)$ in (1.18) is precisely $\#Q_R\,\nu_R([0,E])$, for any $E\in[0,\infty)$. Let $\mathcal{L}(\nu_R,t)$ be the Laplace transform of ν_R evaluated at $t\ge 0$,

$$\mathcal{L}(\nu_R,t) = \int \nu_R(d\lambda)\,e^{-\lambda t} = \frac{1}{\#Q_R}\sum_k e^{t\lambda_k}. \tag{4.42}$$

Adapting Theorem VI.1.1. in [CL90] to our discrete setting, the existence of the limit (1.19) is proved by establishing the a.s. convergence of ν_R to some non-random ν, which in turn is done by proving that $\mathcal{L}(\nu_R,\cdot)$ has a.s. a non-random limit. In our case, the argument is so short that we find it convenient to reproduce it here.

Invoking (3.11) and (3.9) for $V=\xi$, we have from (4.42) that

$$\mathcal{L}(\nu_R,t) = \frac{1}{\#Q_R}\sum_{z\in Q_R}\mathbb{E}_z\Big\{\exp\Big[\int_0^t \xi\big(X(s)\big)ds\Big]\mathbb{1}\{\tau_R>t\}\mathbb{1}\big\{X(t)=z\big\}\Big\}. \tag{4.43}$$

Next, writing $\mathbb{1}\{\tau_R>t\} = 1-\mathbb{1}\{\tau_R\le t\}$ we arrive at two terms, the second of which tends to zero as $R\to\infty$ for any fixed t by the estimate

$$0 \le \frac{1}{\#Q_R}\sum_{z\in Q_R}\mathbb{E}_z\Big\{e^{\int_0^t \xi(X(s))ds}\mathbb{1}\{\tau_R\le t\}\mathbb{1}\{X(t)=z\}\Big\} \le \frac{1}{\#Q_R}\sum_{z\in Q_R}\mathbb{P}_z(\tau_R\le t), \tag{4.44}$$

where we used that $\xi\le 0$. Indeed, $\mathbb{P}_z(\tau_R\le t)\le \mathbb{P}_0(\tau_{R(z)}\le t)$ with $R(z)=\mathrm{dist}(z,Q_R^{\mathrm{c}})$, which by (4.27) means that $\mathbb{P}_z(\tau_R\le t)$ decays exponentially with $\mathrm{dist}(z,Q_R^{\mathrm{c}})$. Thus, $\mathcal{L}(\nu_R,t)$ is asymptotically given by the right-hand side of (4.43) with $\mathbb{1}\{\tau_R>t\}$ omitted. But then the right-hand side is the average of an L^1 function over the translates in the box Q_R, so by the Ergodic Theorem,

$$\lim_{R\to\infty} \mathcal{L}(\nu_R, t) = \Big\langle \mathbb{E}_0\Big\{\exp\Big[\int_0^t \xi\big(X(s)\big)ds\Big]\mathbb{1}\{X(t)=0\}\Big\}\Big\rangle \tag{4.45}$$

ξ-almost surely for every fixed $t \ge 0$ (the exceptional null set is *a priori* t-dependent). Both the right-hand side of (4.45) and $\mathcal{L}(\nu_R, t)$ for every R are continuous and decreasing in t. Consequently, with probability one (4.45) holds for all $t \ge 0$.

The right-hand side of (4.45) inherits the complete monotonicity property from $\mathcal{L}(\nu_R, t)$; it thus equals $\mathcal{L}(\nu, t)$ where ν is some measure supported in $[0,\infty)$. Moreover, this also implies that $\nu_R \to \nu$ weakly as $R \to \infty$. In particular, we have $n(E) = \nu([0,E])$ for any $E \ge 0$.

Proof of Theorem 1.3. From (4.45) we immediately have

$$e^{o(t/\alpha_t^2)}\big\langle e^{t\lambda^{\rm d}_{R\alpha(t)}}\big\rangle \le \mathcal{L}(\nu,t) \le \langle u(t,0)\rangle, \qquad R \ge 0, \tag{4.46}$$

where $\lambda^{\rm d}_{R\alpha(t)}$ is as in (3.10). Here, for the upper bound we simply neglected $\mathbb{1}\{X(t)=0\}$ in (4.45), whereas for the lower bound we first wrote (4.45) as a normalized sum of the right-hand side of (4.45) with the walk starting and ending at all possible $z \in Q_{R\alpha_t}$, and then inserted $\mathbb{1}\{\operatorname{supp}(\ell_t) \subset Q_{R\alpha(t)}\}$, applied (3.9) and (3.11), and then recalled (4.24). The factor $e^{o(t/\alpha_t^2)}$ comes from the normalization by $\#Q_{R\alpha(t)}$ in the first step. Using subsequently (4.23) for $p=1$, the left-hand side of (4.46) is further bounded from below by $e^{(t/\alpha_t^2)(-4C/R^2+o(1))}\langle u(t,0)\rangle$. Then Theorem 1.2 and the limit $R\to\infty$ enable us to conclude that

$$\lim_{t\to\infty} \frac{\alpha_t^2}{t}\log\mathcal{L}(\nu,t) = -\chi. \tag{4.47}$$

In the remainder of the proof, we have to convert this statement into the appropriate limit for the IDS. This is a standard problem in the theory of Laplace transforms and, indeed, there are theorems that can after some work be applied (e.g., de Bruijn's Tauberian Theorem, see Bingham, Goldie and Teugels [BGT87]). However, for the sake of both completeness and convenience we provide an independent proof below.

Suppose that H is the γ-class. We begin with an upper bound. Clearly,

$$\mathcal{L}(\nu,t) \ge e^{-tE}n(E) \quad \text{for any } t, E \ge 0. \tag{4.48}$$

Let $t_E = \alpha^{-1}(\sqrt{(1-2\nu)\chi\, E^{-1}})$ and insert this for t in the previous expression. The result is

$$\log n(E) \le t_E E + \log\mathcal{L}(\nu,t_E) = -t_E E\tfrac{2\nu}{1-2\nu}\big(1+o(1)\big), \qquad E\downarrow 0, \tag{4.49}$$

where we applied (4.47) and the definition of t_E. In order to finish the upper bound, we first remark that from the first assertion in (1.7) it can be deduced that

$$\lim_{E\downarrow 0}\frac{t_E}{\alpha^{-1}(E^{-\frac12})} = \big[(1-2\nu)\chi\big]^{-\frac{1}{2\nu}}. \tag{4.50}$$

Indeed, define $t'_E = \alpha^{-1}(E^{-1/2})$ and consider the quantity $p_E = t_E/t'_E$. Clearly,

$$\alpha(p_E t'_E) = \alpha(t'_E)\sqrt{(1-2\nu)\chi}. \tag{4.51}$$

Let $\widetilde{p} = [(1-2\nu)\chi]^{-1/(2\nu)}$. Since $t'_E \to \infty$ as $E\downarrow 0$, there is no $\varepsilon > 0$ such that $p_E \ge \widetilde{p}+\varepsilon$ for infinitely many E with an accumulation point at zero, because otherwise the left-hand side (4.51) would, by (1.7), eventually exceed the right-hand side. Similarly we prove that $\liminf_{E\downarrow 0} p_E$ cannot be smaller than $\widetilde{p}-\varepsilon$. Therefore, $p_E \to \widetilde{p}$ as $E\downarrow 0$, which is (4.50).

Using (4.50), we have from (4.49) that

$$\limsup_{E\downarrow 0} \frac{\log n(E)}{E\alpha^{-1}(E^{-\frac12})} \le -\frac{2\nu}{1-2\nu}\big[(1-2\nu)\chi\big]^{-\frac{1}{2\nu}}. \tag{4.52}$$

The lower bound is slightly harder, but quite standard. First, introduce the probability measure on $[0,\infty)$ defined by

$$\mu_E(d\lambda) = \frac{e^{-t_E\lambda}}{\mathcal{L}(\nu,t_E)}\nu(d\lambda), \quad E \ge 0. \tag{4.53}$$

We claim that, for any $\varepsilon > 0$, all mass of μ_E gets eventually concentrated inside the interval $[E-\varepsilon E, E+\varepsilon E]$ as $E\downarrow 0$. Indeed, for any $0 \le t < t_E$ we have

$$\mu_E\big((E+\varepsilon E,\infty)\big) \le \mathcal{L}(\nu,t_E)^{-1}\int_{E+\varepsilon E}^\infty \nu(d\lambda)\, e^{-t_E\lambda + t(\lambda - E - \varepsilon E)} \le e^{-t\varepsilon E}\frac{\mathcal{L}(\nu,t_E-t)}{\mathcal{L}(\nu,t_E)}e^{-tE}. \tag{4.54}$$

Pick $0<\delta<1$ and set $t=\delta t_E$. Then we have

$$\mu_E\big((E+\varepsilon E,\infty)\big) \le \exp\Big\{-\delta\varepsilon t_E E - \delta t_E E - \chi \tfrac{t_E}{\alpha(t_E)^2}\big[(1-\delta)^{1-2\nu} - 1 + o(1)\big]\Big\}, \tag{4.55}$$

where we again used (4.47) and (1.7). Applying that $(1-\delta)^{1-2\nu}-1 = -\delta(1-2\nu)+o(\delta)$, using

$$t_E E - \chi(1-2\nu)\tfrac{t_E}{\alpha(t_E)^2} = 0, \tag{4.56}$$

and noting that $\alpha(t_E)^{-2} = O(E)$, we have

$$\mu_E\big((E+\varepsilon E,\infty)\big) \le \exp\big[-t_E E\big(\delta\varepsilon + o(\delta)\big)\big]. \tag{4.57}$$

Choosing δ small enough, the right-hand side vanishes as $E\downarrow 0$. Similarly we proceed in the case $[0, E-\varepsilon E)$.

Now we can finish the lower bound on Lifshitz tails. Indeed, using Jensen's inequality

$$\begin{aligned}\nu\big([0,E+\varepsilon E]\big) &= \mathcal{L}(\nu,t_E)\int_0^{E+\varepsilon E}\mu_E(d\lambda)\, e^{t_E\lambda}\\ &\ge \mathcal{L}(\nu,t_E)\mu_E\big([0,E+\varepsilon E]\big)\exp\Big\{\tfrac{t_E}{\mu_E([0,E+\varepsilon E])}\int_0^{E+\varepsilon E}\mu_E(d\lambda)\,\lambda\Big\}.\end{aligned} \tag{4.58}$$

But $\int_0^\infty \mu_E(d\lambda)\lambda$ tends to E, by what we have proved about the concentration of the mass of μ_E (note that (4.57) and the similar bound for $[0,E-\varepsilon E)$ are both exponential in ε) and, by the same token, so does $\int_0^{E+\varepsilon E}\mu_E(d\lambda)\lambda$. By putting all this together, dividing both sides of (4.58) by $E'\alpha^{-1}((E')^{-1/2})$ with $E' = E+\varepsilon E$, interpreting E' as a new variable tending to 0 as $E\downarrow 0$, and invoking (4.49) and the subsequent computation, we get

$$\liminf_{E\downarrow 0}\frac{\log n(E)}{E\alpha^{-1}(E^{-\frac12})} \ge -(1+\varepsilon)^{\frac{1-2\nu}{2\nu}}\frac{2\nu}{1-2\nu}\big[(1-2\nu)\chi\big]^{-\frac{1}{2\nu}}, \tag{4.59}$$

where we also used that $t_E/t_{E+\varepsilon E} \to (1+\varepsilon)^{1/(2\nu)}$. Since ε was arbitrary, the claim is finished by taking $\varepsilon\to 0$. $\square$

5. Proof of Theorem 1.5

Again, we divide the proof in two parts: the upper bound and the lower bound.

5.1 The upper bound

Proof of Theorem 1.5, upper bound. Let $r(t) = t \log t$ and let $K \in (0,\infty)$. We want to apply Proposition 4.4 with the random potential $V = \xi$ and with R replaced by $R\alpha(Kb_t)$ for some fixed $R, K > 0$. (Later we shall let $R \to \infty$ and pick K appropriately.)

Recall the definition of $B_R(t)$ in Proposition 4.4 and abbreviate $B(t) = B_{R\alpha(Kb_t)}(t)$. Take logarithms in (4.20), multiply by $\alpha_{b_t}^2/t$ and use (1.7) to obtain

$$\limsup_{t\to\infty} \frac{\alpha_{b_t}^2}{t} \log u(t,0) \le \frac{C}{K^{2\nu}R^2} + \limsup_{t\to\infty} \Big[\alpha_{b_t}^2 \max_{z\in B(t)} \lambda^{\mathrm{d}}_{z;2R\alpha(Kb_t)}(\xi)\Big], \tag{5.1}$$

almost surely w.r.t. the field ξ. Thus, we just need to evaluate the almost sure behavior of the maximum of the random variables on the right-hand side. This will be done by showing that

$$\limsup_{R\to\infty} \limsup_{t\to\infty} \Big[\alpha_{b_t}^2 \max_{z\in B(t)} \lambda^{\mathrm{d}}_{z;2R\alpha(Kb_t)}(\xi)\Big] \le -\widetilde{\chi} \tag{5.2}$$

almost surely w.r.t. the field ξ, provided $K > 0$ is chosen appropriately.

For any $t > 0$, let $(\lambda_i(t))_{i=1,\dots,N(t)}$ be an enumeration of the random variables $\lambda^{\mathrm{d}}_{z;2R\alpha(Kb_t)}(\xi)$ with $z \in B(t)$. Note that $N(t) \le 3^d t^d (\log t)^d$ for t large. Clearly, $(\lambda_i(t))$ are identically distributed but not independent. By (4.6), the tail of their distribution is bounded by

$$\limsup_{t\to\infty} \frac{\alpha_{b_t}^2}{b_t} \log\big\langle \exp\{Kb_t \lambda^{\mathrm{d}}_{2R\alpha(Kb_t)}(\xi)\}\big\rangle \le -K^{1-2\nu}\chi_{6R}, \qquad K, R > 0, \tag{5.3}$$

where χ_R is defined in (3.30).

The assertion (5.2) will be proved if we can verify that, with probability one,

$$\max_{i=1,\dots,N(t)} \lambda_i(t) \le -\frac{\widetilde{\chi} - \varepsilon}{\alpha^2(b_t)}\big(1 + o(1)\big), \quad t \to \infty, \tag{5.4}$$

for any $\varepsilon > 0$ and sufficiently large $R > 0$, as $t \to \infty$. To that end, note first that the left-hand side of (5.4) is increasing in t since the maps $t \mapsto \alpha(Kb_t)$, $R \mapsto \lambda^{\mathrm{d}}_R(\xi)$ and $t \mapsto r(t)$ are all increasing. As a consequence, it suffices to prove the assertion (5.4) only for $t \in \{e^n : n \in \mathbb{N}\}$, because also $\alpha(b_s)^{-2} - \alpha(b_{e^n})^{-2} = o(\alpha(b_{e^n})^{-2})$ as $n \to \infty$ for any $e^{n-1} \le s < e^n$. Let

$$p_n = \mathrm{Prob}\left(\max_{i=1,\dots,N(e^n)} \lambda_i(e^n) \ge -\frac{\widetilde{\chi} - \varepsilon}{\alpha^2(b_{e^n})}\right). \tag{5.5}$$

Abbreviating $t = e^n$ and recalling $b_t \alpha_{b_t}^{-2} = \log t = n$, the exponential Chebyshev inequality and (5.3) allow us to write for any $K > 0$ and n large that

$$\begin{aligned} p_n &\le N(e^n)\, \mathrm{Prob}\Big(e^{Kb_t\lambda_1(e^n)} \ge e^{-Kb_t\alpha^{-2}(b_t)(\widetilde{\chi}-\varepsilon)}\Big) \\ &\le 3^d n^d e^{nd} \exp\big\{Kb_t\alpha^{-2}(b_t)(\widetilde{\chi} - \varepsilon)\big\}\Big\langle e^{Kb_t\lambda^{\mathrm{d}}_{2R\alpha(Kb_t)}(\xi)}\Big\rangle \\ &= \exp\Big\{n\big[-\varepsilon K + d + K\widetilde{\chi} - K^{1-2\nu}\chi_{6R} + o(1)\big]\Big\}. \end{aligned} \tag{5.6}$$

Now set K to be a positive solution to $K\widetilde{\chi} - K^{1-2\nu}\chi = -d$. As follows by Proposition 1.4, $K = [(1-2\nu)\chi/\widetilde{\chi}\,]^{1/(2\nu)}$ will do. Substituting this into (5.6), we obtain

$$p_n \le \exp\bigl\{-n\bigl[\varepsilon K - K^{1-2\nu}(\chi - \chi_{6R}) + o(1)\bigr]\bigr\}, \tag{5.7}$$

which is clearly summable on n provided R is sufficiently large. The Borel-Cantelli lemma then guarantees the validity of (5.4), which in turn proves (5.2). The limit $R \to \infty$ then yields the upper bound in Theorem 1.5. $\square$

5.2 The lower bound

Recall the notation of Subsection 3.1. Let $Q_{\gamma_t} = [-\gamma_t, \gamma_t]^d \cap \mathbb{Z}^d$ denote the "macrobox", where γ_t is the time scale defined by

$$\gamma_t = \frac{t}{\alpha_{b_t}^3}, \qquad t > 0. \tag{5.8}$$

We assume without loss of generality that $t \mapsto \gamma_t$ is strictly increasing. Since we assumed $\mathrm{Prob}(\xi(0) > -\infty) > p_{\rm c}(d)$ for $d \ge 2$, there is a $K \in (0,\infty)$ such that $\mathrm{Prob}(\xi(0) \ge -K) > p_{\rm c}(d)$. Consequently, $\{z \in \mathbb{Z}^d \colon \xi(z) \ge -K\}$ contains almost-surely a unique infinite cluster $\mathcal{C}_\infty^*$.

Given a $\psi \in C^-([-R,R]^d)$, let $\psi_t \colon \mathbb{Z}^d \to (-\infty, 0]$ be the function $\psi_t(\cdot) = \psi(\cdot/\alpha(b_t))/\alpha(b_t)^2$. Suppose H is in the γ-class. Abbreviate

$$Q^{(t)} = \begin{cases} Q_{R\alpha(b_t)} & \text{if } \gamma \ne 0,\\ Q_{R\alpha(b_t)} \cap \operatorname{supp}\psi_t & \text{if } \gamma = 0. \end{cases} \tag{5.9}$$

The main point of the proof of the lower bound in Theorem 1.5 is the existence of a microbox of diameter of order α_{b_t} in Q_{γ_t} (which is contained in $\mathcal{C}_\infty^*$ for $d \ge 2$) where the field is bounded from below by ψ_t:

Proposition 5.1 *Let $R > 0$ and fix a function $\psi \in C^-(R)$ satisfying $\mathcal{L}_R(\psi) < d$. Let $\varepsilon > 0$ and let H is in the γ-class with $\gamma \in [0,1)$. Then the following holds almost surely: There is a $t_0 = t_0(\xi, \psi, \varepsilon, R) < \infty$ such that for each $t \ge t_0$, there exists a $y_t \in Q_{\gamma_t}$ such that*

$$\xi(z + y_t) \ge \frac{1}{\alpha_{b_t}^2}\psi\left(\frac{z}{\alpha_{b_t}}\right) - \frac{\varepsilon}{\alpha_{b_t}^2} \qquad \forall z \in Q^{(t)}. \tag{5.10}$$

In addition, whenever $d \ge 2$, y_t can be chosen such that $y_t \in \mathcal{C}_\infty^*$.

The proof of Proposition 5.1 is deferred to Subsection 5.3. In order to make use of it, we need that the walk can get to $y_t + Q^{(t)}$ in a reasonable time. In $d \ge 2$, this will be possible whenever the above microbox can be reached from any point in $\mathcal{C}_\infty^* \cap Q_{\gamma_t}$ by a path in $\mathcal{C}_\infty^*$ whose length is comparable to the lattice distance between the path's end-points. Given $x, z \in \mathcal{C}_\infty^*$, let $\mathrm{d}_*(x,z)$ denote the length of the shortest path in $\mathcal{C}_\infty^*$ connecting x and z. Let $|x - z|_1$ be the lattice distance of x and z. The following lemma is the site-percolation version of Lemma 2.4 in Antal's thesis [A94], page 72. While the proof is given there in the bond-percolation setting, its inspection shows that it carries over to our case. Therefore, we omit it.

Lemma 5.2 *Suppose $d \geq 2$. Then, with probability one,*

$$\varrho(x) := \sup_{z \in \mathcal{C}_\infty^* \setminus \{x\}} \frac{\mathrm{d}_*(x,z)}{|x-z|_1} < \infty \qquad \textit{for all } x \in \mathcal{C}_\infty^*. \tag{5.11}$$

We proceed with the proof of Theorem 1.5 in the case $d \geq 2$. In $d = 1$, Lemma 5.2 will be substituted by a different argument.

Proof of Theorem 1.5 ($d \geq 2$), lower bound. Let $R, \varepsilon > 0$ and let $\psi \in C^-(R)$ be twice continuously differentiable with $\mathcal{L}_R(\psi) < d$. If $\gamma = 0$, let $\operatorname{supp} \psi$ be a non-degenerate ball in Q_R centered at 0. Suppose that $\xi = (\xi(z))_{z \in \mathbb{Z}^d}$ does not belong to the exceptional null sets of the preceding assertions. In particular, there are unique infinite clusters $\mathcal{C}_\infty$ in $\{z \in \mathbb{Z}^d \colon \xi(z) > -\infty\}$ and $\mathcal{C}_\infty^*$ in $\{z \in \mathbb{Z}^d \colon \xi(z) \geq -K\}$, and ξ satisfies the claims in Proposition 5.1 and Lemma 5.2. Clearly, $\mathcal{C}_\infty^* \subset \mathcal{C}_\infty$. Assume $0 \in \mathcal{C}_\infty$ and pick a $z^* \in \mathcal{C}_\infty^*$. For each $t \geq t_0$ choose a $y_t \in Q_{\gamma_t} \cap \mathcal{C}_\infty^*$ such that (5.10) holds. We assume that t is so large that $z^* \in Q_{\gamma_t}$.

The lower bound on $u(t,0)$ will be obtained by restricting the random walk $(X(s))_{s \geq 0}$ (which starts at 0) to be at z^* at time 1, at y_t at time γ_t (staying within $\mathcal{C}_\infty^*$ in the meantime) and to remain in $y_t + Q^{(t)}$ until time t. Introduce the exit times from $\mathcal{C}_\infty^*$ and $y_t + Q^{(t)}$, respectively,

$$\tau_\infty^* = \inf\bigl\{s > 0 \colon X(s) \notin \mathcal{C}_\infty^*\bigr\} \qquad \text{and} \qquad \tau_{y_t,t} = \inf\bigl\{s > 0 \colon X(s) \notin y_t + Q^{(t)}\bigr\}. \tag{5.12}$$

Let $t \geq t_0(\xi)$. Inserting the indicator on the event described above and using the Markov property twice at times 1 and γ_t, we get

$$u(t,0) \geq \mathrm{I} \times \mathrm{II} \times \mathrm{III}, \tag{5.13}$$

where the three factors are given by

$$\begin{aligned}
\mathrm{I} &= \mathbb{E}_0\Bigl[\exp\Bigl\{\int_0^1 \xi\bigl(X(s)\bigr)\,ds\Bigr\} 1\bigl\{X(1) = z^*\bigr\}\Bigr],\\
\mathrm{II} &= \mathbb{E}_{z^*}\Bigl[\exp\Bigl\{\int_0^{\gamma_t - 1} \xi\bigl(X(s)\bigr)\,ds\Bigr\} 1\bigl\{\tau_\infty^* > \gamma_t - 1, X(\gamma_t - 1) = y_t\bigr\}\Bigr],\\
\mathrm{III} &= \mathbb{E}_{y_t}\Bigl[\exp\Bigl\{\int_0^{t-\gamma_t} \xi\bigl(X(s)\bigr)\,ds\Bigr\} 1\bigl\{\tau_{y_t,t} > t - \gamma_t\bigr\}\Bigr].
\end{aligned} \tag{5.14}$$

Clearly, the quantity I is independent of t and is non-vanishing because $0, z^* \in \mathcal{C}_\infty$. Our next claim is that $\mathrm{II} \geq e^{o(t\alpha_{b_t}^{-2})}$ as $t \to \infty$. Indeed,

$$\mathrm{II} \geq e^{-K\gamma_t} \mathbb{P}_{z^*}\bigl(\tau_\infty^* > \gamma_t - 1, X(\gamma_t - 1) = y_t\bigr), \tag{5.15}$$

since there is at least one path connecting z^* to y_t within $\mathcal{C}_\infty^*$ (recall that the field ξ is bounded from below by $-K$ on $\mathcal{C}_\infty^*$). Denote by $\mathrm{d}_t = \mathrm{d}_*(z^*, y_t)$ the minimal length of such a path and abbreviate $\varrho(z^*) = \varrho$, where $\varrho(z^*)$ is as in (5.11). Then, for $t \geq t_0$,

$$\mathrm{d}_t \leq \varrho |z^* - y_t|_1 \leq 2d\varrho\gamma_t \leq 3d\varrho(\gamma_t - 1), \tag{5.16}$$

by Lemma 5.2 and the fact that the both $z^*, y_t \in Q_{\gamma_t}$. Hence, using also that $\mathrm{d}_t! \leq \mathrm{d}_t^{\mathrm{d}_t}$,

$$\begin{aligned}
\mathbb{P}_{z^*}\bigl(\tau_\infty^* > \gamma_t - 1, X(\gamma_t - 1) = y_t\bigr) &\geq e^{-(\gamma_t - 1)} \frac{(\gamma_t - 1)^{\mathrm{d}_t}}{\mathrm{d}_t!} (2d)^{-\mathrm{d}_t}\\
&\geq e^{-\gamma_t} \exp\bigl[-\mathrm{d}_t \log(2d\,\mathrm{d}_t / (\gamma_t - 1))\bigr] \geq \exp\bigl[-\gamma_t\bigl(1 + 3d\varrho \log(6d^2\varrho)\bigr)\bigr].
\end{aligned} \tag{5.17}$$

In order to see that II $\geq e^{o(t\alpha_{b_t}^{-2})}$, recall that $\gamma_t = o(t\alpha_{b_t}^{-2})$ as $t \to \infty$ by (5.8) and that z^* does not depend on t.

We turn to the estimate of III. By spatial homogeniety of the random walk, we have

$$\mathrm{III} = \mathbb{E}_0\Big[\exp\Big\{\int_0^{t-\gamma_t} \xi\big(y_t + X(s)\big)\, ds\Big\} 1\{\tau_{0,t} > t-\gamma_t\}\Big], \tag{5.18}$$

where $\tau_{0,t}$ is the first exit time from $Q^{(t)}$. Using (5.10), we obtain the estimate

$$\mathrm{III} \geq e^{-\varepsilon(t-\gamma_t)\alpha_{b_t}^{-2}}\, \mathbb{E}_0\Big[\exp\Big\{\int_0^{t-\gamma_t} \psi_t\big(X(s)\big)\, ds\Big\} 1\{\tau_{0,t} > t-\gamma_t\}\Big], \tag{5.19}$$

By invoking (3.5) and (3.12), the expectation on the right-hand side is bounded from below by

$$\exp\big\{(t-\gamma_t)\lambda^{\mathrm{d}}(t)\big\}\, \mathrm{e}_t(0)^2, \tag{5.20}$$

where $\lambda^{\mathrm{d}}(t)$ resp. e_t denote the principal Dirichlet eigenvalue resp. the ℓ^2-normalized principal eigenfunction of $\kappa\Delta^{\mathrm{d}} + \psi_t$ in $Q^{(t)}$. For $e_t(0)$ and $\lambda^{\mathrm{d}}(t)$ we have the following bounds, whose proofs will be given subsequently:

Lemma 5.3 *We have*

$$\liminf_{t\to\infty} \frac{\alpha_{b_t}^2}{t} \log \mathrm{e}_t(0)^2 \geq 0, \tag{5.21}$$

$$\liminf_{t\to\infty} \alpha_{b_t}^2 \lambda^{\mathrm{d}}(t) \geq \lambda_R(\psi). \tag{5.22}$$

Summarizing all the preceding estimates and applying (5.21) and (5.22), we obtain

$$\liminf_{t\to\infty} \frac{\alpha_{b_t}^2}{t} \log u(t,0) \geq \lambda_R(\psi) - \varepsilon, \tag{5.23}$$

where we also noted that $t - \gamma_t = t(1+o(1))$. In the case $\gamma > 0$, let $\varepsilon \downarrow 0$, optimize over $\psi \in C^-(R)$ with $\mathcal{L}_R(\psi) < d$ (clearly, the supremum in (1.23) may be restricted to the set of twice continuously differentiable functions $\psi \in C^-(R)$ such that $\mathcal{L}_R(\psi) < d$) and let $R \to \infty$ to get the lower bound in Theorem 1.5. In the case $\gamma = 0$, recall that $\mathcal{L}_R(\psi) = \mathrm{const.}\,|\{\psi < 0\}|$. It is classical (see, e.g., [DV75], Lemma 3.13, or argue directly by Faber-Krahn's inequality) that the supremum (1.23) can be restricted to ψ whose support is a ball. The proof is therefore finished by letting $\varepsilon \downarrow 0$, optimizing over such ψ and letting $R \to \infty$. $\square$

Proof of Lemma 5.3. We begin with (5.21). Recall that e_t is also an eigenfunction for the transition densities of the random walk in $Q^{(t)}$ with potential $\psi_t - \lambda^{\mathrm{d}}(t)$. Using this observation at time 1, we can write

$$\mathrm{e}_t(0) = \mathbb{E}_0\Big[\exp\Big\{\int_0^1 \big[\psi_t\big(X(s)\big) - \lambda^{\mathrm{d}}(t)\big]\, ds\Big\} 1\{\tau_{0,t} > 1\} \mathrm{e}_t\big(X(1)\big)\Big], \tag{5.24}$$

Since $\lambda^{\mathrm{d}}(t)$ is nonpositive and ψ is bounded from below, we have

$$\mathrm{e}_t(0) \geq \exp\big[\alpha(b_t)^{-2} \inf \psi\big] \sum_{z\in Q^{(t)}} \mathbb{P}_0\big(\tau_{0,t} > 1, X(1) = z\big) \mathrm{e}_t(z). \tag{5.25}$$

Using the same strategy as in (5.17), we have $\mathbb{P}_0(\tau_{0,t} > 1, X(1) = z) \ge e^{-O(\alpha(b_t)\log\alpha(b_t))}$. Since e_t is nonnegative and satisfies $\|\mathrm{e}_t\|_2 = 1$, we have $\sum_z \mathrm{e}_t(z) \ge \|\mathrm{e}_t\|_2^2 = 1$. From these estimates, (5.21) is proved by noting that $\alpha(b_t)\log\alpha(b_t)) = o(t/\alpha(b_t)^2)$.

In order to establish (5.22), we shall restrict the supremum in (3.10) to a particular choice of g. Let $Q_R(\psi) = [-R,R]^d$ if $\gamma \neq 0$ and $Q_R(\psi) = \operatorname{supp}\psi$ if $\gamma = 0$. Let $\widehat{g}\colon [-R,R]^d \to [0,\infty)$ be the L^2-normalized principal eigenfunction of the (continuous) operator $\kappa\Delta + \psi$ on $Q_R(\psi)$ with Dirichlet boundary conditions. Let us insert $\widehat{g}_t(z) = \widehat{g}(z/\alpha(b_t))/\alpha(b_t)^{d/2}$ into (3.10) in the place of g. Thus we get

$$\alpha(b_t)^2\lambda^{\mathrm{d}}(t)(\psi_t) \ge \alpha(b_t)^{-d}\sum_{z\in Q^{(t)}}\Bigl[(\psi\widehat{g}^2)\bigl(\tfrac{z}{\alpha(b_t)}\bigr) - \kappa\alpha(b_t)^2\sum_{y\colon y\sim z}\Bigl(\widehat{g}\bigl(\tfrac{z}{\alpha(b_t)}\bigr) - \widehat{g}\bigl(\tfrac{y}{\alpha(b_t)}\bigr)\Bigr)^2\Bigr], \qquad (5.26)$$

where $y \sim z$ denotes that y and z are nearest neighbors.

Since ψ is smooth, standard theorems guarantee that $\widehat{g}$ is continuously differentiable on $Q_R(\psi)$ and, hence, $\|\nabla\widehat{g}\|_\infty < \infty$. (This fact is derived using regularity properties of Green's function of the Poisson equation, see, e.g., Theorem 10.3 in Lieb and Loss [LL96].) Then

$$\widehat{g}\bigl(z/\alpha(b_t)\bigr) - \widehat{g}\bigl(y/\alpha(b_t)\bigr) = \alpha(b_t)^{-1}(y-x)\cdot\nabla\widehat{g}\bigl(z_\eta/\alpha(b_t)\bigr), \qquad z,y\in Q^{(t)}, \qquad (5.27)$$

where $z_\eta = \eta z + (1-\eta)y$ for some $\eta\in[0,1]$. For the pairs $z\sim y$ with $y\notin Q^{(t)}$ we only get a bound $|\widehat{g}(z/\alpha(b_t)) - \widehat{g}(y/\alpha(b_t))| \le (1+\|\nabla\widehat{g}\|_\infty)/\alpha(b_t)$ (note that $\widehat{g}(y/\alpha(b_t)) = 0$ in this case). Since the total contribution of these boundary terms to (5.26) is clearly bounded by $(1+\|\nabla\widehat{g}\|_\infty)/\alpha(b_t)$, we see that the right-hand side of (5.26) converges to $(\psi,\widehat{g}^2) - \kappa\|\nabla\widehat{g}\|_2$ as $t\to\infty$. By our choice of $\widehat{g}$, this limit is equal to the eigenvalue $\lambda_R(\psi)$, which proves (5.22). $\square$

Proof of Theorem 1.5 ($d=1$), lower bound. Suppose that $\langle\log(-\xi(0)\vee 1)\rangle > -\infty$. This implies that $\mathcal{C}_\infty = \mathbb{Z}$ almost surely and, by the law of large numbers,

$$K_\xi := \sup_{y\in\mathbb{Z}\setminus\{0\}}\frac{1}{|y|}\sum_{x=0}^{|y|}\log\bigl(-\xi(x)\vee 1\bigr) < \infty \quad \text{almost surely.} \qquad (5.28)$$

Suppose that $\xi = (\xi(z))_{z\in\mathbb{Z}}$ does not belong to the exceptional sets of (5.28) and Proposition 5.1. For sufficiently large t, let $y_t \in Q_{\gamma_t}$ be such that (5.10) holds.

Let $r_x = (-1/\xi(x))\wedge 1$. The strategy for the lower bound on $u(t,0)$ is that the random walk performs $|y_t|$ steps toward y_t, resting at most time r_x at each site x between 0 and y_t, so that y_t is reached before time γ_t. Afterwards the walk stays at y_t until γ_t. Use $E^{(t)}$ to denote the latter event. Then $u(t,0)\ge \mathrm{II}\times\mathrm{III}$, where III is as in (5.14) and $\mathrm{II} = \mathbb{E}_0\bigl[e^{\int_0^{\gamma_t}\xi(X(s))\,ds}\mathbf{1}_{E^{(t)}}\bigr]$.

The lower bound on III is identical to the case $d\ge 2$. To estimate the term II, suppose that $y_t > 0$ (clearly, if $y_t = 0$ no estimate on II is needed; $y_t < 0$ is handled by symmetry) and abbreviate $|y_t| = n+1$. Using the shorthand $[s]_n = s_0 + \dots + s_n$, we have

$$\begin{aligned}\mathrm{II} &= \int_0^{r_0} ds_0\cdots\int_0^{r_n} ds_n\int_0^{\gamma_t-[s]_n} ds_{n+1}\,\exp\Bigl\{-\sum_{x=0}^{n+1}s_x\bigl(\kappa-\xi(x)\bigr)\Bigr\}\\ &\ge e^{O(\gamma_t)}\prod_{x=0}^n\Bigl[r_x\exp\bigl(r_x\xi(x)\bigr)\Bigr] \ge e^{O(\gamma_t)}\exp\Bigl\{-\sum_{x=0}^n\log\bigl(-\xi(x)\vee 1\bigr)\Bigr\}. \qquad (5.29)\end{aligned}$$

Indeed, in the first line we noted that $[s]_n \le \gamma_t$ because $r_x \le 1$. Then we took out the terms $\exp(-\kappa s_x)$ as well as $\exp(s_{n+1}\xi(y_t))$, recalling that $\xi(y_t) \ge \inf \psi_t = \inf \psi/\alpha(b_t)^2 = O(1)$ and that $|y_t| = O(\gamma_t)$. The last inequality follows by the fact that $r_x \exp(r_x\xi_x) \ge r_x/e$. Invoking (5.28), the sum in the exponent is bounded above by $K_\xi |y_t| = O(\gamma_t)$, whereby we finally get that $\mathrm{II} \ge e^{-O(\gamma_t)}$. $\square$

5.3 Technical claims

For the proof of Proposition 5.1, we need to introduce some notation and prove two auxiliary lemmas. For $y \in \mathbb{Z}^d$, define the event

$$A_y^{(t)} = \{y + Q^{(t)} \subset \mathcal{C}_\infty^*\} \cap \bigcap_{z\in Q^{(t)}} \Big\{\xi(y+z) \ge \psi_t(z) - \tfrac{\varepsilon}{2\alpha(b_t)^2}\Big\}. \tag{5.30}$$

Note that the distribution of $A_y^{(t)}$ does not depend on y. By $\partial(Q)$ we denote the outer boundary of a set $Q \subset \mathbb{Z}^d$. To estimate $\mathrm{Prob}(A_y^{(t)})$, it is convenient to begin with the first event on the right-hand side of (5.30).

Lemma 5.4 *Let $d \ge 2$ and let $\psi \in C^-(R)$ be such that $\psi \not\equiv 0$. Then there is a $c \in (0,\infty)$ such that, for t large enough,*

$$\mathrm{Prob}\big(\partial Q^{(t)} \cap \mathcal{C}_\infty^* = \emptyset\big) \le e^{-c\alpha(b_t)}. \tag{5.31}$$

Proof. Since $\psi \not\equiv 0$ is continuous, there is a ball $B_{\alpha(b_t)}$ of radius of order $\alpha(b_t)$ such that $B_{\alpha(b_t)} \subset Q^{(t)}$. If t is so large that $\psi_t \ge \inf \psi/\alpha(b_t)^2 \ge -K$, then $B_{\alpha(b_t)} \subset \{z \colon \xi(z) \ge -K\}$ and the left-hand side of (5.31) is bounded from above by $\mathrm{Prob}(\partial B_{\alpha(b_t)} \cap \mathcal{C}_\infty^* = \emptyset)$. The proof now proceeds in a different way depending whether $d \ge 3$ or $d = 2$. In the following, the words "percolation", "infinite cluster", etc., refer to site-percolation on $\mathbb{Z}^d$ with parameter $p = \mathrm{Prob}(\xi(0) > -K)$. Recall that $p > p_{\mathrm{c}}(d)$ by our choice of K.

Let $d \ge 3$. Then, by equality of $p_{\mathrm{c}}(d)$ and the limit of slab-percolation thresholds, there is a width k such that the slab $S_k = \mathbb{Z}^{d-1} \times \{1, \dots, k\}$ contains almost surely an infinite cluster. Pick a lattice direction and decompose $\mathbb{Z}^d$ into a disjoint union of translates of S_k. There is $c' > 0$ such that, for t large, at least $\lfloor c'\alpha(b_t)/k \rfloor$ slabs are intersected by $\partial B_{\alpha(b_t)}$. Then $\{\partial B_{\alpha(b_t)} \cap \mathcal{C}_\infty^* = \emptyset\}$ is contained in the event that in none of the slabs intersecting $\partial B_{\alpha(b_t)}$ the respective infinite cluster reaches $\partial B_{\alpha(b_t)}$. Let $P_\infty(k)$ be minimum probability that a site in S_k belongs to an infinite cluster. Combining the preceding inclusions, we have

$$\mathrm{Prob}(\partial B_{\alpha(b_t)} \cap \mathcal{C}_\infty^* = \emptyset) \le P_\infty(k)^{c'\alpha(b_t)/k}. \tag{5.32}$$

Now the claim follows by putting $c = -c'k^{-1} \log P_\infty(k)$.

In $d = 2$, suppose without loss of generality that $B_{\alpha(b_t)}$ is centered at the origin. Recall that x and y are $*$-connected if their Euclidean distance is not more than $\sqrt{2}$. On the event $\{\partial B_{\alpha(b_t)} \cap \mathcal{C}_\infty^* = \emptyset\}$, the origin is encircled by a $*$-connected circuit of size at least $c\alpha(b_t)$ for some $c > 0$, not depending on t. Denote by x the nearest point of this circuit in the first coordinate direction. Call sites z with $\xi(z) \ge -K$ "occupied", the other sites are "vacant".

Note that percolation of occupied sites rules out percolation of vacant sites, e.g., by the result of Gandolfi, Keane, and Russo [GKR88]. Moreover, using the site-perolation version of the famous "$p_c = \pi_c$" result (see e.g., Grimmett [G89]), the probability that a given site is contained in a vacant $*$-cluster of size n is bounded by $e^{-\sigma(p)n}$, where $\sigma(p)>0$ since $p>p_c(d)$. If the ball $B_{\alpha(b_t)}$ has diameter at least $r\alpha(b_t)$, then by taking the above circuit for such a cluster we can estimate the probability of its occurrence:

$$\operatorname{Prob}\big(\partial Q^{(t)}\cap\mathcal{C}^*_\infty=\emptyset\big)\le\sum_{n=\lfloor r\alpha(b_t)\rfloor}^{\infty} ne^{-\sigma(p)n}\le e^{-\sigma(p)r\alpha(b_t)/2}, \tag{5.33}$$

for t large enough. Here "n" in the sum accounts for the possition of the circuit's intersection with the positive part of the first coordinate axes. The minimal size of the circuit is at least $\lfloor r\alpha(b_t)\rfloor$, since it has to stay all outside $B_{\alpha(b_t)}$. The claim follows by putting $c=r\sigma(p)/2$. □

Lemma 5.5 *For any $\varepsilon>0$,*

$$\operatorname{Prob}(A_0^{(t)})\ge t^{-\mathcal{L}_R(\psi)+o(1)},\qquad t\to\infty. \tag{5.34}$$

Let H be in the γ-class. Let $\psi\not\equiv 0$ (otherwise there is nothing to prove because $\mathcal{L}_R(0)=\infty$). Consider the event

$$\widetilde{A}^{(t)}=\bigcap_{z\in Q^{(t)}}\Big\{\xi(z)\ge\psi_t(z)-\tfrac{\varepsilon}{2\alpha(b_t)^2}\Big\}. \tag{5.35}$$

Note that both events on the right-hand side of (5.30) are increasing in the partial order $\xi\succeq\xi'$ $\Leftrightarrow\xi(x)\ge\xi'(x)$ for all x. Therefore, by the FKG-inequality,

$$\operatorname{Prob}(A_0^{(t)})\ge\operatorname{Prob}\big(\{y+Q^{(t)}\subset\mathcal{C}^*_\infty\}\big)\operatorname{Prob}(\widetilde{A}^{(t)})\ge\operatorname{Prob}(0\in\mathcal{C}^*_\infty)\operatorname{Prob}(\widetilde{A}^{(t)}). \tag{5.36}$$

Hence, we only need to prove the assertion for $A_0^{(t)}$ replaced by $\widetilde{A}^{(t)}$. The proof proceeds in three steps, depending on γ and on whether there is an atom at 0.

Proof for $\gamma\in(0,1)$. Let $f\in C^+(R)$ be the solution to $\psi-\frac{3}{8}\varepsilon=\widetilde{H}'\circ f$ and let $f_t\colon\mathbb{Z}^d\to(0,\infty)$ be its scaled version: $f_t(z)=(b_t/\alpha(b_t)^d)f(z/\alpha(b_t))$. Define the tilted probability measure

$$\operatorname{Prob}_{t,z}(\,\cdot\,)=\big\langle e^{f_t(z)\xi(z)}1\{\xi(z)\in\,\cdot\,\}\big\rangle e^{-H(f_t(z))}. \tag{5.37}$$

We denote expectation with respect to $\operatorname{Prob}_{t,z}$ by $\langle\,\cdot\,\rangle_{t,z}$. Consider the event

$$D_t(z)=\Big\{-\frac{\varepsilon}{4\alpha(b_t)^2}\ge\xi(z)-\psi_t(z)\ge-\frac{\varepsilon}{2\alpha(b_t)^2}\Big\}. \tag{5.38}$$

Then $\operatorname{Prob}(\widetilde{A}^{(t)})$ can be bounded as

$$\operatorname{Prob}\big(\widetilde{A}^{(t)}\big)\ge\prod_{z\in Q^{(t)}}\Big[e^{H(f_t(z))}\big\langle e^{-f_t(z)\xi(z)}1\{D_t(z)\}\big\rangle_{t,z}\Big]. \tag{5.39}$$

Applying the left inequality in (5.38), we obtain

$$\operatorname{Prob}\big(\widetilde{A}^{(t)}\big)\ge\exp\Big\{\sum_{z\in Q^{(t)}}\big[H(f_t(z))-f_t(z)\big(\psi_t(z)-\tfrac{\varepsilon}{4\alpha(b_t)^2}\big)\big]\Big\}\prod_{z\in Q^{(t)}}\operatorname{Prob}_{t,z}\big(D_t(z)\big). \tag{5.40}$$

Since $\gamma > 0$ and f is continuous and bounded, we can use our Scaling Assumption and the fact that $b_t\alpha(b_t)^{-2} = \log t$ to turn the sum over $z \in Q^{(t)}$ into a Riemann integral over $[-R, R]^d$:

$$\mathrm{Prob}(\widetilde{A}^{(t)}) \geq t^{-\int[f\psi - \widetilde{H}\circ f] + \frac{\varepsilon}{4}\int f + o(1)} \prod_{z\in Q^{(t)}} \mathrm{Prob}_{t,z}\big(D_t(z)\big). \tag{5.41}$$

where we also used that $Q^{(t)} = Q_{R\alpha(b_t)}$ in this case. In order to finish the proof of the lower bound in (5.34), we thus need to show that

$$\int \big[f\psi - \widetilde{H}\circ f\big] \leq \mathcal{L}_R(\psi), \tag{5.42}$$

and that

$$\prod_{z\in Q^{(t)}} \mathrm{Prob}_{t,z}\big(D_t(z)\big) \geq t^{o(1)}, \qquad t\to\infty. \tag{5.43}$$

Let us begin with (5.42). For simplicity, we restrict ourselves to the case when $\widetilde{H}(1) = -1$. Then $\mathcal{L}_R(\psi) = \gamma^{1/(1-\gamma)}(\gamma^{-1} - 1)\int |\psi|^{-\gamma/(1-\gamma)}$ and $f = \gamma^{1/(1-\gamma)}|\psi - \frac{3}{8}\varepsilon|^{-1/(1-\gamma)}$. Hence,

$$\int \big[f\psi - \widetilde{H}\circ f\big] - \mathcal{L}_R(\psi) = \gamma^{\frac{1}{1-\gamma}} \int |\psi|^{-\frac{\gamma}{1-\gamma}} \zeta_\gamma\Big(\big|\tfrac{\psi}{\psi - \frac{3}{8}\varepsilon}\big|^{\frac{1}{1-\gamma}}\Big), \tag{5.44}$$

where $\zeta_\gamma(x) = 1 - x - \frac{1}{\gamma}(1 - x^\gamma)$. Since $\zeta_\gamma(x) \leq 0$ for any $x \geq 0$, (5.42) is proved.

In order to prove (5.43), note that

$$\begin{aligned}\mathrm{Prob}_{t,z}\big(D_t(z)\big) \geq 1 - \mathrm{Prob}_{t,z}\Big(\xi(z) \geq \psi_t(z) - \frac{\varepsilon}{4\alpha(b_t)^2}\Big)& \\ - \mathrm{Prob}_{t,z}\Big(\xi(z) \leq \psi_t(z) - \frac{\varepsilon}{2\alpha(b_t)^2}\Big)&.\end{aligned} \tag{5.45}$$

We concentrate on estimating the second term; the first term is handled analogously. By the exponential Chebyshev inequality, we have for any $g_t(z) \in (0, f_t(z))$ that

$$\begin{aligned}&\mathrm{Prob}_{t,z}\Big(\xi(z) \leq \psi_t(z) - \frac{\varepsilon}{2\alpha(b_t)^2}\Big)\\ &\qquad \leq e^{-H(f_t(z))}\Big\langle \exp\Big\{f_t(z)\xi(z) - g_t(z)\big[\xi(z) - \psi_t(z) + \tfrac{\varepsilon}{2\alpha(b_t)^2}\big]\Big\}\Big\rangle \\ &\qquad = \exp\Big\{H\big(f_t(z) - g_t(z)\big) - H\big(f_t(z)\big) + g_t(z)\psi_t(z) - g_t(z)\frac{\varepsilon}{2\alpha(b_t)^2}\Big\}.\end{aligned} \tag{5.46}$$

Note that $\widetilde{H}_t' \to \widetilde{H}'$ (recall (3.13)) as $t\to\infty$ uniformly on compact sets in $(0,\infty)$. Also note that f is bounded away from 0. Choose $g_t(z) = \delta_t f_t(z)$, where $\delta_t \downarrow 0$ is still to be chosen appropriately. Then the exponent in the third line of (5.46) can be bounded from above by

$$\begin{aligned}&-\delta_t\frac{b_t}{\alpha(b_t)^{d+2}} f\Big(\frac{z}{\alpha(b_t)}\Big)\Big\{\widetilde{H}_t'\Big[f\Big(\frac{z}{\alpha(b_t)}\Big)(1-\delta_t)\Big] - \psi\Big(\frac{z}{\alpha(b_t)}\Big) + \frac{\varepsilon}{2}\Big\}\\ &\qquad\qquad = -\delta_t\frac{b_t}{\alpha(b_t)^{d+2}} f\Big(\frac{z}{\alpha(b_t)}\Big)\Big[\frac{\varepsilon}{8} + o(1)\Big],\end{aligned} \tag{5.47}$$

where we replaced $\widetilde{H}_t'$ by $\widetilde{H}'+o(1)$ and used the definition relation for f. Pick $\delta_t=(\alpha_{b_t}^{d+2}/b_t)^{1/2}$ for definiteness. Taking the product over $z\in Q^{(t)}$ in (5.45) and using that $[\frac{\varepsilon}{8}+o(1)]f\ge C>0$, we obtain for t large that

$$\prod_{z\in Q^{(t)}}\operatorname{Prob}_{t,z}\bigl(D_t(z)\bigr)\ge\Bigl[1-2\exp\Bigl\{-C\delta_t\frac{b_t}{\alpha(b_t)^{d+2}}\Bigr\}\Bigr]^{\#Q^{(t)}}$$
$$\ge\exp\Bigl\{-4\#Q^{(t)}\exp\Bigl\{-C\delta_t\frac{b_t}{\alpha(b_t)^{d+2}}\Bigr\}\Bigr\}=t^{-C'(\alpha_{b_t}^{d+2}/b_t)\exp\bigl(-C\delta_t\frac{b_t}{\alpha(b_t)^{d+2}}\bigr)},\quad(5.48)$$

where also used that $b_t\alpha(b_t)^{-2}=\log t$ and $\#Q^{(t)}\le\alpha(b_t)^dC'/4$ for some C' as $t\to\infty$. By our choice of δ_t, (5.43) is clearly satisfied, which finishes the proof in the case $\gamma\in(0,1)$. $\square$

Proof for $\gamma=0$*, atom at* 0. Suppose $\operatorname{Prob}(\xi(0)\in\cdot)$ has an atom at 0 with mass $p>0$. Then, noting that $Q^{(t)}$ are only the sites with $\psi_t<0$, we have

$$\operatorname{Prob}(\widetilde{A}^{(t)})\ge\operatorname{Prob}\bigl(\xi(0)=0\bigr)^{\#Q^{(t)}}=\exp\bigl\{\alpha(b_t)^d(|\operatorname{supp}\psi|+o(1))\log p\bigr\},\quad t\to\infty.\quad(5.49)$$

Since $\alpha_t=t^{1/(d+2)}$ and $\widetilde{H}(1)=\log p$, we have $\mathcal{L}_R(\psi)=-\widetilde{H}(1)|\operatorname{supp}\psi|$ and $\alpha(b_t)^d=\log t$, whereby (5.34) immediately follows. $\square$

Proof for $\gamma=0$*, no atom at* 0. Suppose $\gamma=0$ and $\operatorname{Prob}(\xi(0)=0)=0$. Set $f_t=b_t\alpha(b_t)^{-d}$ and consider the probability measure $\operatorname{Prob}_t(\xi(0)\in\cdot)$ with density $\exp[f_t\xi(0)-H(f_t)]$ with respect to $\operatorname{Prob}(\xi(0)\in\cdot)$. Invoking that $\xi(0)\le0$, we obtain

$$\operatorname{Prob}(\widetilde{A}^{(t)})\ge\operatorname{Prob}\Bigl(\xi(0)\ge-\frac{\varepsilon}{2\alpha(b_t)^2}\Bigr)^{\#Q^{(t)}}\ge e^{\#Q^{(t)}H(f_t)}\operatorname{Prob}_t\Bigl(\xi(0)\ge-\frac{\varepsilon}{2\alpha(b_t)^2}\Bigr)^{\#Q^{(t)}}.\quad(5.50)$$

Now use the Scaling Assumption and the fact that $\#Q^{(t)}=\alpha(b_t)^d(|\operatorname{supp}\psi|+o(1))$ as $t\to\infty$ to extract the term $t^{-\mathcal{L}_R(\psi)}$ from the exponential on the right-hand side (here we recalled that $\mathcal{L}_R(\psi)=-\widetilde{H}(1)|\operatorname{supp}\psi|$). Moreover, by an argument similar to (5.46), the last term on the right-hand side is no smaller than $t^{o(1)}$ as $t\to\infty$. To that end we noted that our choice of f_t corresponds to $f\equiv1$ and then we used again that $\lim_{t\to\infty}b_t\alpha(b_t)^{-(d+2)}=\infty$, which follows from the fact that $\xi(0)$ has no atom at zero. $\square$

Proof of Proposition 5.1. Fix $R>0$ and $\psi\in C^-(R)$ with $\mathcal{L}_R(\psi)<d$. Recall the notation (5.9) and (5.30). Let $t_1=t_1(\psi,\varepsilon,R)$ be such that for all $t\ge t_1$ and for all $s\in[0,e)$

$$\psi_{et}(z)-\frac{\varepsilon}{2\alpha(b_{et})^2}\ge\psi_{st}(z)-\frac{\varepsilon}{\alpha(b_{st})^2},\qquad z\in Q^{(st)}.\quad(5.51)$$

Such a $t_1<\infty$ indeed exists, since $\alpha(b_{st})/\alpha(b_{et})\to1$ as $t\to\infty$ and since ψ is uniformly continuous on $[-R,R]^d$. This implies that to prove Proposition 5.1 it suffices to find an almost-surely finite $n_0=n_0(\xi,\psi,\varepsilon,R)$ such that for each $n\ge n_0$ there is a $y_n\in Q_{\gamma_{e^n}}$ for which the event $A_{y_n}^{(e^{n+1})}$ occurs. Indeed, for any $t=se^n$ with $n\ge n_0$ and $s\in[0,e)$ we have that $Q_{\gamma_{e^n}}\subset Q_{\gamma_t}$ and $y_n+Q_{R\alpha(b_t)}\subset y_n+Q_{R\alpha(b_{e^{n+1}})}$, as follows by monotonicity of the maps $t\mapsto\gamma_t$ and $t\mapsto\alpha(b_t)$ and, consequently,

$$\bigcap_{z\in Q^{(t)}}\bigl\{\xi(y_n+z)\ge\psi_t(z)-\tfrac{\varepsilon}{\alpha(b_t)^2}\bigr\}\supset A_{y_n}^{(e^{n+1})},\quad(5.52)$$

by invoking (5.51). Then Proposition 5.1 would follow with the choice $t_0 = t_1 \vee e^{n_0}$.

Based on the preceding reduction argument, let $t \in \{e^n : n \in \mathbb{N}\}$ for the remainder of the proof. Let $M_t = Q_{\gamma_t} \cap \lfloor 3R\alpha(b_{et})\rfloor \mathbb{Z}^d$. We claim that, to prove Proposition 5.1 for $t \in \{e^n : n \in \mathbb{N}\}$, it suffices to show the summability of

$$p_t = \mathrm{Prob}\Big(\sum_{y \in M_t} \mathbb{1}_{A_y^{(et)}} \le \tfrac{1}{2}\#M_t \,\mathrm{Prob}\big(A_0^{(et)}\big)\Big), \qquad t \in \{e^n : n \in \mathbb{N}\}. \tag{5.53}$$

Indeed, since $\#M_t \ge t^{d+o(1)}$ we have by Lemma 5.5

$$\#M_t \,\mathrm{Prob}(A_0^{(et)}) \ge t^{d-\mathcal{L}_R(\psi)+o(1)}, \qquad t \to \infty. \tag{5.54}$$

Since we assumed $\mathcal{L}_R(\psi) < d$, summability of p_t would imply the existence of at least one site $y \in Q_{\gamma_t}$ (in fact, at least $t^{d-\mathcal{L}_R(\psi)+o(1)}$ sites) with $A_y^{(et)}$ satisfied.

To prove a suitable bound on p_t we invoke Chebyshev's inequality to find that

$$p_t \le \frac{4}{\#M_t \,\mathrm{Prob}(A_0^{(et)})} + \frac{4 \max_{y \ne y'} \mathrm{cov}(A_y^{(et)}, A_{y'}^{(et)})}{\mathrm{Prob}(A_0^{(et)})^2}. \tag{5.55}$$

As follows from (5.54), the first term on the right-hand side is summable on $t \in \{e^n : n \in \mathbb{N}\}$. In order to estimate $\mathrm{cov}(A_y^{(et)}, A_{y'}^{(et)})$ for $y \ne y'$, let $\mathbb{H}$ and $\mathbb{H}'$ be two disjoint half spaces in $\mathbb{R}^d$ which contain $y + Q^{(et)}$ and $y' + Q^{(et)}$, respectively, including the outer boundaries. By our choice of M_t, $\mathbb{H}$ can be chosen such that $\mathrm{dist}(y + Q^{(et)}), \mathbb{H}^{\mathrm{c}}) \ge R\alpha(b_t)/3$, and similarly for $\mathbb{H}'$. We introduce the event F_y that the outer boundary of $y + Q^{(et)}$ is connected to infinity by a path in $\mathcal{C}^*_\infty \cap \mathbb{H}$, and the analogous event $F_{y'}$ with y' and $\mathbb{H}'$ instead of y and $\mathbb{H}$. By splitting $A_y^{(et)}$ into $A_y^{(et)} \cap F_y$ and $A_y^{(et)} \cap F_y^{\mathrm{c}}$ (and analogously for y') and invoking the independence of $A_y^{(et)} \cap F_y$ and $A_{y'}^{(et)} \cap F_{y'}$ we see that

$$\begin{aligned} \mathrm{cov}\big(A_y^{(et)}, A_{y'}^{(et)}\big) &= \mathrm{cov}\big(A_y^{(et)} \cap F_y^{\mathrm{c}}, A_{y'}^{(et)}\big) + \mathrm{cov}\big(A_y^{(et)} \cap F_y, A_{y'}^{(et)} \cap F_{y'}^{\mathrm{c}}\big) \\ &\le \mathrm{Prob}\big(\widetilde{A}^{(et)}\big)^2 \big[\mathrm{Prob}(F_y^{\mathrm{c}}) + \mathrm{Prob}(F_{y'}^{\mathrm{c}})\big], \end{aligned} \tag{5.56}$$

where we recalled (5.35) for the definition of $\widetilde{A}^{(et)}$.

In order to estimate the last expression, let us observe that

$$F_y^{\mathrm{c}} \subset \big\{\partial(y + Q^{(et)}) \cap \mathcal{C}^*_\infty = \emptyset\big\} \cup \bigcup_{x \in \partial(y+Q^{(et)})} G_x \tag{5.57}$$

where G_x is the event that x is in a finite component of $\{z \colon \xi(z) \ge -K\} \cap \mathbb{H}$ which reaches up to $\mathbb{H}^{\mathrm{c}}$. By Lemma 5.4, the probability of the first event is bounded by $e^{-c\alpha(b_t)/2}$ and, as is well known (see, e.g., Grimmett [G89], proof of Theorem 6.51), $\mathrm{Prob}(G_x)$ is exponentially small in $\mathrm{dist}(x, \mathbb{H}^{\mathrm{c}})$, which is at least $R\alpha(b_t)/3$. Since $\#\partial(y + Q^{(et)}) = O(\alpha(b_t)^{d-1})$, we have

$$\mathrm{Prob}(F_y^{\mathrm{c}}) \le e^{-c_* \alpha(b_t)} \tag{5.58}$$

for some $c_* > 0$. Since $\alpha(b_t) = n^{\nu/(1-2\nu)+o(1)}$ for $t = e^n$, also the second term is thus summable on $t \in \{e^n : n \in \mathbb{N}\}$, because by Lemma 5.4, $\mathrm{Prob}(\widetilde{A}^{(et)}) \le (1 + o(1))\mathrm{Prob}(A^{(et)})$. Combining all the preceding reasoning, the proof of Proposition 5.1 is finished. □

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