Note on the Hardness of Bounded Budget Betweenness Centrality Game with Path Length Constraints

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1 Introduction

In this technical report, we generalize the betweenness definition in Bounded Budget Betweenness Centrality Game (called B^{3}C game) introduced in [1] to only count shortest paths with a length limit ℓ. We denote this game ℓ-B^{3}C game. We prove that the hardness results in [1] about nonuniform game still hold in this generalized version. In Section 2, we provide the detailed definition of the ℓ-B^{3}C game. In Section 3, we prove that there exists an instance of ℓ-B^{3}C game such that it does not have any maximal Nash equilibrium. In Section 4, we prove that it is NP-hard to decide whether an instance of ℓ-B^{3}C game has a maximal or strict Nash equilibrium.

2 Problem Definition

The definition of Bounded Budget Betweenness Centrality game can be found in [1]. The ℓ-B^{3}C game is a natural extension of B^{3}C game. For any natural number $\ell \geq 2$, an ℓ-B^{3}C game with parameters (n, b, c, w) is a network formation game defined as follows. We consider a set of n players $V = \{1, 2, \ldots, n\}$, which are also nodes in a network. Function $b : V \to \mathbb{N}$ specifies the budget $b(i)$ for each node $i \in V$ ($\mathbb{N}$ is the set of natural numbers). Function $c : V \times V \to \mathbb{N}$ specifies the cost $c(i, j)$ for the node i to establish a link to node j, for $i, j \in V$. Function $w : V \times V \to \mathbb{N}$ specifies the weight $w(i, j)$ from node i to node j for $i, j \in V$, which can be interpreted as the amount of traffic i sends to j, or the importance of the communication from i to j.[4]

The strategy space of player i in an ℓ-B^{3}C game is $S_i = \{s_i \subseteq V \setminus \{i\} \mid \sum_{j \in s_i} c(i, j) \leq b(i)\}$, i.e., all possible subsets of outgoing links of node i within i's budget. A strategy profile $s = (s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \ldots \times S_n$ is referred to as a *configuration* in this paper. The graph induced by configuration s is denoted as $G_s = (V, E)$, where $E = \{(i, j) \mid i \in V, j \in s_i\}$. For convenience, we will also refer G_s as a configuration.

The utility of a node i in configuration s is defined by the *ℓ-betweenness centrality* of i in the graph G_s as follows:

$$btw_i(G_s, \ell) = \sum_{u \neq v \neq i \in V,\, m(u,v,\ell) > 0} w(u, v) \frac{m_i(u, v, \ell)}{m(u, v, \ell)}, \tag{1}$$

[4] We may also define a distance function specifying distances between every pair of nodes, but it is not needed throughout our paper.

where $m(u, v, \ell)$ is the number of shortest paths from u to v in G_s with length at most ℓ, and $m_i(u, v, \ell)$ is the number of shortest paths from u to v that passes i in G_s with length at most ℓ. We can see from the formal definition that *ℓ-betweenness centrality* extends the definition of betweenness centrality by only considering shortest paths with length at most ℓ in computing node betweenness. For convenience, we sometimes use $btw_i(G_s)$ instead of $btw_i(G_s, \ell)$ if the parameter ℓ is clear.

In a configuration s, if no node can increase its own utility by changing its own strategy unilaterally, we say that s is a *(pure) Nash equilibrium*, and we also say that s is *stable*. Moreover, if in configuration s any strategy change of any node strictly decreases the utility of the node, we say that s is a *strict* Nash equilibrium.

The following Lemmata show the basic property of the game and motivate our definition of maximal Nash equilibrium. Betweenness centrality is monotonic in terms of adding edges to a node, as stated below.

Lemma 1. *Adding an outgoing edge to a node i does not decrease i's betweenness. That is, for any graph $G = (V, E)$ with $i \in V$ and $(i, j) \notin E$ for some $j \in V$. Let $G' = (V, E \cup \{(i, j)\})$. Then $btw_i(G, \ell) \leq btw_i(G', \ell)$.*

Given an ℓ-B^{3}C game with parameters (n, b, c, w), a *maximal strategy* of a node v is a strategy with which v cannot add any outgoing edges without exceeding its budget. We say that a graph (configuration) is *maximal* if all nodes use maximal strategies in the configuration. By the monotonicity of betweenness centrality, it makes sense to study maximal graphs where no node can add more edges within its budget limit. Moreover, some trivial non-maximal graphs are trivial Nash equilibria, e.g. empty graphs with no edges. However, when nodes add more edges into such a graph allowed by their budgets, other nodes may have chance of improving their utilities by changing their strategies. Therefore, for the rest of the paper, we focus on Nash equilibria in maximal graphs. In particular, we say that a configuration is a *maximal Nash equilibrium* if it is a maximal graph and it is a Nash equilibrium.

The following lemma states the relationship between maximal Nash equilibria and strict Nash equilibria, a direct consequence of the monotonicity of betweenness centrality.

Lemma 2. *Given an ℓ-B^{3}C game with parameters (n, b, c, w), any strict Nash equilibrium in the game is a maximal Nash equilibrium.*

Based on the above lemma, our results may refer to strict Nash equilibria when it is approriate and makes the result stronger.

3 Nonexistence of Maximal Nash Equilibrium

In this section, we show that maximal Nash equilibria may not exist in some version of ℓ-B^{3}C games where edge costs are not uniform.

First for the cases of $\ell \geq 3$, the follow lemma shows that the ℓ-B^{3}C game based on the gadget presented in [1] (Figure 1) has no maximal Nash equilibria for all $\ell \geq 3$.

Lemma 3. *For any $\ell \geq 3$, the ℓ-B^{3}C game based on the gadget in Figure 1 of [1] does not have any maximal Nash equilibrium. This implies that for any $n \geq 6$, there is an instance of ℓ-B^{3}C game with n players that does not have any maximal Nash equilibrium, and in the game only the edge costs are nonuniform.*

Proof. Theorem 1 in [1] already shows that the B^{3}C game (without path length constraint) based on the gadget in Figure 1 of [1] does not have any maximal Nash equilibrium. It is easy to verify that, in the proof of Theorem 1 in [1], in any configuration where a node v uses a best response, all shortest paths passing through v have length at most 3. Therefore, we have in any configuration, a best response of a node v in

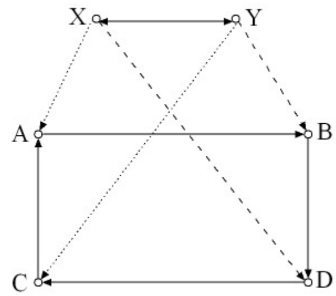

Fig. 1. Main structure of the gadget that has no maximal Nash equilibrium for ℓ-B^{3}C games with $\ell \geq 3$. Solid arrows represent fixed edges, while dotted arrows and dashed arrows represent conflicting choices of flexible edges from a node.

the original B^{3}C game without path length constraint must also be a best response of v in the ℓ-B^{3}C game with $\ell \geq 3$ with the same betweenness value. Together with the fact that the ℓ-betweenness value is no greater than the betweenness value without path length constraint, we know that the ℓ-B^{3}C game based on the gadget in in Figure 1 of [1] does not have a Nash equilibrium. □

However, the gadget in Figure 1 of [1] does not work for the case of $\ell = 2$. We now construct a separate gadget for $\ell = 2$ in Figure 1. The outgoing edges for nodes A, B, C, D and the two edges from X and Y point to each other are fixed as shown in the gadget. Node X can establish at most one edge to a node in $\{A, D\}$, while node Y can establish at most one edge to a node in $\{B, C\}$.

We classify nodes and edges as follows. Nodes X and Y are *flexible* nodes since they can choose to connect one node in $\{A, D\}$ and $\{B, C\}$ respectively. Nodes A, B, C, D are *rectangle* nodes. Edges $(X, A), (X, D), (Y, B), (Y, C)$ are *flexible* edges (in the figure dotted arrows and dashed arrows represent conflicting choices of flexible edges, e.g. (X, A) and (X, D) cannot be selected at the same time). Other edges shown in the figure are *fixed* edges. The remaining pairs with no edge connected (e.g. (X, B), (X, C), etc.) are referred to as *forbidden* edges.

We use the parameters (n, b, c, w) of a 2-B^{3}C game to realize the gadget. In particular, (a) $n = 6$; (b) $b(i) = 1$ for all $i \in V$; (c) $c(i, j) = 0$ if (i, j) is a fixed edge, $c(i, j) = 1$ if (i, j) is a flexible edge, $c(i, j) = M > 1$ if (i, j) is a forbidden edge; and (d) $w(i, j) = 1$ for all $i, j \in V$.

With the above construction, we can show the following theorem.

Lemma 4. *The 2-B^{3}C game based on the gadget in Figure 1 does not have any maximal Nash equilibrium. This implies that for any $n \geq 6$, there is an instance of ℓ-B^{3}C game with n players that does not have any maximal Nash equilibrium, and in the game only the edge costs are nonuniform.*

Proof. Note that in a maximal graph all fixed edges are included, and nodes X and Y each selects one edge to connect to one node in $\{A, D\}$ and $\{B, C\}$ respectively. We now show that this maximal graph is not stable, by discussing the following cases separately.

(1) Node X connects to A and node Y connects to B. In this case, the only path that can contribute betweenness to node Y is $X \to Y \to B$. But there is another shortest path $X \to A \to B$. So we have $btw_Y(G, 2) = 1/2$. However, if Y changes its strategy to connect to node C, it can gain betweenness 1 from the unique shortest path $X \to Y \to C$. So Y is not at its best response position.
(2) Node X connects to D and node Y connects to B. Here the only path that can contribute betweenness to node X is $Y \to X \to D$. But there is another shortest path $Y \to B \to D$ from Y to D. Thus $btw_X(G, 2) = 1/2$. Now if X changes its strategy to connect to node A, it can gain betweenness 1 from the unique shortest path $Y \to X \to A$. So X is not at its best response position.

(3) Node X connects to A and node Y connects to C. This case is equivalent to case (2), thus is not stable.
(4) Node X connects to D and node Y connects to C. This case is equivalent to case (1), which is also not stable.

In summary, each of X and Y uses the strategy such that its outgoing neighbor points to the outgoing neighbor of the other node, making an endless dynamic in the game.

Therefore, we know that none of the maximal graphs is stable, so the gadget of Figure 1 does not have any maximal Nash equilibrium.

For $n > 6$, we can use 6 nodes of them to build the above gadget and make all other nodes' outgoing edges forbidden edges. It is easy to see that there is still no maximal Nash equilibrium in this graph, thus the theorem holds. □

An important remark is that for the gadget in Figure 1, when $\ell \geq 4$, all maximal graphs become maximal Nash equilibria for the ℓ-B^{3}C game. Therefore, we need both Lemma 3 and Lemma 4 to show the following theorem.

Theorem 1. *For any $\ell \geq 2$ and $n \geq 6$, there is an instance of ℓ-B^{3}C game with n players that does not have any maximal Nash equilibrium.*

4 Hardness Result

In this section we show that determining the existence of maximal Nash equilibria given an ℓ-B^{3}C game is NP-hard. In fact, we can combine the definition of strict Nash equilibria to obtain a stronger result.

We define a problem TWOEXTREME as follows. The input of the problem is (n, b, c, w) as the parameter of an ℓ-B^{3}C game. The output of the problem is Yes or No, such that (a) if the game has a strict Nash equilibrium, the output is Yes; (b) if the game has no maximal Nash equilibrium, the output is No; and (c) for other cases, the output could be either Yes or No. It is easy to see that both deciding the existence of maximal Nash equilibria and deciding the existence of strict Nash equilibria is a stronger problem than TWOEXTREME, because their outputs are valid outputs for the TWOEXTREME problem by Lemma 2. The following theorem shows that even the weaker problem TWOEXTREME is NP-hard.

Theorem 2. *The problem of* TWOEXTREME *is NP-hard.*

The immediate consequence of the above theorem is:

Corollary 1. *Both deciding the existence of maximal Nash equilibria and deciding the existence of strict Nash equilibria are NP-hard.*

Proof. For the case $\ell \geq 3$, we can almost follow the proof in Theorem 2 of [1]. All of the shortest path used in the proof have length at most 3 except in Lemma 9, the shortest path from A to F_j is 4. It is easy to prove that in that case, if we only consider length 3 path, the proof still holds.

Henceforth we focus on the case $\ell = 2$.

We reduce the problem from the 3-SAT problem. Each 3-SAT instance has k variables $\{x_1, x_2, \ldots, x_k\}$ and m clauses $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$. Each variable x has two literals x and $\bar{x}$. Each clause has three literals from three different variables. We use the following construction to obtain an instance of a 2-B^{3}C game with parameters (n, b, c, w) from the 3-SAT instance, which is illustrated by Figure 2.

The overall idea of the reduction is as follows. First, each clause $\mathcal{C}_j$ is mapped to the gadget similar to the gadget in Figure 1 while each literal x_i and $\bar{x}_i$ are mapped to the gadget containing nodes $L_i, \bar{L}_i, P_i, Q_i$. We call nodes L_i's and $\bar{L}_i$'s *literal nodes*. Nodes L_i and $\bar{L}_i$ can either point to node Q_i or all of the nodes X_j. We make sure that those literal nodes pointing to nodes X_j's correspond to an assignment. Next, if the 3-SAT

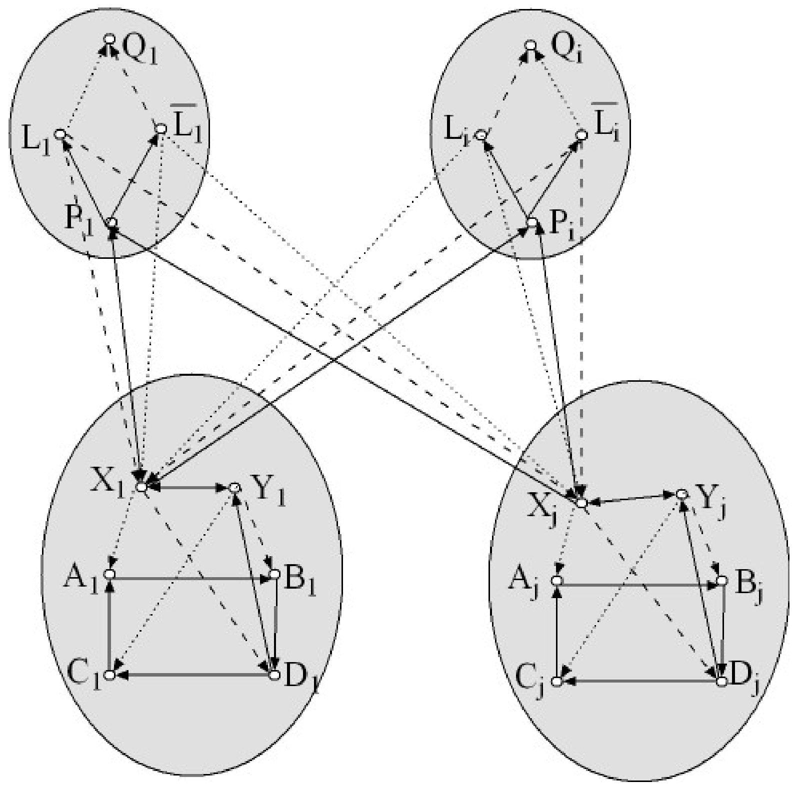

Fig. 2. The structure of the instance of a 2-B^{3}C game corresponding to an instance of a 3-SAT problem. Solid arrows represent fixed edges, while dotted arrows and dashed arrows represent conflicting choices of flexible edges from a node.

instance has a satisfying assignment, we show that for each clause $\mathcal{C}_j$, there exist shortest paths from some literal nodes to A_j with significant weights. We show that these paths make the gadget for clause $\mathcal{C}_j$ stable. Thus all gadgets are stable and the configuration is a maximal Nash equilibrium. We further argue that it is a strict Nash equilibrium by examining all other alternatives of all nodes and showing that they strictly decrease nodes' betweenness. Finally, if the 3-SAT instance has no satisfying assignment, there must exist at least one clause $\mathcal{C}_j$ such that there is no path from the literal nodes to A_j with nonzero weights. When this is the case, the gadget corresponding to $\mathcal{C}_j$ will not be stable and thus the game has no Nash equilibrium.

All of the solid arrows in the graph are called *fixed edges*. They are $\{(P_i, L_i), (P_i, \bar{L}_i), (X_j, Y_j), (Y_j, X_j), (A_j, B_j), (B_j, D_j), (D_j, C_j), (C_j, A_j), (X_j, P_i), (D_j, Y_j) \mid \forall 1 \leq i \leq k, 1 \leq j \leq m\}$. All of the dashed arrows and dotted arrows represent conflicting choices of *flexible edges* starting from one node (e.g. edge (L_1, Q_1) cannot be selected together with any edge (L_1, X_j)). They are $\{(L_i, Q_i), (\bar{L}_i, Q_i), (L_i, X_j), (\bar{L}_i, X_j), (X_j, A_j), (X_j, D_j), (Y_j, B_j), (Y_j, C_j) \mid \forall 1 \leq i \leq k, 1 \leq j \leq m\}$.

We set the parameters (n, b, c, w) of the ℓ-B^{3}C game as follows. First, $n = 4k + 6m$. The budgets of all nodes are 0 except $b(L_i) = b(\bar{L}_i) = m$ and $b(X_j) = b(Y_j) = 1$. The costs of all fixed edges are 0. The costs of all flexible edges are 1 except $c(L_i, Q_i) = c(\bar{L}_i, Q_i) = m$. The costs of all other edges (which is forbidden edges) are larger than m. Finally, the weight function has to be carefully set as follows to make the reduction work. For all $1 \leq i \leq k, 1 \leq j \leq m, w(X_j, L_i) = w(X_j, \bar{L}_i) = w(Y_j, P_i) = w(L_i, Y_j) = w(\bar{L}_i, Y_j) = 1$,; for all $1 \leq i \leq k, 1 \leq j \leq m, w(P_i, Q_i) = ma, w(P_i, X_j) = w(P_i, Y_j) = a$ for some constant a; for all $1 \leq j \leq m, w(X_j, B_j) = w(X_j, C_j) = w(Y_j, A_j) = w(Y_j, D_j) = w(C_j, B_j) = w(B_j, C_j) = w(A_j, D_j) = w(D_j, A_j) = w(B_j, Y_j) = w(D_j, X_j) = 1$; for all $i \in \{1, \ldots, k\}$ and all $j \in \{1, \ldots, m\}$, if literal x_i (or $\bar{x}_i$) is in clause $\mathcal{C}_j$, then $w(L_i, A_j) = b$ (or $w(\bar{L}_i, A_j) = b$), for some constant $b > 1$. For all other pairs (u, v) not included above, $w(u, v) = 0$.

We consider maximal graphs of the game in which all nodes exhaust their budget. Then, for all nodes L_i and $\bar{L}_i$, they point to Q_i or the nodes X_j for all $1 \leq j \leq m$ in G. We call the second case *pointing to the clause nodes*. We say that a maximal graph G of the game is an *assignment graph* if for all $1 \leq i \leq k$, there is exactly one node from $\{L_i, \bar{L}_i\}$ pointing to Q_i in G. Thus, the other node points to the clause nodes.

Lemma 5. *If a maximal graph G of the game is stable, G must be an assignment graph.*

Proof. Suppose, for a contradiction, that G is not an assignment graph. Then for some $i \in \{1, \ldots, k\}$, both L_i and $\bar{L}_i$ connect to Q_i or to X_j. Suppose they both connect to Q_i. The only shortest paths that pass through L_i and $\bar{L}_i$ and have nonzero weights are $\langle P_i, L_i, Q_i \rangle$ and $\langle P_i, \bar{L}_i, Q_i \rangle$. Since $w(P_i, Q_i) = ma$, we have $btw_{L_i}(G) = btw_{\bar{L}_i}(G) = ma/2$. In this case, L_i can change its strategy to connect to the clause nodes instead of Q_i to obtain G'. In G', L_i is on the only shortest path from P_i to X_j, and thus $btw_{L_i}(G') = m \times a > btw_{L_i}(G)$. Therefore, G is not stable, contradicting to the assumption of the lemma.

Now suppose that both L_i and $\bar{L}_i$ connect to the clause nodes. They split the shortest paths from P_i to X_j, which contributes $ma/2$ to the betweenness of L_i and $\bar{L}_i$ each. By the same reason, L_i can change its strategy to connect to Q_i instead of X_j to obtain betweenness value ma. Therefore, G is not stable, again contradicting to the assumption of the lemma. Hence, G must be an assignment graph. $\square$

Lemma 6. *If the 3-SAT instance does not have a satisfying assignment, then for any maximal assignment graph G, there always exists a $j \in \{1, \ldots, m\}$ such that for all $i \in \{1, \ldots, k\}$ and all literals $v \in \{L_i, \bar{L}_i\}$, edge (v, X_j) being in G implies $w(v, A_j) = 0$.*

Proof. Suppose that the 3-SAT instance does not have a satisfying assignment and G is a maximal assignment graph. The edges pointing to the clause nodes in G correspond to a truth assignment to variables in the 3-SAT instance: If the node L_i points to the clause nodes in G, assign variable x_i to be true; otherwise, assign variable x_i to be false. Since the 3-SAT instance is not satisfiable, for the above assignment, there exists a clause $\mathcal{C}_j$ that is evaluated to false. For any variable x_i not in $\mathcal{C}_j$ we have $w(L_i, A_j) = w(\bar{L}_i, A_j) = 0$ by our definition of the weight function. So we only consider a variable x_i appearing in $\mathcal{C}_j$. If the node L_i points to the clause nodes in G, we assign x_i to true, and since $\mathcal{C}_j$ is evaluated to false, we know that literal $\bar{x}_i$ is in $\mathcal{C}_j$. Then by our definition, $w(\bar{L}_i, A_j) = b$ but $w(L_i, A_j) = 0$. The case when $\bar{L}_i$ points to the clause nodes in G has a symmetric argument. Therefore, the lemma holds. $\square$

Lemma 7. *For a maximal assignment graph G, if there exists a $j \in \{1, \ldots, m\}$ such that for all $i \in \{1, \ldots, k\}$ and all literals $v \in \{L_i, \bar{L}_i\}$, node v pointing to the clause nodes in G implies $w(v, A_j) = 0$, then G is not a Nash equilibrium.*

Proof. Consider such a graph G with $j \in \{1, \ldots, m\}$ satisfying the condition given in the lemma. Consider the shortest paths that pass through X_j and Y_j. Since all literal nodes that connect to the clause nodes have zero weights to A_j, the only shortest paths passing through X_j and Y_j that have nonzero weights are paths from X_j to B_j, C_j, from Y_j to A_j, D_j, from $L_i, \bar{L}_i$ to Y_j and from D_j to X_j. The betweenness of pairs from $L_i, \bar{L}_i$ to Y_j and from D_j to X_j are only affected by whether X_j points to Y_j and vice verse. Since these two edges are cost 0, they are always connected in a stable graph. For other pairs, it essentially reduces the gadget corresponding to $\mathcal{C}_j$ to the gadget in Figure 1. The only difference is that here we have an additional edge (D_j, Y_j) compare to Figure 1. But the addtional edge does not have any infection to the betweenness value of node X_j and node Y_j. It only helps to make the graph a strict Nash equilibrium when needed. We will explain this later in Lemma 10. Therefore, by an argument similar to the one in the proof of Theorem 1, no matter how X_j and Y_j currently connect to nodes in $\{A_j, B_j, C_j, D_j\}$, one of them will always want to change its strategy to increase its utility. Therefore, G is not a Nash equilibrium. $\square$

Lemma 8. *If the 3-SAT instance does not have a satisfying assignment, then the constructed 2-B^{3}C game instance does not have maximal Nash equilibrium.*

Proof. Suppose, for a contradiction, that the 2-B^{3}C game instance has a maximal Nash equilibrium. Then there exists a maximal graph G that is stable. By Lemma 5, G must be an assignment graph. Since the 3-SAT instance does not have a satisfying assignment, by Lemmata 6 and 7, G is not stable, a contradiction. □

Lemma 9. *If the 3-SAT instance has a satisfying assignment, then there exists a maximal assignment graph G of the game in which for all $j \in \{1, \ldots, m\}$, there exists $i \in \{1, \ldots, k\}$ and literal $v \in \{L_i, \bar{L}_i\}$ such that the node v points to the clause nodes in G and $w(v, A_j) = b$.*

Proof. Suppose that the 3-SAT instance has a satisfying assignment f. construct a maximal assignment graph G such that for all $i \in \{1, \ldots, k\}$, if variable x_i is assigned to true in the assignment f, then L_i connects to the clause nodes; otherwise, $\bar{L}_i$ connects to the clause nodes. For all $j \in \{1, \ldots, m\}$, since clause $\mathcal{C}_j$ is evaluated to true under assignment f, there exists variable x_i whose corresponding literal in $\mathcal{C}_j$ is evaluated to true. If literal x_i is in $\mathcal{C}_j$, x_i is assigned to true. By the above construction of G, L_i points to the clause nodes in G, and by the definition of the weight function, $w(L_i, A_j) = b$. The same argument applies to the case when literal $\bar{x}_i$ is in $\mathcal{C}_j$. Therefore, the lemma holds. □

Lemma 10. *Given a maximal assignment graph G in which for all $j \in \{1, \ldots, m\}$, there exists $i \in \{1, \ldots, k\}$ and literal $v \in \{L_i, \bar{L}_i\}$ such that the node v points to the clause nodes in G and $w(v, A_j) = b$, we construct a graph G' such that G' is the same as G except that for all $j \in \{1, \ldots, m\}$, X_j connects to A_j and Y_j are connected to C_j in G'. The maximal graph G' must be a strict Nash equilibrium.*

Proof. We prove that in G' any strategy change strictly decreases the changers betweenness, and thus G' must be a strict Nash equilibrium.

We go through all nodes and check all possible strategy changes in the following list.

- For each node Q_i, $i \in \{1, \ldots, k\}$, it has only the empty strategy so there is no strategy change for Q_i.
- For nodes other than $L_i, \bar{L}_i, X_j, Y_j (1 \le i \le k, 1 \le j \le m)$, they only have fixed edge to choose, so we only need to prove that for each fixed edge, there exists a pair with nonzero weight such that if the node removes this fixed edge, the betweenness value will decrease. We call this pair *pushes* such fixed edge.
  For node P_i, pair (X_j, L_i) pushes edge (P_i, L_i) while pair $(X_j, \bar{L}_i)$ pushes edge $(P_i, \bar{L}_i)$.
  For node A_j, pair (C_j, B_j) pushes edge (A_j, B_j). For node B_j, pair (A_j, D_j) pushes edge (B_j, D_j). For node C_j, pair (D_j, A_j) pushes edge (C_j, A_j). For node D_j, pair (B_j, C_j) pushes edge (D_j, C_j) while pair (B_j, Y_j) pushes edge (D_j, Y_j).
- For each node L_i, $i \in \{1, \ldots, k\}$, its strategy change is either removing its flexible edge or changing its flexible edge. If it removes its flexible edge, it loses the shortest path from P_i to Q_i or X_j, and since $w(P_i, Q_i) = a$ and $w(P_i, X_j) = a/m$, its betweenness strictly decreases. If it changes its flexible edge, then both L_i and $\bar{L}_i$ connects to Q_i or X_j. By the same argument as in the proof of Lemma 5, its betweenness strictly decreases. For each node $\bar{L}_i$, $i \in \{1, \ldots, k\}$, the argument is the same as the argument for L_i.
- For each node X_j, $j \in \{1, \ldots, m\}$, it can remove its fixed edge or remove its flexible edge or change its flexible edge. For the fixed edge, pair (Y_j, P_i) pushes edge (X_j, P_i) and pair (L_i, Y_j) or $(\bar{L}_i, Y_j))$ pushes edge (X_j, Y_j). Then, we only consider the betweenness value caused by the flexible edge. By the assumption of the Lemma, there exists $i \in \{1, \ldots, k\}$ and literal node $v \in \{L_i, \bar{L}_i\}$ such that the node v points to the clause nodes G and $w(v, A_j) = b$. Suppose that there are t such literal nodes v. By the definition of w, we know that $t \le 3$. Since X_j splits the shortest paths from v to A_j and Y_j to A_j $btw_{X_j}(G', 2) = tb + 1/2 \ge b + 1/2$. If X_j removes its flexible edge (X_j, A_j), it will not connect to any node and its betweenness will decrease to zero. If X_j changes its flexible edge to (X_j, D_j) to obtain

a graph G'', it does not connect nodes v and A_j but gain the full share on the shortest paths from Y_j to D_j. Then $btw_{X_j}(G'', 2) = 1 < b + 1/2 \leq btw_{X_j}(G', 2)$ since $b > 1$. So X_j's betweenness strictly decreases. Therefore, all strategy changes on X_j strictly decreases X_j's betweenness.
- For each node Y_j, $j \in \{1, \ldots, m\}$, it can remove its fixed edge or remove its flexible edge or change its flexible edge. For the fixed edge, pair (D_j, X_j) pushes edge (Y_j, X_j). For the flexible edge, by the same argument in Theorem 1, all strategy changes on Y_j strictly decreases Y_j's betweenness.

By the above argument exhausting all possible cases, we show that graph G' is indeed a strict Nash equilibrium. □

Lemma 11. *If the 3-SAT instance has a satisfying assignment, then the constructed 2-B^{3}C game instance has a strict Nash equilibrium.*

Proof. This is immediate from Lemmata 9 and 10. □

The entire proof for the case $\ell = 2$ of Theorem 2 is now complete with Lemmata 8 and 11. □

References

1. Wei Chen, Shang-Hua Teng, and Jiajie Zhu, *The betweenness centrality game for strategic network formations*, Tech. Report MSR-TR-2008-167, Microsoft Research, November 2008.