The Betweenness Centrality Game for Strategic Network Formations

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Microsoft Research Technical Report
MSR-TR-2008-167
November 2008

Abstract. In computer networks and social networks, the *betweenness centrality* of a node measures the amount of information passing through the node when all pairs are conducting shortest-path exchanges. In this paper, we introduce a strategic network-formation game based on the betweenness centrality. In this game, nodes are selfish players, each of which has some budget to build connections. The goal of each player is to fully utilize her budget to strategically build connections so that her betweenness centrality is as large as possible. We refer to this game as the *bounded budget betweenness centrality* game or the B^3C game.

We present both theoretical and experimental results about this game. Theoretically, we show that a general B^3C game may not have any nontransient Nash equilibrium and it is in fact NP-hard to determine whether nontransient Nash equilibria exist. Experimentally, we studied the family of uniform B^3C games, in which there is an equal amount of information exchange between every pair of nodes, every connection has the same construction cost and every node has enough budget to build k connections. We have discovered several interesting Nash equilibria when $k = 2$. Our experiments have also inspired us to establish some theoretical results about uniform B^3C games. For example, we show that, when k is a variable, it is NP-hard to compute a best response of a node in uniform B^3C games; we also prove that a family of symmetric graphs called Abelian Cayley graphs cannot be Nash equilibria for $k = 2$ when the graph is large enough, but there is a unique nontransient Nash equilibrium when $k = 1$. We conjecture, based on our experiments, that every uniform B^3C game has a Nash equilibrium, and more strongly, has a Nash equilibrium that is Eulerian.

1 Introduction

Many network structures in real life are not designed by central authorities. Instead, they are formed by autonomous agents who often have selfish motives [12]. Some typical examples of such networks include the Internet where autonomous systems linked together to achieve global connection, peer-to-peer networks where peers connect to one another for online file sharing (e.g. [2, 14]), and social networks where individuals connect to one another for information exchange and other social functions [13]. Since these autonomous agents have their selfish motives and are not under any centralized control, they often act strategically in deciding whom to connect to in order to improve their own benefits. This gives rise to the field of *network formation games*, which studies the game-theoretic properties of the networks formed by these selfish agents as well as the process in which all agents dynamically adjust their strategies [4, 1, 10, 9, 8].

A key measure of importance of a node is its betweenness centrality. The *betweenness centrality* (or betweenness for short) is introduced originally from social network analysis as one of the measures on how central an individual is in a social network [6, 11]. If we view a network as a graph $G = (V, E)$ (directed or undirected), the betweenness of a node (or vertex) i in G is

$$btw_i(G) = \sum_{u \neq v \neq i \in V,\, m(u,v)>0} w(u, v) \frac{m_i(u, v)}{m(u, v)} \tag{1}$$

⋆ Supported by NSF grants CCR-0635102 and ITR CCR-0325630. Part of this work was done while visiting ITCS at Tsinghua University and Microsoft Research Asia Lab.

where $m(u, v)$ is the number of shortest paths from u to v in G, $m_i(u, v)$ is the number of shortest paths from u to v that passes i in G, and $w(u, v)$ is the weight on pair (u, v). Intuitively, if the amount of information from u to v is $w(u, v)$, and the information is passed along the shortest paths from u to v (which is often the case in computer networks with optimal routing algorithms) and all shortest paths split the traffic equally, then the betweenness of node i measures the amount of information passing through i incurred by all pair-wise exchanges.

In a decentralized network with autonomous agents, each agent may have incentive to maximize its betweenness in the network. For example, in computer networks and peer-to-peer networks, a node in the network may be able to charge on the traffic that it helps relaying, in which case the revenue of the node is proportional to its betweenness in the network. So the maximization of revenue is consistent with the maximization of the betweenness. In a social network, an individual may want to gain or control the most amount of information travelling in the network by maximizing her betweenness.

In this paper, we introduce a network formation game in which agents' objectives are to maximize their betweenness centrality. In this game, nodes are selfish players, each of which has some budget to build connections. The goal of each player is to fully utilize her budget to strategically build connections so that her betweenness centrality is as large as possible. We refer this game to as the *bounded budget betweenness centrality* game or the B^3C game. In particular, we focus on one variant of such games, while other variants are subjects of future research. We consider that links in the network are directed and nodes can establish outgoing links to other nodes. This is suitable for computer networks and peer-to-peer networks that relay traffics, and some type of social networks where information flows between connected pairs are often one-directional. Each link has a cost to be established, and each node has a bounded budget such that the total cost of establishing the outgoing links it selects cannot exceed its budget. This reflects the practical situation in computer and peer-to-peer networks where the number of connections are often restricted due to resource constraint, and in social networks where each individual only has a finite amount of time and energy to create and maintain relationships.

We focus on the existence and the properties of Nash equilibria in B^3C games. In an equilibrium configuration, no player can improves her betweenness by unilaterally changing her connections, i.e., her current connection is the best response to the connections of others. Since the game allows some trivial Nash equilibria (such as a network with no links at all), we study a stronger form of Nash equilibria called *nontransient* Nash equilibria. Informally, a Nash equilibrium is nontransient if after any finite sequence of strategy changes in which each change keeps the changer's utility unmodified, the resulting configuration is still a Nash equilibrium.

We present both theoretical and experimental results about this game. We first show that a general B^3C game may not have any nontransient Nash equilibria. A general B^3C game is specified by several parameters concerning the node budgets, link costs, and pair-wise exchange weights (see Section 2 for a formal definition). Moreover, given these parameters as input, we show that it is NP-hard to determine whether the game has a nontransient Nash equilibrium. This indicates that finding Nash equilibria in general B^3C games is a difficult task.

Next, we turn our attention to *uniform* B^3C games where all pair weights are 1, all link costs are 1, and all node budgets are given as an integer k, that is, each node is allowed to create at most k outgoing links. By experiments, we have discovered several interesting Nash equilibria when $k = 2$. Our experiments have also inspired us to establish some theoretical results about uniform B^3C games. For example, we show that, when k is a variable, it is NP-hard to compute a best response of a node in uniform B^3C games (similar to a result in [4]). This result indicates that large-scale computer simulations of the game dynamics is infeasible. We investigate a class of symmetric graphs call Abelian Cayley graphs, where nodes are from an Abelian group and edges are determined by a generating set of size k. We prove that, when $k = 2$ and the number of nodes is large enough, no Abelian Cayley graph is a Nash equilibrium, but there is a unique nontransient Nash equilibrium when $k = 1$. Like the result of [9], this is an indication that Nash equilibria for uniform

B^3C games may also have some symmetry breaking properties, and we conjecture that the result is also true for $k > 2$.

Finally, based on our experiments, we conjecture that every uniform B^3C game has a Nash equilibrium, and more strongly, has a Nash equilibrium that is Eulerian.

Related work. There are a number of studies on network formation games with Nash equilibrium as the solution concept [4, 1, 10, 9, 8]. However, to the best of our knowledge, we are not aware of any work that uses betweenness centrality as the objective function in network formation games. Most of the above work belong to a class of games in which nodes try to minimize their average shortest distances to other nodes in the network [4, 1, 10, 9], which is call *closeness centrality* in social network analysis [6]. The game in [4] considers undirected edges and the cost of links in the network are part of the objective function to minimize. It focuses on the study of price of anarchy of the game and also presents results on the structure of Nash equilibria. In [1], Albers et al. extend the research of [4] by disproving a conjecture made in [4] that all Nash equilibria have a tree structure, and studying other variants of the game including the cost of an edge being shared by two end nodes. The game in [10] instead considers minimizing the average stretch of each node, where stretch is defined as the ratio between the shortest path distance of two nodes in the graph versus the geometric distance in the underlying space.

Our game is inspired by the BBC game of Laoutaris *et al* [9]. This game considers directed links and bounded budgets on nodes, using minimization of average shortest distances to others as the objective for each node. It shows hardness results in determining the existence of Nash equilibria in general games, and provides tree-like structures as Nash equilibria for the uniform version of the game. It also shows that Abelian Cayley graphs cannot be Nash equilibria in large networks, similar to the result in our paper.

In [8], Kleinberg et al. study a different type of network formation games related to the concept of structural holes in organizational social network research. In this game, each node tries to bridge other pairs of nodes that are not directed connected. In a sense, this is a restricted type of betweenness where only length-two shortest paths are considered. There are also a number of other differences between their work and our paper. They consider undirected links, they do not consider bounded budget but include link costs as part of the objective function, and they also include the number of direct neighbors as part of the benefit in the objective function. They show the structures of Nash equilibria in their game, and also show that polynomial-time algorithm exists for computing a best response for a node, in contrast to the hardness result in our paper.

Solution concept other than Nash equilibrium is also used in the study of network formation games. Authors in [7, 3] consider games in which two endpoints of a link have to jointly agree on adding the link, and they use pairwise stability as an alternative to Nash equilibrium.

Paper organization. Section 2 provides the detailed definition of the B^3C game and the related concepts. Section 3 studies the nonuniform games, while Section 4 studies uniform games. We conclude the paper and list open problems and future directions in Section 5. Due to space constraints, proofs not fitting into the main text are included in the appendix.

2 Problem definition

A *Bounded-Budget Betweenness Centrality* game (or B^3C game for short) is one class of network formation games defined as follows. We consider a set of n players $V = \{1, 2, \ldots, n\}$, which are also nodes in a network. Function $b : V \rightarrow \mathbb{N}$ specifies the budget $b(i)$ for each node $i \in V$ ($\mathbb{N}$ is the set of natural numbers). Function $c : V \times V \rightarrow \mathbb{N}$ specifies the cost $c(i, j)$ for the node i to establish a link to node j, for

$i, j \in V$. Function $w : V \times V \rightarrow \mathbb{N}$ specifies the weight $w(i, j)$ from node i to node j for $i, j \in V$, which can be interpreted as the amount of traffic i sends to j, or the importance of the communication from i to j.[1]

The strategy space of player i in B^{3}C game is $S_i = \{s_i \subseteq V \setminus \{i\} \mid \sum_{j \in s_i} c(i, j) \leq b(i)\}$, i.e., all possible subsets of outgoing links of node i within i's budget. The combination of all players' strategies $s = (s_1, s_2, \ldots, s_n) \in S_1 \times S_2 \times \ldots \times S_n$ is a *configuration*. The graph induced by configuration s is denoted as $G_s = (V, E)$, where $E = \{(i, j) \mid i \in V, j \in s_i\}$. For convenience, we will also refer G_s as a configuration.

The utility function that each node wants to maximize is defined by the *betweenness centrality* of the graph. The betweenness centrality of node i in configuration s (or in graph G_s) is

$$btw_i(s) = btw_i(G_s) = \sum_{u \neq v \neq i \in V,\, m(u,v) > 0} w(u, v) \frac{m_i(u, v)}{m(u, v)} \tag{2}$$

where $m(u, v)$ is the number of shortest paths from u to v in G_s and $m_i(u, v)$ is the number of shortest paths from u to v that passes i in G_s.

In a configuration s, if no node can increase its own utility by changing its own strategy unilaterally, we say that s is a *(pure) Nash equilibrium*, and we also say that s is *stable*. A Nash equilibrium is a *transient* Nash equilibrium if there exists a finite sequence of single-player strategy changes such that the last strategy change strictly increases the changer's utility while each of the rest changes leaves the changer's utility unchanged. A *nontransient* Nash equilibrium is a Nash equilibrium that is not transient.

A B^{3}C game is *uniform* if $b(i) = k$ for all $i \in V$ and some parameter $k \in \mathbb{N}$, and $c(i, j) = w(i, j) = 1$ for all $i, j \in V$. As a contrast, the general form is called *nonuniform* games. Therefore, a uniform B^{3}C game is determined by parameters (n, k), while a nonuniform B^{3}C game is determined by parameters (n, b, c, w).

3 Nonuniform games

In this section, we study the existence of Nash equilibria in nonuniform B^{3}C games. We first show through the following lemma that the game allows some trivial transient Nash equilibria.

Lemma 1. *In any nonuniform B^{3}C game, the following configurations are all Nash equilibria: (i) the empty graph with no edges; and (ii) any Hamiltonian cycle, i.e., any cycle that passes through every node exactly once, provided that the budget constraint allows such configurations. Moreover, they are transient Nash equilibria under the following conditions: (a) for the empty graph, if there exist nodes i, j, and k such that $\{j\} \in S_i$, $\{k\} \in S_j$ and $w(i, k) > 0$; (b) for a Hamiltonian cycle, if there exist nodes i, j and k such that (b.1) j is on the shortest path from i to k, (b.2) the length of the shortest path from k to i is at least 2, (b.3) k is allowed to add an edge to j, (b.4) j is allowed to add an edge to i, and (b.5) $w(k, i) > 0$.*

The above lemma shows that many trivial Nash equilibria may exist for a B^{3}C game because a node can only change its own outgoing edges, which may not directly make it sitting on more shortest paths. However, these trivial Nash equilibria are usually transient, since when several nodes add more edges, some node may start to receive benefit by adding new edges. Therefore, for B^{3}C games, it is more meaningful to study nontransient Nash equilibria, which is our focus from now on.

3.1 Nonexistence of nontransient Nash equilibrium

In this section, we show that nontransient Nash equilibria may not exist in some version of B^{3}C games where edge costs are not uniform. We first show that adding edges can only increase the betweenness of a node.

[1] We may also define a distance function specifying distances between every pair of nodes, but it is not needed throughout our paper.

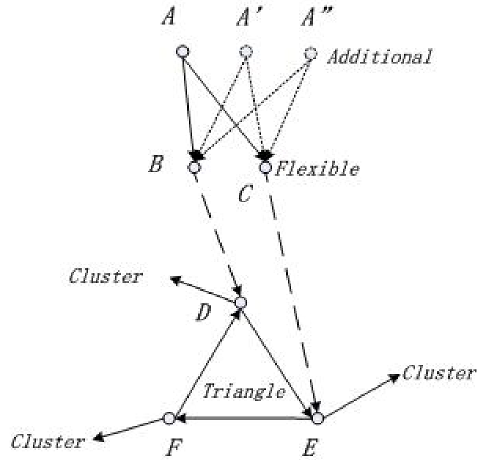

Fig. 1. Main structure of the gadget that has no nontransient Nash Equilibrium.

Lemma 2. *Adding an outgoing edge to a node i does not decrease i's betweenness. That is, for any graph $G = (V, E)$ with $i \in V$ and $(i, j) \notin E$ for some $j \in V$. Let $G' = (V, E \cup \{(i, j)\})$. Then $btw_i(G) \leq btw_i(G')$.*

Given a nonuniform B^{3}C game with parameters (n, b, c, w), we say that a graph (configuration) is *maximal* if no node can add any edges to the graph without exceeding its budget. We say that a graph $G_1 = (V, E_1)$ is a subgraph of graph $G_2 = (V, E_2)$ if $E_1 \subseteq E_2$. The above lemma implies that when studying nontransient Nash equilibria, we only need to look at maximal graphs, which is formalized by the following lemma.

Lemma 3. *Given a nonuniform B^{3}C game with parameters (n, b, c, w), if a maximal graph G of the game is not stable, then any subgraph of G cannot be a nontransient Nash equilibrium. If none of the maximal graphs of the game is stable, then the game does not have any nontransient Nash equilibrium.*

We now construct a family of graphs, which we refer to as the gadget, and show that B^{3}C games based on the gadget do not have any nontransient Nash equilibrium. The gadget is shown in Figure 1. There are $5 + 3t + r$ nodes in the gadget, where $t \in \mathbb{N}$ and $r = 1, 2, 3$. The values of t and r allow us to construct a graph of any size great than 5. There are r nodes, denoted as A, A', A'' in the figure, which establish edges to B and C. Both B, C can establish at most one edge to a node in $\{D, E, F\}$ respectively. Each node in $\{D, E, F\}$ connects to a cluster of size t each (not shown in the figure). The only requirement for each cluster is that it is strongly connected so D, E, F can reach all nodes in their corresponding clusters. Nodes in the three clusters do not establish edges to the other clusters or to $A, A', A'', B, C, D, E, F$.

We classify nodes and edges as follows. Nodes B and C are *flexible* nodes since they can choose to connect one node in $\{D, E, F\}$. Nodes D, E, F are *triangle* nodes, nodes in the clusters are *cluster* nodes, and nodes A, A', A'', are *additional* nodes. Edges (i, j) with $i \in \{B, C\}$ and $j \in \{D, E, F\}$ *flexible* edges. Other edges shown in the figure plus the edges in the clusters are *fixed* edges. The remaining pairs with no edge connected (e.g. (A, D), (A, E), etc.) are referred to as *forbidden* edges.

We use the parameters (n, b, c, w) of a B^{3}C game to realize the gadget. In particular, (a) $n = 5 + 3t + r$; (b) $b(i) = 1$ for all $i \in V$; (c) $c(i, j) = 0$ if (i, j) is a fixed edge, $c(i, j) = 1$ if (i, j) is a flexible edge, $c(i, j) = M > 1$ if (i, j) is a forbidden edge; and (d) $w(i, j) = 1$ for all $i, j \in V$.

With the above construction, we can show the following theorem.

Theorem 1. *The B^3C game based on the gadget of Figure 1 does not have any nontransient Nash equilibrium. This implies that for any $n \geq 6$, there is an instance of B^3C game with n players that does not have any nontransient Nash equilibrium, and in the game only the edge costs are nonuniform.*

Proof. By Lemma 3, we only need to show that none of the maximal graphs of the game is stable. Note that in a maximal graph all fixed edges are included, and nodes B and C each selects one edge to connect to one node in $\{D, E, F\}$. Consider one maximal graph G in which B connects to D and C connects to E (as in Figure 1). Node B is on all shortest paths from nodes in $\{A, A', A''\}$ to D and the cluster D points to, but it is not on any shortest paths from nodes in $\{A, A', A''\}$ to E and F and the two clusters they point to (these shortest paths all pass through C). Thus $btw_B(G) = r(t + 1)$. In this case, B can change its strategy to connect to F instead of D, so that it will be on all shortest paths from those additional nodes to F and D and their clusters, and thus its betweenness is increased to $2r(t + 1)$. Therefore, maximal graph G is not stable.

The second case to consider is that both B and C connect to the same node, say E. In this case, they split equally among all shortest paths from the additional nodes to the triangle nodes and the clusters nodes, giving each of them a betweenness $3r(t + 1)/2$. In this case, each of them could improve their betweenness to $2r(t + 1)$ by connecting to F instead of E. Hence, this maximal graph is not stable either.

All other maximal graphs are rotationally equivalent to one of the above two graphs. Therefore, we know that none of the maximal graphs is stable, and the theorem holds. □

3.2 Complexity of determining the existence of nontransient Nash equilibrium

In this section we use the gadget given in Figure 1 as a building block to show that determining the existence of nontransient Nash equilibrium given a nonuniform B^{3}C game is NP-hard.

Theorem 2. *It is NP-hard to determine whether there is a nontransient Nash equilibrium in a nonuniform B^3C game nonuniform weights, and edge costs.*

Proof. We reduce the problem from the 3-SAT problem. Each 3-SAT instance has ℓ variables $\{x_1, x_2, \ldots, x_\ell\}$ and m clauses $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m\}$. Each variable x has two literals x and $\bar{x}$. Each clause has three literals from three different variables. We use the following construction to obtain an instance of a B^3C game with parameters (n, b, c, w) from the 3-SAT instance, which is illustrated by Figure 2.

Each clause $\mathcal{C}_j$ is mapped to the core of gadget of Figure 1, which is the substructure of the gadget excluding the additional nodes and the cluster nodes. We use B_j and C_j to represent the flexible nodes in the gadget and D_j, E_j and F_j to represent the triangle nodes in the gadget, all corresponding to the clause $\mathcal{C}_j$. This leads to $5m$ nodes in the graph. There is a sepcial node A called the *assignment node*, with fixed edges pointing to all flexible nodes B_j and C_j in all gadgets corresponding to all clauses.

Each variable x_i is mapped to a structure with four nodes P_i, Q_i, L_i, and $\bar{L}_i$. Node P_i has two fixed edges pointing to L_i and $\bar{L}_i$. Node L_i and $\bar{L}_i$, called *literal nodes*, each may have one flexible edge pointing to either Q_i or the assignment node A. For each clause $\mathcal{C}_j$ with three variables x_{i_1}, x_{i_2} and x_{i_3}, we add one fixed edge from D_j to each of P_{i_1}, P_{i_2} and P_{i_3} respectively.

To realize the above structure, we set the parameters (n, b, c, w) of the B^{3}C game as follows. First, $n = 1 + 4\ell + 5m$ and $b(i) = 1$ for all $i \in V$. Next, same as in Figure 1, each fixed edge has cost 0, each flexible edge has cost 1 (so that the corresponding starting node can choose at most one flexible edge), and each forbidden edge has cost $M > 1$. Finally, the weight function has to be carefully set as follows to make the reduction work. For all $j \in \{1, \ldots, m\}$, $w(A, D_j) = w(A, E_j) = w(A, F_j) = w(D_j, F_j) = w(E_j, D_j) = w(F_j, E_j) = 1$; for all $i \in \{1, \ldots, \ell\}$, $w(P_i, A) = w(P_i, Q_i) = a$ for some constant $a > 2m$; for all $i \in \{1, \ldots, \ell\}$ and all $j \in \{1, \ldots, m\}$, (a) if clause $\mathcal{C}_j$ contains variable x_i, then $w(P_i, B_j) =$

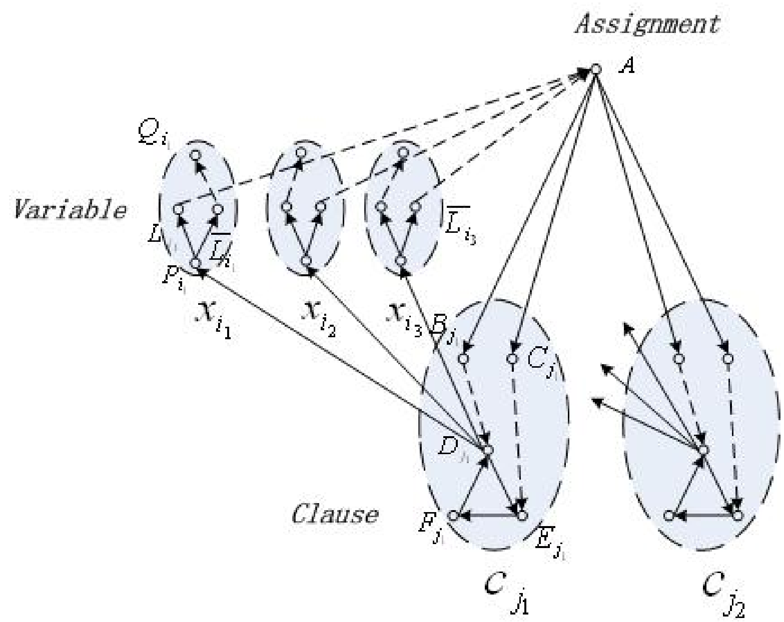

Fig. 2. The strucutre of the instance of a B^3C game corresponding to an instance of a 3-SAT problem.

$w(P_i, C_j) = w(F_j, L_i) = w(F_j, \bar{L}_i) = 1$; and (b) if literal x_i (or $\bar{x}_i$) is in clause $\mathcal{C}_j$, then $w(L_i, D_j) = b$ (or $w(\bar{L}_i, D_j) = b$), for some constant $b > 1$. For all other pairs (u, v) not included above, $w(u, v) = 0$.

The overall idea of the reduction is as follows. First, we make sure that edges pointing to the assignment node A correspond to an assignment to all variables. Next, if the 3-SAT instance has a satisfying assignment, we show that for each clause $\mathcal{C}_j$, there exist shortest paths from some literal nodes to D_j with significant weights. We show that these paths make the gadget for clause $\mathcal{C}_j$ stable. Thus all gadgets are stable and the configuration is a nontransient Nash equilibrium. Finally, if the 3-SAT instance has no satisfying assignment, there must exist at least one clause $\mathcal{C}_j$ such that there is no path from the literal nodes to D_j with nonzero weights. When this is the case, the gadget corresponding to $\mathcal{C}_j$ will not be stable and thus the game has no nontransient Nash equilibrium.

We consider maximal graphs of the game in which all fixed edges are present and exactly one flexible edge from each node in $\{L_i, \bar{L}_i \mid i = 1, 2, \ldots, \ell\} \cup \{B_j, C_j \mid j = 1, 2, \ldots, m\}$ is present. We say that a maximal graph G of the game is an *assignment graph* if for all $i \in \{1, \ldots, \ell\}$, there is exactly one edge from $\{L_i, \bar{L}_i\}$ to A in G. We show the follow sequence of Lemmata to prove the theorem, the proofs of which are included in the appendix.

Lemma 4. *If a maximal graph G of the game is stable, G must be an assignment graph.*

Lemma 5. *If the 3-SAT instance does not have a satisfying assignment, then for any maximal assignment graph G, there always exists a $j \in \{1, \ldots, m\}$ such that for all $i \in \{1, \ldots, \ell\}$ and all literals $v \in \{L_i, \bar{L}_i\}$, edge (v, A) being in G implies $w(v, D_j) = 0$.*

Lemma 6. *For a maximal assignment graph G, if there exists a $j \in \{1, \ldots, m\}$ such that for all $i \in \{1, \ldots, \ell\}$ and all literals $v \in \{L_i, \bar{L}_i\}$, edge (v, A) being in G implies $w(v, D_j) = 0$, then G is not a Nash equilibrium.*

Lemma 7. *If the 3-SAT instance does not have a satisfying assignment, then the constructed B^3C game instance does not have nontransient Nash equilibrium.*

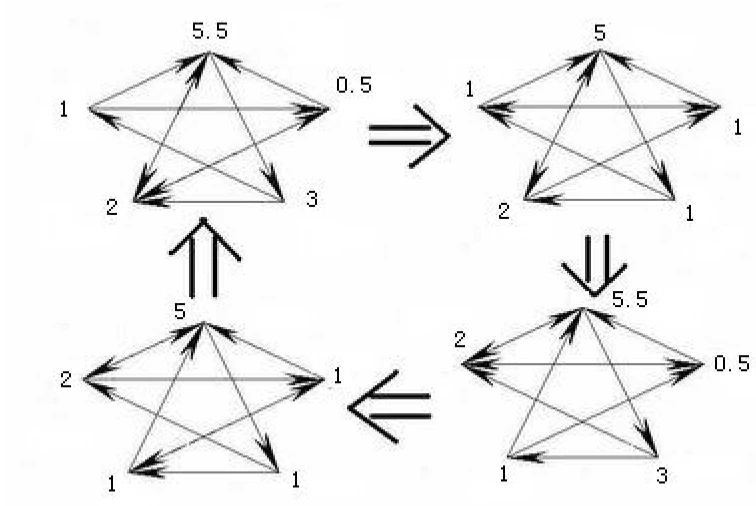

Fig. 3. A cyclic sequence of better responses in the uniform B^{3}C game with $n = 5$ and $k = 2$.

Proof. Suppose, for a contradiction, that the B^{3}C game instance has a nontransient Nash equilibrium. By Lemma 3, there exists a maximal graph G that is stable. By Lemma 4 G must be an assignment graph. Since the 3-SAT instance does not have a satisfying assignment, by Lemmata 5 and 6 G is not stable, a contradiction. □

Lemma 8. *If the 3-SAT instance has a satisfying assignment, then there exists a maximal assignment graph G of the game in which for all $j \in \{1, \ldots, m\}$, there exists $i \in \{1, \ldots, \ell\}$ and literal $v \in \{L_i, \bar{L}_i\}$ such that the edge (v, A) is in G and $w(v, D_j) = b$.*

Lemma 9. *Given a maximal assignment graph G in which for all $j \in \{1, \ldots, m\}$, there exists $i \in \{1, \ldots, \ell\}$ and literal $v \in \{L_i, \bar{L}_i\}$ such that the edge (v, A) is in G and $w(v, D_j) = b$, we construct a graph G' such that G' is the same as G except that for all $j \in \{1, \ldots, m\}$, both B_j and C_j are connected to D_j in G'. The maximal graph G' must be a nontransient Nash equilibrium.*

Lemma 10. *If the 3-SAT instance has a satisfying assignment, then the constructed B^{3}C game instance has a nontransient Nash equilibrium.*

Proof. This is immediate from Lemmata 8 and 9. □

The entire proof for Theorem 2 is now complete with Lemmata 7 and 10. □

4 Uniform games

In this section, we investigate uniform B^{3}C games and report both experimental results and theoretical results on uniform games. Experimentally, we focus on the case of $k = 2$ with small values of n and we study both the Nash equilibria structures and better response dynamics. Our experiments have also inspired us to establish some theoretical results reported in this section.

4.1 Experimental results

We conduct many experiments on uniform B^{3}C games with small n and $k = 2$. The experiments are computationally intensive, so we cannot afford to run experiments for large n yet. We obtained several

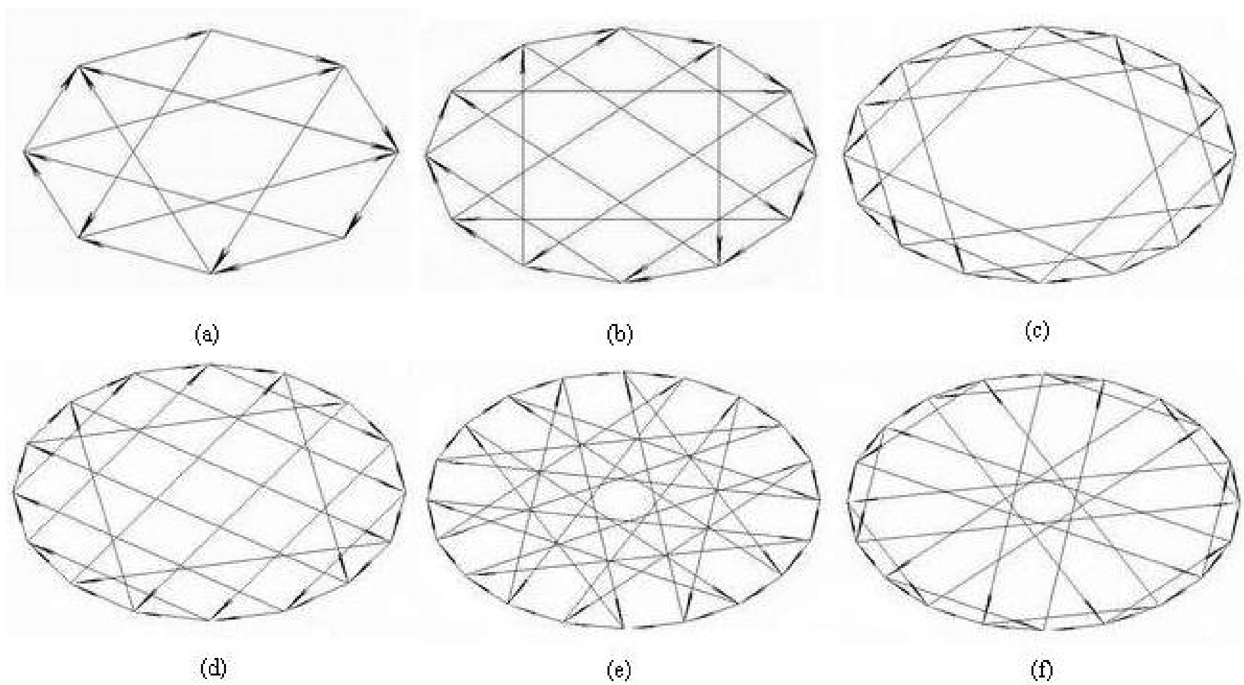

Fig. 4. (a) A counter example with $n = 8$ and $k = 2$ showing that not all Nash equilibria are Eulerian graphs; (b) Nash equilibrium for $n = 12$; (c)(d) Nash equilibrium for $n = 16$; (e)(f) Nash equilibrium for $n = 20$.

results from the experiments. First, we find cyclic sequences of better responses even for $n = 5$ and $k = 2$ (Figure 3). This means that the uniform game is not a potential game.

Next for $k = 2$ and different n values, we try to find the structures of Nash equilibria. When $n > 5$, exhaustive search of Nash equilibria becomes infeasible, and we need to use some conjectures to narrow the search space. The conjectures we use are: for every (n, k), there exist Nash equilibria that have Hamiltonian cycles and are Eulerian (i.e. each node has the same in-degree and out-degree). So far, we have found Nash equilibria satisfying this conjecture for all $n \leq 12$ and $n = 16, 20$. Figure 4 (b)-(f) show examples we found for $n = 12, 16$, and 20, and more examples are included in the appendix. We also found that not all Nash equilibria are Eulerian, and an example is shown in Figure 4 (a). Since it becomes infeasible to exhaustively search all graphs satisfying the above conjecture when $n > 12$, we have not found any n and $k = 2$ that does not satisfy the conjecture.

4.2 Hardness in computing the best response

The *best response* of a node in a configuration of the uniform game is the strategy of the node that gives the node the best utility (i.e. best betweenness). In a uniform game with parameters (n, k), one can exhaustively search all $\binom{n-1}{k}$ and find the one with the largest betweenness. Computing the betweenness of nodes given a fixed graph can be done by all-pair shortest paths algorithms in polynomial time (e.g. [5]). Therefore, the entire brute-force computation takes polynomial time if k is a constant. However, if k is not a constant, we show by the following theorem that computing the best response of a node (reformulated as a decision problem) is NP-hard.

Theorem 3. *Consider the following betweenness problem instance: (a) a directed graph $G = (V, E)$ with n nodes and each node has k outgoing edges; (b) a natural number b; (c) one node v in G. It is NP-hard to* *decide whether there is a strategy of* v *(i.e. a set of* k *nodes in* $V \setminus \{v\}$*) such that* v*'s betweenness is at least* b *using the strategy.*

4.3 On Abelian Cayley graphs

In this section, we investigate a class of symmetric graphs called Abelian Cayley graphs. A Cayley graph is defined by a tuple $\langle G, S \rangle$, where G is a group and S is a generating set of G. Each element in group G corresonds to a node in the graph, and for all $g, h \in G$, there is an edge (g, h) in the graph if any only if $h = g \cdot s$ for some $s \in S$, where $\cdot$ is the operation of the group.

An Abelian Cayley Graph is a Cayley Graph whose group is an Abelian Group. It can be denoted as $\langle \mathbb{Z}_n, A \rangle$, where $\mathbb{Z}_n$ is the additive group containing $\{0..n-1\}$, A is a generating set of $\mathbb{Z}_n$ of size k, and the operation is addition module n. Thus, any Abelian Cayley graph with parameters (n, k) is a configuration of the uniform B^3C game with the same parameters.

We prove below that when $k = 2$, Abelian Cayley graphs cannot be Nash equilibria for the corresponding uniform B^3C game, when the size of the graph is large enough. The intuition is that when nodes are all symmetric to each other, a node may change some of its edge to break the symmetry and gain a better betweenness. We suspect that the result is also true for larger k.

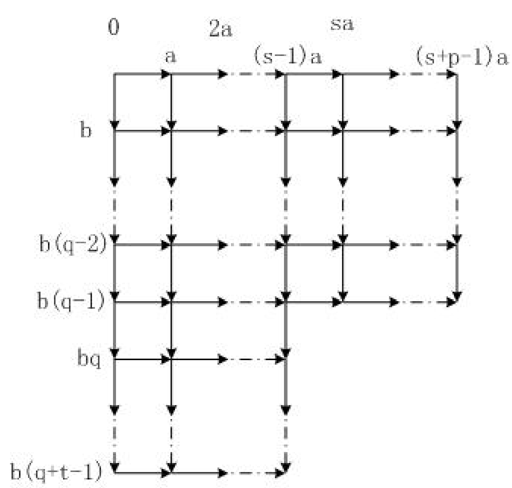

Fig. 5. Grid representation of an Abelian Cayley Graph, with $\{a, b\}$ as the generating set. The graph as four parameters (p, q, s, t) shown. Wrap-around edges from the right border to the left border and from the bottom border to the top border are not shown.

When $k = 2$, the generating set $A = \{a, b\}$, where $a \neq 0, b \neq 0$, and $(n, a, b) = 1$. We use several results about these graphs from [15]. First, we represent the graph as a two-dimensional grid (Figure 5), following the construction in [15]. Each node in the graph has a coordinate (x, y), where x and y are integers. Node 0 is put at the origin $(0, 0)$, but since the graph is symmetric any node could be put at the origin and the resulting graph structure is the same. The node with coordinate (x, y) is $v \equiv x + yb \pmod n$. In the first round, we fix nodes that are one step away from the origin, which are $(1, 0)$ and $(0, 1)$, and we start with $(1, 0)$. In the second round, we fix nodes that are two steps away from the origin, which are $(2, 0)$, $(1, 1)$, and $(0, 2)$, in this order. The x-axis edges $(x, y) \rightarrow (x + 1, y)$ are drawn horizontally toward right, and y-axis edges $(x, y) \rightarrow (x, y + 1)$ are drawn vertically downwards. When we encounter a node (x, y) whose value already appear earlier, we ignore this node. The construction process is complete when all values of $\mathbb{Z}_n$ have appeared in the graph. Figure 5 shows the graph with all nodes and rightward edges and downward edges. There are wrap-around edges from the right border to the left border and from the bottom border to the top border, which are omitted in the figure. Let $d(u,v)$ be the shortest distance from node u to node v. The following are results proven in [15].

Lemma 11 (Wong and Coppersmith [15]). [2] *The following results hold for the constructed Abelian Cayley graph $\langle \mathbb{Z}_n, \{a,b\}\rangle$.*

(1) The graph has the structure shown in Figure 5, which is a rectangle with a missing rectangle corner, with four parameters s,t,p,q such that $st+sq+pq=n$ and $s,q>0, t,p\geq 0$.
(2) $sa+qb\equiv 0 \pmod{n}$, $tb\equiv (s+p)a \pmod{n}$, and $pa\equiv (q+t)b \pmod{n}$.
(3) The average distance of the graph $A_n(b)$ is given below:

$$A_n(b)=\frac{\sum_{u\neq v\in\mathbb{Z}_n} d(u,v)}{n(n-1)}=\frac{s^2q+sq^2+2stq+s^2t+st^2+2spq+p^2q+pq^2}{2(st+sq+pq)}-1 \tag{3}$$

We now show how to calculate the betweenness of node 0 in the graph.

Lemma 12. *For any graph $G=(V,E)$, we have*

$$\sum_{i\in V} btw_i(G)=\sum_{u\neq v\in V, m(u,v)>0}(d(u,v)-1) \tag{4}$$

Proof. By the definition of $btw_i(G)$ (equation (1)), we have

$$\begin{aligned}\sum_{i\in V} btw_i(G) &= \sum_{i\in V}\sum_{u\neq v\neq i\in V, m(u,v)>0}\frac{m_i(u,v)}{m(u,v)}\\ &= \sum_{u\neq v\in V, m(u,v)>0}\sum_{i\neq u,v\in V}\frac{m_i(u,v)}{m(u,v)}\\ &= \sum_{u\neq v\in V, m(u,v)>0}(d(u,v)-1)\end{aligned}$$

□

Lemma 13. *In the Abelian Cayley Graph $\langle \mathbb{Z}_n, \{a,b\}\rangle$, the betweenness of each node is*

$$btw_{orig}=(n-1)(A_n(b)-1) \tag{5}$$

where $A_n(b)$ is given in Lemma 11.

Proof. According to Lemma 12:

$$\sum_{i\in\mathbb{Z}_n} btw_i=\sum_{u\neq v\in\mathbb{Z}_n}(d(u,v)-1)=n(n-1)(A_n(b)-1)$$

Since Abelian Cayley graphs are symmetric, every node has the same betweenness. Therefore,

$$btw_{orig}=\frac{\sum_{i\in\mathbb{Z}_n} btw_i}{n}=(n-1)(A_n(b)-1)$$

□

[2] The results in [15] are for the case $\langle \mathbb{Z}_n, \{1,b\}\rangle$, but the same results apply to $\langle \mathbb{Z}_n, \{a,b\}\rangle$ with the same proofs.

In the remaining analysis, we show that in almost all cases, node 0 can improve its betweenness by changing either its edge to node a (coordinate $(1,0)$) or node b (coordinate $(0,1)$) to the edge connecting node $a+b$ (coordinate $(1,1)$). The two exceptions are (a) when $a+b \equiv 0 \pmod{n}$, in which case the graph is a bidirectional Hamiltonian cycle, and (b) when $2a+b \equiv 0 \pmod{n}$, in which case the graph has one Hamiltonian cycle in one direction, and a reverse cycle made of every other node in the Hamiltonian cycle. We show that in these two cases, node 0 can increase its betweenness by connecting to the opposite end of the cycle. The detailed analysis is given in the appendix. As a result, we have the following theorem.

Theorem 4. *There exists a number $N > 0$ such that for all $n \geq N$, no Abelian Cayley graph $G_{n,a,b}$ is a Nash equilibrium.*

On the contrary, when $k = 1$, we show below that the Abelian Cayley graph, which is the Hamiltonian cycle, is the only nontransient Nash equilibrium, and from any maximal graph, there is a sequence of better responses that leads to the Hamiltonian cycle.

Theorem 5. *In the uniform B^{3}C game with $k = 1$, the Hamiltonian cycle is the unique nontransient Nash equilibrium, and from every maximal graph, there is a sequence of better responses that leads to the Hamiltonian cycle.*

In general, we conjecture that for all fixed $k \geq 2$, no Abelian Cayley graph with a generating set of size k is Nash equilibrium when the graph size is large enough.

5 Conclusion and future work

In this paper, we present results on bounded budget betweenness centrality (B^{3}C) game, a type of network formation games in which nodes in the network try to strategically select other nodes to connect subject to the budget constraint in order to maximize their betweenness centrality in the network. There are a number of directions to continue the study of B^{3}C games.

First, it is still an open question on whether Nash equilibrium always exists in uniform games, and what are the possible structures in these Nash equilibria. Second, we can study various properties of Nash equilibria, such as strong connectivity, diameter, and fairness in betweenness among different nodes. We may also look into different variants of the game, such as the undirected version, or the version without budget constraint but minimizing link cost as part of the objective function. Given that computing exact best responses for nodes is NP-hard, we may study approximate algorithms that node may employ in finding near-optimal responses and the corresponding Nash equilibria with such dynamics, which may also be related to approximate Nash equilibria.

Acknowledgment. We would like to thank Heiko Röglin for his helpful discussions on this research topic.

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Appendix

A Complete proofs for results in Section 3

Lemma 1 *In any nonuniform B^3C game, the following configurations are all Nash equilibria: (i) the empty graph with no edges; and (ii) any Hamiltonian cycle, i.e., any cycle that passes through every node exactly once, provided that the budget constraint allows such configurations. Moreover, they are transient Nash equilibria under the following conditions: (a) for the empty graph, if there exist nodes i, j, and k such that $\{j\} \in S_i$, $\{k\} \in S_j$ and $w(i,k) > 0$; (b) for a Hamiltonian cycle, if there exist nodes i, j and k such that (b.1) j is on the shortest path from i to k, (b.2) the length of the shortest path from k to i is at least 2, (b.3) k is allowed to add an edge to j, (b.4) j is allowed to add an edge to i, and (b.5) $w(k,i) > 0$.*

Proof. For the empty graph, the betweenness centrality of every node is zero. Since each node i can only add its own outgoing edges, there is no path between any pair of nodes different from i, no matter how i add its own outgoing edges. Thus, the betweenness centrality of i is still zero with any strategy change, which means the empty graph is a Nash equilibrium. Moreover, with condition (a), we can consider the strategy change sequence in which i first changes to strategy $\{j\}$ (adding an edge from i to j), and then j changes to strategy $\{k\}$ (adding an edge from j to k). After j's change, its betweenness improves from 0 to $w(i,k) > 0$, since it is now on the shortest path from i to k. Thus, the empty graph is a transient Nash equilibrium.

Now consider a configuration with a Hamiltonian cycle. Consider a node i. For any pair of nodes $j,k \neq i$, if i is on the shortest path from j to k, i gain 1 in its betweenness since the shortest path is unique on the Hamiltonian cycle. To improve i's betweenness, the only possibility is for i to find a pair of nodes $j,k \neq i$ such that i was not on the shortest path from j to k but i is on the shortest path from j to k after i's strategy change. However, since i's strategy change can only change its own outgoing edges, and the above implies that the shortest distance from j to i is less than the shortest distance from j to k before i's strategy change, which means i is already on the shortest path from j to k because it is a Hamiltonian cycle. Therefore, the Hamiltonian cycle is a Nash equilibrium.

Moreover, with conditions (b.1) to (b.5), we can have a strategy change sequence in which k adds an edge to j and then j adds an edge to i. After k adds the edge to j, k's utility does not change, but after j adds the edge to i, path $\langle k,j,i\rangle$ becomes a shortest path from k to i since the shortest distance from k to i was at least two before the change. Then j's utility increases at least $w(k,i)/2 > 0$. Therefore, in this case, the Hamiltonian cycle is a transient Nash equilibrium. □

Lemma 2 *Adding an outgoing edge to a node i does not decrease i's betweenness. That is, for any graph $G = (V,E)$ with $i \in V$ and $(i,j) \notin E$ for some $j \in V$. Let $G' = (V, E \cup \{(i,j)\})$. Then $btw_i(G) \leq btw_i(G')$.*

Proof. For each pair of nodes $u,v \neq i$, the length of the shortest path from u to v does not increase after adding edge (i,j). If the length decreases, the new shortest path must pass through (i,j), and thus pair (u,v) contributes 1 to $btw_i(G')$, which is no less than the contribution of (u,v) to $btw_i(G)$. If the length remains the same, all the additional shortest paths between u and v pass through (i,j). Thus, the contribution of (u,v) to $btw_i(G')$ is at least the same as the contribution to $btw_i(G)$. Therefore, when summing over the contributions from all pairs, we have $btw_i(G) \leq btw_i(G')$. □

Lemma 3 *Given a nonuniform B^3C game with parameters (n,b,c,w), if a maximal graph G of the game is not stable, then any subgraph of G cannot be a nontransient Nash equilibrium. If none of the maximal graphs of the game is stable, then the game does not have any nontransient Nash equilibrium.*

Proof. Consider any subgraph G' of G. We can construct a sequence of strategy changes that change the graph G' to G, where only new edges are added in all strategy changes. By Lemma 2, all these changes will either keep the changer's utility or increase its utility. Since G is unstable, some node can have a strategy change that increases its utility. Therefore, G' cannot be a nontransient Nash equilibrium.

If none of the maximal graphs of the game is stable, by the above argument we know that no configuration (graph) is a nontransient Nash equilibrium, since any configuration is a subgraph of some maximal graph. □

Lemma 4 *If a maximal graph G of the game is stable, G must be an assignment graph.*

Proof. Suppose, for a contradiction, that G is not an assignment graph. Then for some $i \in \{1, \ldots, \ell\}$, both L_i and $\bar{L}_i$ connect to Q_i or to A. Suppose they both connect to Q_i. The only shortest paths that pass through L_i and $\bar{L}_i$ and have nonzero weights are $\langle P_i, L_i, Q_i\rangle$ and $\langle P_i, \bar{L}_i, Q_i\rangle$. Since $w(P_i, Q_i) = a$, we have $btw_{L_i}(G) = btw_{\bar{L}_i}(G) = a/2$. In this case, L_i can change its strategy to connect to A instead of Q_i to obtain G'. In G', L_i is on the only shortest path from P_i to A, and thus $btw_{L_i}(G') = a > btw_{L_i}(G)$. Therefore, G is not stable, contradicting to the assumption of the lemma.

Now suppose that both L_i and $\bar{L}_i$ connect to A. They split the shortest paths from P_i to A, which contributes $a/2$ to the betweenness of L_i and $\bar{L}_i$ each. Among other possible shortest paths that pass through L_i or $\bar{L}_i$, the only nonzero weight ones are from P_i to B_j and C_j for all $j \in \{1, \ldots, m\}$. Since L_i and $\bar{L}_i$ equally split these shortest paths, we have $btw_{L_i}(G) \le a/2 + \sum_{j=1}^{m}(w(P_i, B_j) + w(P_i, C_j))/2 = a/2 + m$. In this case, L_i can change its strategy to connect to Q_i instead of A to obtain G'. In G', L_i is on the only shortest path from P_i to Q_i, so $btw_{L_i}(G') = a > a/2 + m$ since $a > 2m$. Therefore, G is not stable, again contradicting to the assumption of the lemma. Hence, G must be an assignment graph. □

Lemma 5 *If the 3-SAT instance does not have a satisfying assignment, then for any maximal assignment graph G, there always exists a $j \in \{1, \ldots, m\}$ such that for all $i \in \{1, \ldots, \ell\}$ and all literals $v \in \{L_i, \bar{L}_i\}$, edge (v, A) being in G implies $w(v, D_j) = 0$.*

Proof. Suppose that the 3-SAT instance does not have a satisfying assignment and G is a maximal assignment graph. The edges pointing to A in G correspond to a truth assignment to variables in the 3-SAT instance: If edge (L_i, A) is in G, assign variable x_i to true; if edge $(\bar{L}_i, A)$ is in G, assign variable x_i to false. Since the 3-SAT instance is not satisfiable, for the above assignment, there exists a clause $\mathcal{C}_j$ that is evaluated to false. For any variable x_i not in $\mathcal{C}_j$ we have $w(L_i, D_j) = w(\bar{L}_i, D_j) = 0$ by our definition of the weight function. So we only consider a variable x_i appearing in $\mathcal{C}_j$. If edge (L_i, A) is in G, we assign x_i to true, and since $\mathcal{C}_j$ is evaluated to false, we know that literal $\bar{x}_i$ is in $\mathcal{C}_j$. Then by our definition, $w(\bar{L}_i, D_j) = b$ but $w(L_i, D_j) = 0$. The case when $(\bar{L}_i, A)$ is in G has a symmetric argument. Therefore, the lemma holds. □

Lemma 6 *For a maximal assignment graph G, if there exists a $j \in \{1, \ldots, m\}$ such that for all $i \in \{1, \ldots, \ell\}$ and all literals $v \in \{L_i, \bar{L}_i\}$, edge (v, A) being in G implies $w(v, D_j) = 0$, then G is not a Nash equilibrium.*

Proof. Consider such a graph G with $j \in \{1, \ldots, m\}$ satisfying the condition given in the lemma. Consider the shortest paths that pass through B_j and C_j. Since all literal nodes that connect to A have zero weights to D_j (and thus also to E_j and F_j), the only shortest paths passing through B_j and C_j that have nonzero weights are paths from A to D_j, E_j and F_j. This essentially reduces the gadget corresponding to $\mathcal{C}_j$ to the gadget in Figure 1 with one additional node A and no cluster nodes. By an argument similar to the one in the proof of Theorem 1, no matter how B_j and C_j currently connect to nodes in $\{D_j, E_j, F_j\}$, one of them will always want to change its strategy to connect to one node in $\{D_j, E_j, F_j\}$ that is next to what the other current connects to (according to the direction of the triangle) to increase its utility. Therefore, G is not a Nash equilibrium. □

Lemma 8 *If the 3-SAT instance has a satisfying assignment, then there exists a maximal assignment graph G of the game in which for all $j \in \{1, \ldots, m\}$, there exists $i \in \{1, \ldots, \ell\}$ and literal $v \in \{L_i, \bar{L}_i\}$ such that the edge (v, A) is in G and $w(v, D_j) = b$.*

Proof. Suppose that the 3-SAT instance has a satisfying assignment f. construct a maximal assignment graph G such that for all $i \in \{1, \ldots, \ell\}$, if variable x_i is assigned to true in the assignment f, then L_i connects to A; otherwise, $\bar{L}_i$ connects to A. For all $j \in \{1, \ldots, m\}$, since clause $\mathcal{C}_j$ is evaluated to true under assignment f, there exists variable x_i whose corresponding literal in $\mathcal{C}_j$ is evaluated to true. If literal x_i is in $\mathcal{C}_j$, x_i is assigned to true. By the above construction of G, (L_i, A) is in G, and by the definition of the weight function, $w(L_i, D_j) = b$. The same argument applies to the case when literal $\bar{x}_i$ is in $\mathcal{C}_j$. Therefore, the lemma holds. □

Lemma 9 *Given a maximal assignment graph G in which for all $j \in \{1, \ldots, m\}$, there exists $i \in \{1, \ldots, \ell\}$ and literal $v \in \{L_i, \bar{L}_i\}$ such that the edge (v, A) is in G and $w(v, D_j) = b$, we construct a graph G' such that G' is the same as G except that for all $j \in \{1, \ldots, m\}$, both B_j and C_j are connected to D_j in G'. The maximal graph G' must be a nontransient Nash equilibrium.*

Proof. We prove that in G' any strategy change strictly decreases the changers betweenness, and thus G' must be a nontransient Nash equilibrium.

We go through all nodes and check all possible strategy changes in the following list.

- For each node Q_i, $i \in \{1, \ldots, \ell\}$, it has only the empty strategy so there is no strategy change for Q_i.
- For each node P_i, $i \in \{1, \ldots, \ell\}$, the only change of the strategy is to remove one or both of the edges (P_i, L_i) and $(P_i, \bar{L}_i)$. Suppose variable x_i appears in clause $\mathcal{C}_j$. Then we know that D_j connects to P_j (since G' is maximal). By the definition of the weight function $w(F_j, L_i) = w(F_j, \bar{L}_i) = 1$. Thus paths from F_j to L_i and $\bar{L}_i$ through P_i contribute positive values to the betweenness of P_i. If P_i were to remove edge (P_i, L_i) or $(P_i, \bar{L}_i)$ or both, P_i's betweenness would strictly decrease.
- For each node L_i, $i \in \{1, \ldots, \ell\}$, its strategy change is either removing its flexible edge or changing its flexible edge. If it removes its flexible edge, it loses the shortest path from P_i to Q_i or A, and since $w(P_i, Q_i) = w(P_i, A) = a$, its betweenness strictly decreases. If it changes its flexible edge, then both L_i and $\bar{L}_i$ connects to Q_i or A. By the same argument as in the proof of Lemma 4, its betweenness strictly decreases.
- For each node $\bar{L}_i$, $i \in \{1, \ldots, \ell\}$, the argument is the same as the argument for L_i.
- For node A, it can remove any of edges (A, B_j) or (A, C_j), for $j \in \{1, \ldots, m\}$. Suppose it removes edge (A, B_j) in G'. Let x_i be a variable in $\mathcal{C}_j$. Since G' is an assignment graph, P_i has a shortest path connecting to B_j through L_i or $\bar{L}_i$ and A. Since $w(P_i, B_j) = 1$, this shortest path contributes 1 to the betweenness of A in G'. If A removes edge (A, B_j) in G', there will be no path from P_i to B_j and A's betweenness will decrease by 1. Therefore, any strategy change of A strictly decrease its betweenness.
- For each node B_j, $j \in \{1, \ldots, m\}$, it can either remove its flexible edge or change its flexible edge. By the assumption of the Lemma, there exists $i \in \{1, \ldots, \ell\}$ and literal node $v \in \{L_i, \bar{L}_i\}$ such that the edge (v, A) is in G and $w(v, D_j) = b$. Suppose that there are t such literal nodes v. By the definition of w, we know that $t \leq 3$. Since B_j at least splits the shortest paths from v and A to D_j, $btw_{B_j}(G') = (tb + 3)/2 \geq (b + 3)/2$. If B_j removes its flexible edge (B_j, D_j), it will not connect to any node and its betweenness will decrease to zero. If B_j changes its flexible edge to (B_j, E_j) to obtain a graph G'', it loses the share on the shortest paths from v and A to D_j but gain the full share on the shortest paths from A to E_j and F_j. Then $btw_{B_j}(G'') = 2 < (b + 3)/2 \leq btw_{B_j}(G')$ since $b > 1$. So B_j's betweenness strictly decreases. If B_j changes its flexible edge to (B_j, F_j), it loses the share on the shortest paths from v and A to D_j and E_j and only gains the full shar on the shortest paths from A to F_j, so it is worse than the above case. Therefore, all strategy changes on B_j strictly decreases B_j's betweenness.
- For each node C_j, $j \in \{1, \ldots, m\}$, the argument is the same as the argument for B_j.
- For each node D_j, $j \in \{1, \ldots, m\}$, it can change its strategy by removing its fixed edge to E_j and/or removing some of its fixed edges to some P_i's. If it removes its edge to E_j, it loses the shortest path from F_j to E_j with weight 1, so its betweenness strictly decreases. If it removes any edge to some node P_i, it

loses shortest paths from F_j to L_i and $\bar{L}_i$ with weight 1, so its betweenness strictly decreases. Therefore, D_i cannot change its strategy.

- For each node E_j, $j \in \{1, \ldots, m\}$, it can change its strategy by removing its fixed edge to F_j. This however will cause E_j losing the shortest path from D_j to F_j with weight 1, so its betweenness strictly decreases.
- For each node F_j, $j \in \{1, \ldots, m\}$, it can change its strategy by removing its fixed edge to D_j. This however will cause F_j losing the shortest path from E_j to D_j with weight 1, so its betweenness strictly decreases.

By the above argument exhausting all possible cases, we show that graph G' is indeed a nontransient Nash equilibrium. □

B More experimental results on Nash equilibria

Figure 6 provides several more Nash equilibria found for $k = 2$ and $n = 16, 20$. All Nash equilibria have Hamiltonian cycles and are Eulerian. More over every four nodes along the Hamiltonian cycle are symmetric, which is the property we used to search for Nash equilibria. However, we have not been able to find Nash equilibria using this property for larger graphs such as $n = 24$ or $n = 28$.

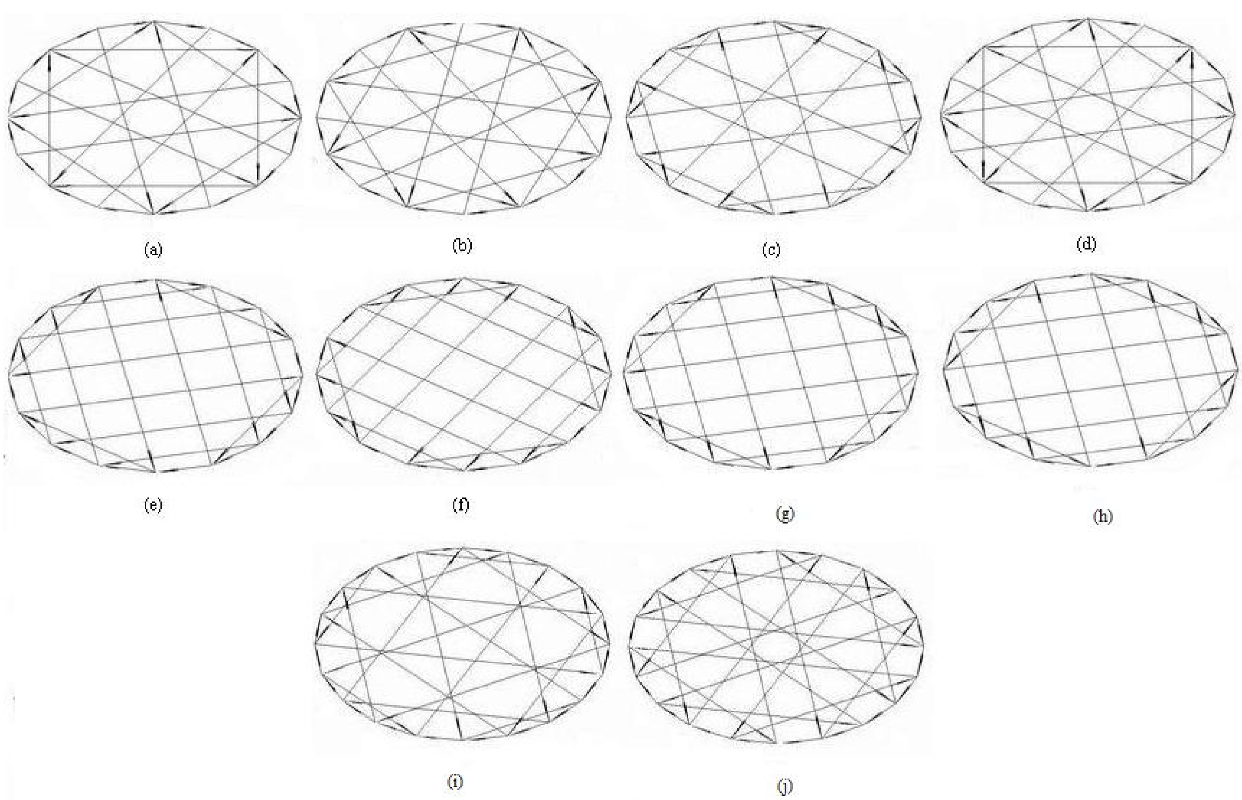

Fig. 6. Several more Nash equilibria for $k = 2$. (a) to (h) have $n = 16$ and (i) (j) have $n = 20$

C Complete proofs for Section 4

C.1 Proofs for the hardness result on finding best responses

Theorem 3 *Consider the following betweenness problem instance: (a) a directed graph $G = (V, E)$ with n nodes and each node has k outgoing edges; (b) a natural number b; (c) one node v in G. It is NP-hard to decide whether there is a strategy of v (i.e. a set of k nodes in $V \setminus \{v\}$) such that v's betweenness is at least b using the strategy.*

Proof. We reduce this problem from the set cover problem. Given an instance of the set cover problem $\langle U, S, t\rangle$, in which U is a universe and S is a family of subsets of U with $|U| = n, |S| = m$, and t is a natural number. The problem is to determine whether there are at most t subsets in S whose union is the universe. We construct an instance of the betweenness problem as follows (see Figure 7).

- Let $k = \min(t, m)$.
- We use $k + 1$ nodes to form a clique so that each node has out degree k. These nodes are used to absorb links from other nodes that would otherwise do not have k outgoing edges.
- We set node B to be the one we need to compute the best response for.
- We set node A to connect to B and another $k - 1$ nodes in the clique;
- We set n element nodes $v_1, \ldots, v_n$ to correspond to n elements in U, and they connect to arbitrary k nodes in the clique;
- We set m subset nodes $s_1, \ldots, s_m$ to correspond to m subsets in S. For a slight abuse of notation, we use v_i to denote both the node in the graph and the element in U, and s_j to denote both the node in the graph and the subset in S. We intend to connect s_j to all node v_i if $v_i \in s_i$, but subset s_j may have more than k elements causing the node s_j to have out-degree larger than k. To deal with this case, for each s_j, we build a complete k-ary tree rooted at s_i with height $h = \lceil \log_k \max_j\{|s_j|\}\rceil - 1$. In the tree of s_j, we connect k^h leaves arbitrarily to all $v_i \in s_j$ as long as each leaf connects to at most k nodes in v_i's. For leaves do not have k connections, we connect them arbitrarily to nodes in the clique to increase their out-degree to k. Let a be the number of nodes in one k-ary tree. Note that $a = (k^{h+1} - 1)/(k - 1) < k \max_j\{|s_j|\}$, so the construction is certainly in polynomial time.

The decision problem in the game is to determine whether node B can choose a set of edges of size at most k that make its betweenness at least $n + ka$.

Lemma 14. *If there is a cover of size at most t whose union is the universe U, then node B can choose a set of edges of size at most k that makes its betweenness to be at least $n + ka$.*

Proof. Suppose that the cover which satisfies the requirement is C. Without loss of generality, we can assume that $|C| = k$. Let node B connect to the root set nodes s_i for all $s_i \in C$. In this case, B stands on the shortest paths from A to the k root set nodes and their corresponding tree nodes, and thus k gains betweenness ka from these shortest paths. Since $\cup_{s_i \in C} s_i = U$, according to the construction of the structure, B can reach all n element nodes $\{v_1, \ldots, v_n\}$ and B stands on all the paths from A to the elements nodes. Hence they contribute n to the betweenness of B. So betweenness of B is at least $n + ka$. This concludes the proof. □

Lemma 15. *If node B can find a set of edges of size at most k that makes its betweenness to be at least $n + ka$, then there is a cover of size at most t whose union is the universe U.*

Proof. We first prove that B can achieve the best betweenness by connecting to k subset nodes s_j's.

Node B's betweenness comes from the shortest paths from A to other nodes. If B connects to a node L not in the clique, it will not gain any betweenness from the paths from A to B to L and then to any clique

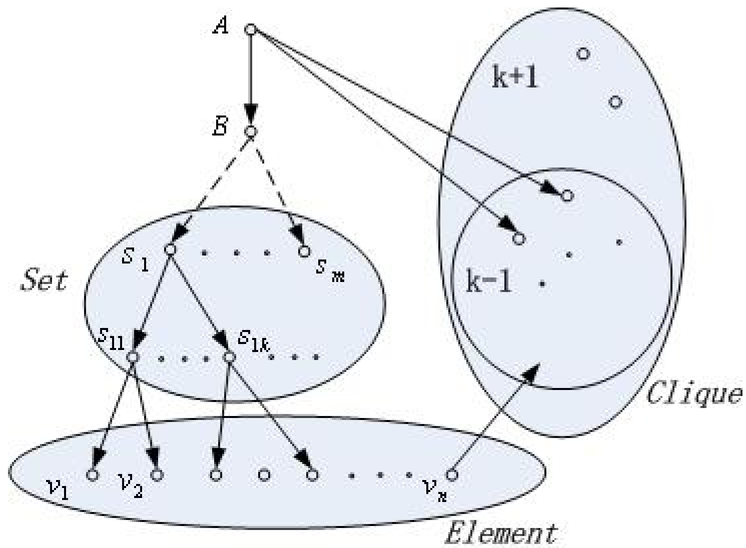

Fig. 7. A structure corresponding to the instance of set cover problem.

node, because A can reach $k-1$ clique nodes directly and the remaining two clique nodes in one more step, but the paths through B and L have length at least 3.

Since $k \leq m$, if B connects to any other node, it will lose a connection to some subset node s_j. We argue case by case below that it will not give a better betweenness than B connecting to some subset node instead.

- Node B connects to node L in the tree rooted at node s_j and $L \neq s_j$. If B already connects to s_j, then connection to L does not contribute any more to the betweenness of B, and B should connect to some other subset node $s_{j'}$ not yet connected to increase its betweenness. If B has no connection to s_j, then replacing the connection to L with connection to s_j will increase B's betweenness since B is at least on one more shortest path from A to s_j.
- Node B connects to some element node v_i. It can gain at most 1 by the shortest path from A to v_i via B, since v_i only connects to clique nodes, and we have already argued that the path from A to B to v_i and then to clique nodes are not shortest paths. In this case, B can instead connect to an available subset node s_j not yet connected, by which it gains betweenness of at least 1, no worse than the connection to v_i.
- Node B connects to some clique node L. If A has a direct connection to L, B will not gain any betweenness by this connection. If A does not have direction connection to L, $\langle A, B, L\rangle$ is a shortest path of length 2, but there are $k-1$ other shortest paths from A to L. Thus B gains betweenness of at most $1/k$. In this case, B is better off connecting to an available subset node s_j.
- Node B connects to node A. This does not contribute any betweenness to B, so B is better off connecting to an available subset node s_j.

Therefore, node B can achieve the best betweenness by connecting to k subset nodes. Let these k subset nodes form a set C. In this case, the betweenness of B is $ka + |\cup_{s_i \in C} s_i|$, because only through B node A can reach all k trees rooted at nodes in C plus nodes in $\cup_{s_i \in C} s_i$, but for the clique nodes A has shorter paths to reach them not through B. Since B can achieve a betweenness of at least $ka + n$, we know that $|\cup_{s_i \in C} s_i| \geq n$, which means that C must be a solution to the set cover instance. □

The proof of Theorem 3 is now complete with Lemmata 14 and 15. □

C.2 Proofs on Abelian Cayley graphs

All lemmata in this section consider the Abelian Cayley Graph $G_{n,a,b}$ generated from $\langle \mathbb{Z}_n, \{a, b\}\rangle$, with parameters (p, q, s, t) dependent on (n, a, b), as shown in Figure 5.

Lemma 16. *For all parameters (p, q, s, t), we have $t \leq s + p$ and $p \leq q + t$.*

Proof. by Lemma 11, $tb \equiv (s + p)a \pmod n$. Since node tb appears in the graph while node $(s + p)a$ does not, the distance from 0 to tb is at most the distance from 0 to $(s + p)a$, which means $t \leq s + p$. Symmetrically, we have $p \leq q + t$. □

Lemma 17. *If $s = q = 1, t, p \geq 0$, the graph is isomorphic to $G_{n,1,-1}$, which is a bidirectional Hamiltonian cycle. There exists number $N_1 > 0$ such that for all $n \geq N_1$, in $G_{n,1,-1}$ node 0 can remove its edge to 1 and add an edge to $\lfloor \frac{n}{2} \rfloor$ to improve its betweenness.*

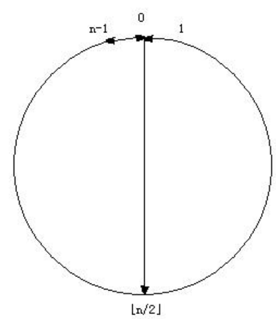

Fig. 8. In undirected circle, node 0 change $(0, 1)$ to $(0, \lfloor \frac{n}{2} \rfloor)$

Proof. By Lemma 11 (2), $sa+qb \equiv 0 \pmod n$, so $b \equiv -a \pmod n$ with $s = q = 1$. Since $(n, a, b) = 1$, we have $(n, a) = 1$ and thus the generating set $\{a, -a\}$ is equivalent to $\{1, -1\}$, which leads to the bidirectional Hamiltonian cycle $G_{n,1,-1}$. The betweenness of node 0 in $G_{n,1,-1}$ has different equations when n is odd or even. To avoid the complexity, we use the asympotic notation. Before node 0 changes its edge,

$$btw_{orig} = 2 \sum_{i=1}^{n/2+o(n)} i = \frac{n^2}{4} + o(n^2)$$

Then we calculate the new betweenness after node 0 removes its edge to 1 and adds an edge to $\lfloor \frac{n}{2} \rfloor$ (Figure 8).

The first part of betweenness comes from the shortest paths to the last quarter of nodes (from node $3n/4$ to node $n - 1$ in Figure 8).

$$btw_{modif1} = \sum_{i=n/4+o(n)}^{n/2+o(n)} i = 3n^2/32 + o(n^2)$$

The second part of betweenness comes from the shortest paths from the first quarter of nodes (from node 1 to node $n/4$) to the third quarter of nodes (from node $n/2$ to node $3n/4$) through the shortcut.

$$btw_{modif2} = (n/4 + o(n))(n/4 + o(n)) = n^2/16 + o(n^2)$$

The third part of betweenness comes from the shortest paths from the first quarter of nodes to the second quarter of nodes (from node $n/4$ to node $n/2$) through the shortcut.

$$btw_{modif3} = \sum_{i=1}^{n/4+o(n)} i = n^2/32 + o(n^2)$$

The fourth part of betweenness comes from the shortest paths from the last quarter of nodes to the third quarter of nodes through the shortcut.

$$btw_{modif4} = \sum_{i=1}^{n/4+o(n)} i = n^2/32 + o(n^2)$$

The fifth part of betweenness comes from the shortest paths from the last quarter of nodes to the first and the second quarter of nodes through the shortcut.

$$btw_{modif5} = (n/4 + o(n))(n/2 + o(n)) = n^2/8 + o(n^2)$$

$$btw_{modif} = btw_{modif1} + btw_{modif2} + btw_{modif3} + btw_{modif4} + btw_{modif5} = 11n^2/32 + o(n^2)$$

In this case, it is obvious that there exists some N_1 such that for all $n \geq N_1$, $btw_{modif} > btw_{orig}$. □

Lemma 18. *If $q = 1, s = 2, t, p \geq 0$, the graph is isomorphic to $G_{n,1,-2}$. There exists number $N_2 > 0$ such that for all $n \geq N_2$, in $G_{n,1,-2}$ node 0 can remove its edge to -2 and add an edge to $\lfloor \frac{n}{2} \rfloor$ to improve its betweenness. Symmetrically, it is also true for $s = 1, q = 2, t, p \geq 0$.*

Proof. By Lemma 11 (2), $sa + qb \equiv 0 \pmod{n}$, so $b \equiv -2a \pmod{n}$ with $s = 2$ and $q = 1$. Since $(n, a, b) = 1$, we have $(n, a) = 1$ and thus the generating set $\{a, -2a\}$ is equivalent to $\{1, -2\}$, which leads to the graph $G_{n,1,-2}$.

By Lemma 11 (1), $n = 2t + p + 2$. By the inequalities of Lemma 16, it is easy to derive that $n/3 - 1 \leq t \leq n/3$, and $n/3 - 3 \leq p \leq n/3 + 1$. Using Lemma 13 and Lemma 11 (3), we plug in $s = 2$, $q = 1$, $t = n/3 + O(1)$, $p = n/3 + O(1)$ and obtain the betweenness of node 0 in $G_{n,1,-2}$:

$$btw_{orig} = \frac{n^2}{6} + O(n)$$

We now caculate the betweenness of node 0 when it removes its edge to -2 and adds an edge to $\lfloor \frac{n}{2} \rfloor$ (Figure 9).

We calculate the contributions to its betweenness from several categories. For convenience, we say that the big cycle with edges $(v, v+1)$ is the clockwise cycle, while the cycle with edges $(v, v-2)$ the counter-clockwise cycle.

First, we consider the contributions of all paths with the clockwise big cycle:

$$btw_{modif1} = \sum_{i=1}^{\lceil (n-1)/3 \rceil - 1} i = \frac{n^2}{18} + O(n)$$

Second, we consider the contributions of paths with the clockwise cycle and the shortcut edge $(0, \lfloor \frac{n}{2} \rfloor)$:

$$btw_{modif2} = \sum_{i=1}^{\lceil \lceil n/2 \rceil /3 \rceil - 1} i = \frac{n^2}{72} + O(n)$$

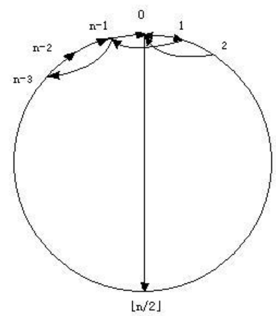

Fig. 9. Node 0 change $(0, -2)$ to $(0, \lfloor \frac{n}{2} \rfloor)$

Third, we consider the contributions of the paths with counter-clockwise cycle and the shortcut edge $(0, \lfloor \frac{n}{2} \rfloor)$:

$$btw_{modif3} = \sum_{i=1}^{\lfloor \lfloor n/2 \rfloor /3 \rfloor} i = \frac{n^2}{72} + O(n)$$

Finally, we consider the contributions of paths that start with the clockwise cycle, then pass through the shortcut and continue with the counter-clockwise cycle, and also the symmetric paths that start with the counter clockwise cycle and end with the clockwise cycle. Each contributes $n^2/18 + O(n)$ to the betweenness of node 0, so together

$$btw_{modif4} = \frac{n^2}{9} + O(n)$$

Therefore, the total betweenness difference after the change is:

$$\begin{aligned} \Delta &\geq btw_{modif1} + btw_{modif2} + btw_{modif3} + btw_{modif4} - btw_{orig} \\ &= \frac{n^2}{36} + O(n) \end{aligned}$$

Therefore, it is clear that there exists N_2 such that for all $n \geq N_2$, node 0 can improve its betweenness when $s = 2$ and $q = 1$. The result also holds for the symmetric case of $s = 1$ and $q = 2$. □

Lemma 19. *If $s, q \geq 2; t, p \geq 0$, there exists number $N_3 > 0$ such that for all $n \geq N_3$, in graph $G_{n,a,b}$ node 0 can remove either its edge to a (coordinate $(1,0)$) or its edge to b (coordinate $(0,1)$) and adds and edge to $a+b$ (coordinate $(1,1)$) to improve its betweenness.*

Proof. We focus on the calculation of the betweenness when node 0 changes its edge to node a to node $a+b$, as shown in Figure 10. We will point out the symmetric change and the corresponding result when necessary. It is not easy to apply Equation (1) to caculate the betweenness of node 0 in the modified graph. Instead, we use the symmetric property of graph $G_{n,a,b}$. We rotate node 0 to all positions in the graph, and for each rotated graph G', we compute the betweenness of node 0 in G' contributed by the path initiated from the node in the origin of G'. We then sum up all these betweenness, which gives the betweenness of node 0 contributed by all possible paths. Figure 11 shows an example of node 0 rotated one position to the right.

Node 0 gains betweenness from several sources after it changes its edge. First, with the diagonal edge from 0 to $a+b$, all shortest paths from the origin to the lower right region of node $a+b$ pass through node 0,

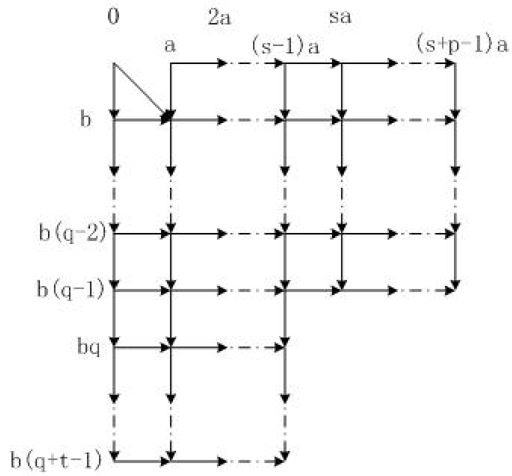

Fig. 10. The configuration in which upleft node 0 changes its strategy by changing one edge.

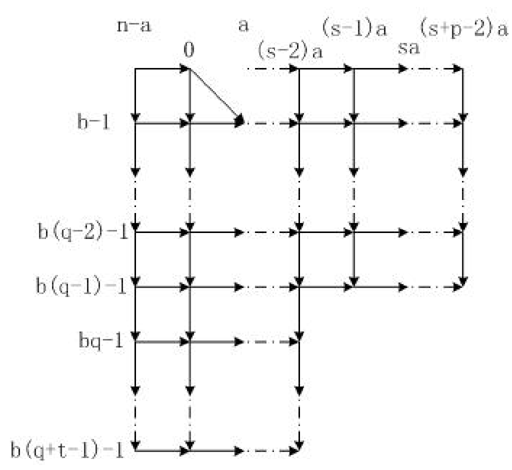

Fig. 11. Change the position of each node in the drawing of the graph

since the diagonal edge makes these path one step shorter than their competing alternative paths not passing through 0. This region is illustrated in Figure 12 (a). For this computation, we consider node 0 rotated into five regions, A, B, C, D and E, as shown in Figure 12 (b). It corresponds to the five summation terms in the follow formula.

$$\begin{aligned}
btw_{modif1} = &\sum_{i=2}^{s-1}((q+t-1)(s-i)+(q-1)p)+\sum_{i=0}^{p-1}(p-i)(q-1)+\\
&\sum_{j=2}^{q-1}(\sum_{i=1}^{s-1}((s-i)(q+t-j)+(q-j)p)+\sum_{i=0}^{p-1}(p-i)(q-j))+\\
&\sum_{j=0}^{t-1}\sum_{i=0}^{s-2}(j+1)(i+1)\\
= &\, p^2q^2/4+ts^2q/2+psq^2/2+t^2s^2/4+q^2s^2/4\\
&-tsq/2-pq^2/4-p^2q/4-pqs/2\\
&-qs^2/4-ts^2/4-t^2s/4-q^2s/4\\
&-3pq/4-3ts/4-3qs/4+s+t+p+q-1\\
= &\, n^2/4-pqst/2-n(p+q+s+t)/4+tp(s+q)/4-3n/4\\
&+p+q+s+t-1\\
= &\, n^2/4-pqst/2-n(p+q+s+t)/4+tp(s+q)/4+O(n)
\end{aligned}$$

Second, consider the case when node 0 is on the top border. By wrap-around edges from the bottom border to the top border, node 0 is still on shortest paths from the origin to some other nodes to the right of 0 in the top border. These shortest paths are better viewed if we attach another copy of the L-shape graph at the corner of the current graph (see Figure 12 (c)). We only consider the shortest paths to the first p nodes in the top border of the attached copy, which gives the following amount of betweenness:

$$\begin{aligned}
btw_{modif2} &= \sum_{i=1}^{p-2} i\\
&= (p-1)(p-2)/2
\end{aligned}$$

Third, consider the case when node 0 is on the left border. Then it is still on all the shortest paths from the origin to nodes that are below node 0, because it still has the edge to node b (see Figure 12 (d)). This gives node 0 the following amount of betweenness:

$$\begin{aligned}
btw_{modif3} &= \sum_{i=1}^{q+t-2} i\\
&= (q+t-2)(q+t-1)/2
\end{aligned}$$

We still need to add some amount of betweenness to node 0 in order to show that its betweenness strictly increases. For this purpose, we finally consider the case when node 0 is on the the second vertical line from the left (see Figure 12 (e)). It still gains partial betweenness from the origin to the nodes below it:

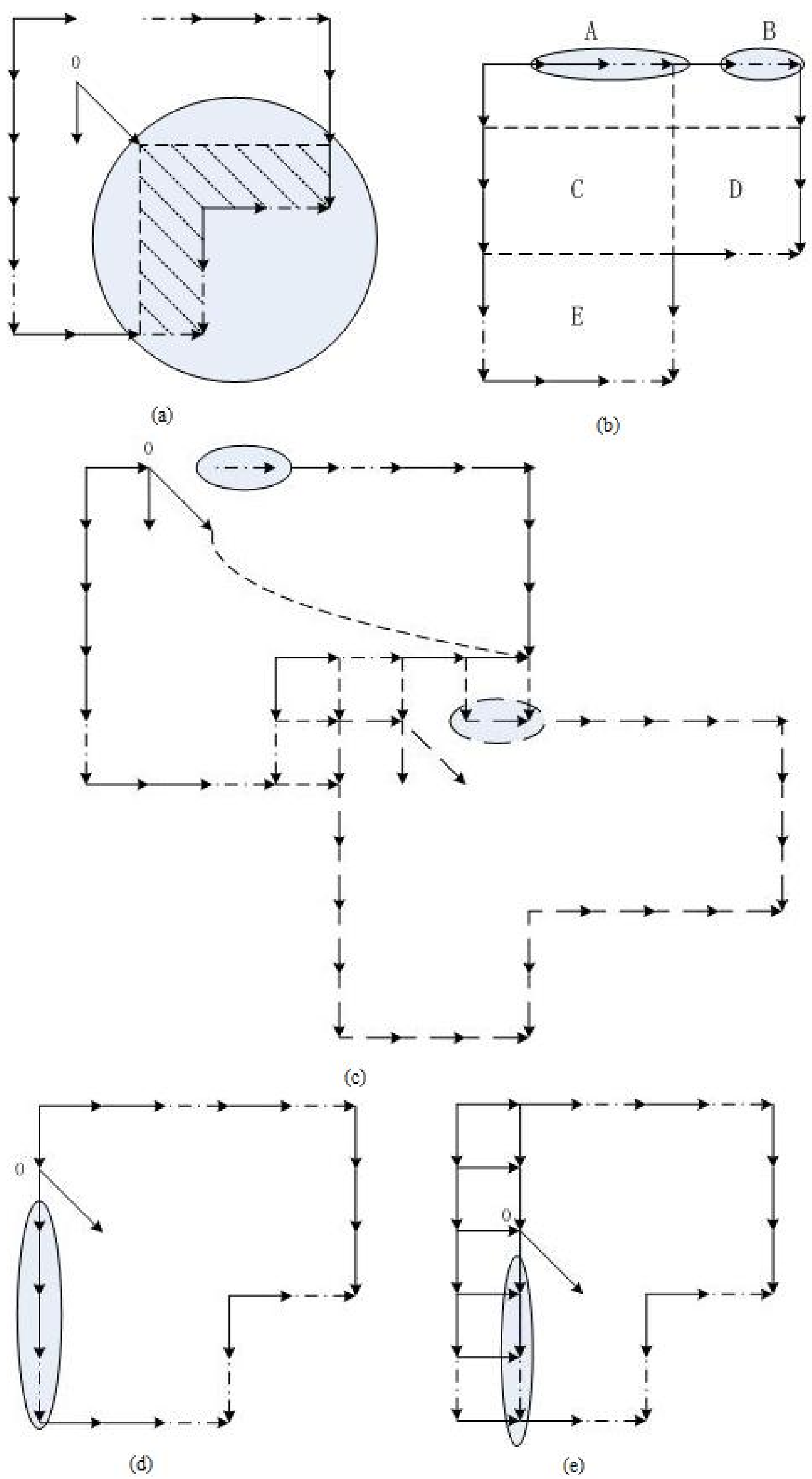

Fig. 12. Illustrations for the calculation of the betweenness of node 0 after its edge change.

$$
\begin{aligned}
btw_{modif4} &= \sum_{i=2}^{q+t} \sum_{j=1}^{i-1} \frac{j}{i} \\
&= \sum_{i=2}^{q+t} \frac{i-1}{2} \\
&= (q+t)(q+t-1)/4
\end{aligned}
$$

There are other contributions to the betweenness of node 0 but we omit them in the calculation. The betweenness of node 0 before changing the edge is

$$
\begin{aligned}
btw_{orig} &= (n-1)(A_n(b)-1) \\
&= (1/2 - 1/(2n))(n(p+q+s+t) - tp(s+q)) - 2(n-1)
\end{aligned}
$$

So the difference of the betweenness before and after changing is

$$
\begin{aligned}
\Delta &\geq btw_{modif1} + btw_{modif2} + btw_{modif3} + btw_{modif4} - btw_{orig} \\
&= \frac{n}{4}(n - 3p - 3q - 3s - 3t) + \frac{pt}{4}(3s + 3q - 2sq) + \\
&\quad \frac{p^2}{2} + \frac{3(q+t)^2}{4} + O(n) \qquad (6)
\end{aligned}
$$

We now separate all possible values of (p, q, s, t) into several cases and argue that in each case node 0 can improve its betweenness when n is large enough.

Case 1. $p, q, s, t \leq n^{4/5}$.

$$
\begin{aligned}
\Delta &\geq \frac{n^2}{4} - \frac{pqst}{2} + o(n^2) \\
&\geq \frac{n^2}{4} - \frac{n^2}{8} + o(n^2) \quad (\text{since } pq + st \leq n) \\
&\geq \frac{n^2}{8} + o(n^2)
\end{aligned}
$$

Thus, there exists $N_{3,1}$ such that for all $n \geq N_{3,1}$ $\Delta > 0$ in this case.

Case 2. $q \geq n^{4/5}$. Since $pq + qs + st = n$, we have $p \leq n^{1/5}$ and $s \leq n^{1/5}$. Since $t \leq s + p$, we have $t \leq 2n^{1/5}$. Moreover, $q = (n - st)/(p + s)$, giving the bound of q as $n/(p+s) \geq q \geq n/(p+s) - st \geq n/(p+s) - 2n^{2/5}$.

$$
\begin{aligned}
\Delta &\geq \frac{n}{4}(n - 3q) + \frac{3}{4}q^2 + o(n^2) \\
&= \frac{(p+s)^2 - 3(p+s) + 3}{4(p+s)^2} n^2 + o(n^2) \quad (\text{using the bound of } q \text{ above}) \\
&\geq \frac{1}{4(p+s)^2} n^2 + o(n^2) \quad (\text{since } s \geq 2)
\end{aligned}
$$

Thus, there exists $N_{3,2}$ such that for all $n \geq N_{3,2}$ $\Delta > 0$ in this case.

Case 3. $s \geq n^{4/5}$. This case is symmetric to Case 2, which means there exists $N_{3,3}$ such that for all $n \geq N_{3,3}$ node 0 gains betweenness if it switches its edge connecting node b to node $a+b$.

Case 4. $p \geq n^{4/5}$. Since $pq + qs + st = n$, we have $q \leq n^{1/5}$. By inequality $t + q \geq p$, and thus $t \geq n^{4/5} - n^{1/5}$. Then again by $pq + qs + st = n$, $s \leq n/(q+t) \leq n/p \leq n^{1/5}$. We simplify Equation (6) in the following way: First, we use $pq + qs + st$ to replace n; second, we remove any term that is $O(n^{7/5})$. This leads to the following:

$$\begin{aligned}\Delta &\geq \frac{t^2(s^2-3s+3)+p^2(q^2-3q+2)}{4} + O(n^{7/5}) \\ &\geq \frac{t^2}{4} + O(n^{7/5}) \quad (\text{since } s, q \geq 2) \\ &\geq \frac{n^{8/5}}{4} + O(n^{7/5}) \quad (\text{since } t \geq n^{4/5} - n^{1/5})\end{aligned}$$

Hence, there exists $N_{3,4}$ such that for all $n \geq N_{3,4}$ $\Delta > 0$ in this case.

Case 5. $t \geq n^{4/5}$. This case is symmetric to Case 4, which means that there exists $N_{3,5}$ such that for all $n \geq N_{3,5}$ node 0 gains betweenness if it switches its edge connecting node b to node $a+b$.

Note that the above five cases cover all possible values of (p, q, s, t) when $s, q \geq 2$. Therefore, we can set $N_3 = \max\{N_{3,1}, N_{3,2}, N_{3,3}, N_{3,4}, N_{3,5}\}$ and the lemma holds for this N_3. □

Lemma 20. *If $q = 1, s > 2; t, p \geq 0$, there exists number $N_4 > 0$ such that for all $n \geq N_4$, in graph $G_{n,a,b}$ node 0 can remove its edge to b (coordinate $(0,1)$) and adds and edge to $a+b$ (coordinate $(1,1)$) and improve its betweenness.*

Proof. We use the same approach as in the proof of the previous lemma to compute the betweenness of node 0 after it changes its edge.

First, we compute the betweenness contributed by the shortest paths from the origin to the lower-left region of the diagonal edge.

$$\begin{aligned}btw_{modif1} &= \sum_{i=2}^{s-1}(t(s-i)) + \sum_{j=0}^{t-1}\sum_{i=0}^{s-2}(j+1)(i+1) \\ &= \frac{t(s-1)(st+3s-4)}{4} = \frac{s^2t^2+3s^2t-st^2}{4} + O(n)\end{aligned}$$

Next, we compute the betweenness contributed by the shortest paths from the origin to the right of node 0 when node 0 is on the top border.

$$\begin{aligned}btw_{modif2} &= \sum_{i=1}^{p+s-2} i \\ &= (p+s-1)(p+s-2)/2\end{aligned}$$

Finally, we compute the betweenness contributed by the shortest paths from the origin to some nodes below node 0, when node 0 is on the left border.

$$\begin{aligned}btw_{modif3} &= \sum_{i=1}^{t-2} i \\ &= (t-1)(t-2)/2\end{aligned}$$

Thus,

$$\begin{aligned}\Delta &\geq btw_{modif1} + btw_{modif2} + btw_{modif3} - btw_{orig} \\ &= \frac{s^2t^2 + s^2t + 2t^2 - 3st^2}{4} + O(n)\end{aligned} \tag{7}$$

Case 1. $s \geq n^{3/4}$. Since $n = st + s + p$, we have $n \geq st$, and thus $t \leq n/s \leq n^{1/4}$. We also have $t \geq 1$, since we already have $q = 1$, and the node b has distance one to node 0 which means it must be added as the node below 0, making $t \geq 1$.

By Equation (7), we have

$$\begin{aligned}\Delta &\geq \frac{s^2 - 3st^2}{4} + O(n) \\ &\geq \frac{s^2 - 3n \cdot t}{4} + O(n) \ \ (\text{since } st \leq n) \\ &\geq \frac{n^{3/2} - 3n^{5/4}}{4} + O(n)\end{aligned}$$

It is clear that there exists $N_{4,1}$ such that for all $n \geq N_{4,1}$ $\Delta > 0$ in this case.

Case 2. $t \geq n^{3/4}$ and $s > 2$. By Equation (7), we have

$$\begin{aligned}\Delta &\geq \frac{(s^2 - 3s + 2)t^2 + s^2t}{4} + O(n) \\ &\geq t^2/2 + O(n) \ \ (\text{since } s > 2) \\ &\geq n^{3/2}/2 + O(n)\end{aligned}$$

It is clear that there exists $N_{4,2}$ such that for all $n \geq N_{4,2}$ $\Delta > 0$ in this case.

Case 3. $p \geq n^{3/4}$ and $s > 2$. Since $t + 1 \geq p$, $t \geq n^{3/4} - 1$. With the similar argument as in Case 3, we can show that there exists $N_{4,3}$ such that for all $n \geq N_{4,3}$ $\Delta > 0$ in this case.

Case 4. $s, t, p \leq n^{3/4}$. By Equation (7), we have

$$\begin{aligned}\Delta &\geq \frac{s^2t^2 - 3st^2}{4} + O(n) \\ &\geq \frac{(n - 2n^{3/4})^2 - 3n^{7/4}}{4} + O(n) \ \ (\text{since } st \geq n - 2n^{3/4}, st \leq n) \\ &\geq n^2/4 + o(n^2)\end{aligned}$$

It is clear that there exists $N_{4,4}$ such that for all $n \geq N_{4,4}$ $\Delta > 0$ in this case.

Note that the four cases above cover all possible parameters of (s, t, p, q) with $q = 1$ and $s > 2$. Let $N_4 = \max\{N_{4,1}, N_{4,2}, N_{4,3}, N_{4,4}\}$. The lemma thus hold with this N_4. □

Lemma 21. *If $s = 1, q > 2; t, p \geq 0$, there exists number $N_5 > 0$ such that for all $n \geq N_5$, in graph $G_{n,a,b}$ node 0 can remove its edge to a (coordinate $(1,0)$) and adds and edge to $a + b$ (coordinate $(1,1)$) and improve its betweenness.*

Proof. This case is symmetric to that of Lemma 20, so it obviously hold too. □

Theorem 4 *There exists a number $N > 0$ such that for all $n \geq N$, no Abelian Cayley graph $G_{n,a,b}$ is a Nash equilibrium.*

Proof. Take $N = \max\{N_1, N_2, N_3, N_4, N_5\}$, where N_1 to N_5 are from Lemmata 17 to 21. Since the five lemmata cover all possible cases of (p, q, s, t), and thus cover all possible Abelian Cayley graph $G_{n,a,b}$, we know that for all $n \geq N$, no Abelian Cayley graph $G_{n,a,b}$ is a Nash equilibrium. □

C.3 Proofs on the case of $k = 1$

Lemma 22. *The Hamiltonian cycle is a nontransient Nash equilibrium in the uniform B^3C game with $k = 1$.*

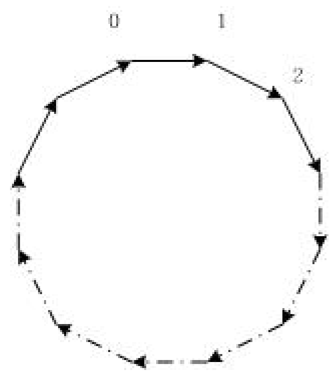

Fig. 13. When $k = 1$, Hamiltonian cycle is the only nontransient Nash equilibrium in the uniform B^3C game.

Proof. Since the Hamiltonian cycle is a symmetric graph, we only consider one node, for example, node 0 in Figure 13.

Betweenness of node 0 contains the contributions of all paths from i to j with $j > i \geq 1$. If 0 chooses to connect to 2 instead of 1, it loses all contributions from node $i \geq 2$ to 1 but it does not gain any new contributions. Connecting to other nodes are evening worse. If 0 simply removes its edge, its betweenness becomes 0. Therefore, all strategies changes of 0 strictly decreases its betweenness. Therefore the Hamiltonian cycle is a nontransient Nash equilibrium. □

Lemma 23. *From any maximal graph, there is a sequence of better responses that leads to the Hamiltonian cycle. This immediately means that the Hamiltonian cycle is the only nontransient Nash equilibrium.*

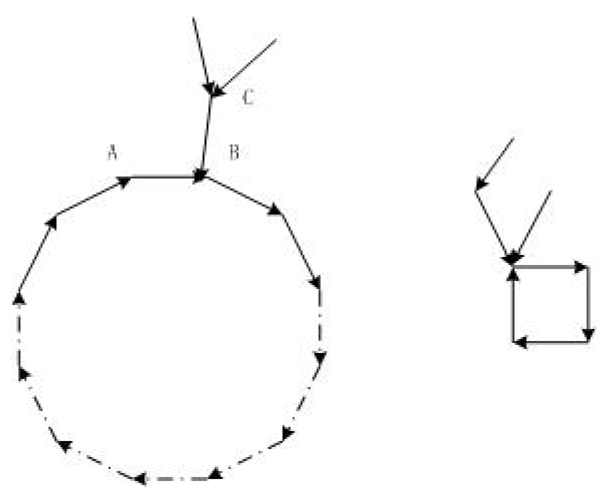

Fig. 14. Node A can benefit from changing (A, B) to (A, C)

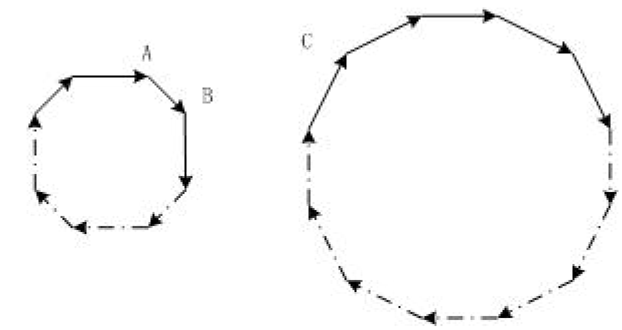

Fig. 15. Node A can benefit from changing (A, B) to (A, C)

Proof. Any maximal graph has several weakly connected components, in each of which there is a cycle and several branches pointing to some nodes on the cycle. (Figure 14)

Consider the case that there is some component which has branches on its cycle. For example, in the left graph of Figure 14, there is a branch on node B, and node C on the branch connects to B while node A on the cycle also connects to B. If node A connect to node C instead of node B, it will gain the betweenness from other nodes in the cycle and the branch nodes except those on B to node C. It is easy to verify that in this case A can change its edge to connect to C to improve its betweenness. The result of this better response is that the component has a larger cycle. This process can continue until that the component has a cycle with no branches.

Now consider the case that there are two components which have no branches on them (e.g. Figure 15). Assume the number of nodes in cycle with A is n_A and that in the cycle with C is n_C. Assume that $n_A \leq n_C$. Betweenness of node A is

$$btw_A = \sum_{i=1}^{n_A-2} i = \frac{(n_A-1)(n_A-2)}{2}$$

If node A change edge (A, B) to (A, C), then the betweenness of node A is

$$btw'_A = n_C(n_A-1) \geq n_A(n_A-1) > btw_A$$

Thus node A will benefit from changing from (A, B) to (A, C).

The above process can continue to merge two components into one, and then make the merged component to be a big cycle with no branches. Therefore, eventually the better response sequence will lead to the Hamiltonian cycle. □

Theorem 5 *In the uniform B^3C game with $k = 1$, the Hamiltonian cycle is the unique nontransient Nash equilibrium, and from every maximal graph, there is a sequence of better responses that leads to the Hamiltonian cycle.*

Proof. From Lemmata 22 and 23. □