A BCJR-like Labeling Algorithm for Tail-Biting Trellises

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I. Introduction

We describe two constructions for tail-biting trellises that are very similar to the well known BCJR construction for conventional trellises. The constructions lead to a simple proof of the fact that there exist linear tail-biting trellises for a linear code and its dual, which have the same state-complexity profiles.

II. Main Results

Let $\mathcal{C}$ be an (n,k) linear code over $\mathbb{F}_q$ with generator matrix $G = \{\mathbf{g}_1, \mathbf{g}_2, \ldots, \mathbf{g}_k\}$. Given a codeword $\mathbf{c} \in \mathcal{C}$, the *linear span* of $\mathbf{c}$, is the semi-open interval $(i,j]$ corresponding to the smallest closed interval $[i,j]$, $j > i$, which contains all the non-zero positions of $\mathbf{c}$. A *circular span* has exactly the same definition with $i > j$. Note that for a given vector, the linear span is unique, but circular spans are not– they depend on the runs of consecutive zeros chosen for the complement of the span with respect to the index set $\mathcal{I}$.

Koetter and Vardy [1] have shown that any linear trellis may be constructed from a generator matrix whose rows have been partitioned into linear span rows and circular span rows. Let G_l and G_c denote the sub-matrices of G containing vectors of linear span and circular span respectively. Let $H = [\ \mathbf{h}_1 \ \ \mathbf{h}_2 \ \ \ldots \ \ \mathbf{h}_n\ ]$ be the parity check matrix for the code. The algorithms BCJR-TBT and BCJR-TBT$^\perp$ respectively, construct *non-mergeable* [3] linear tail-biting trellises T and $T^\perp$ for $\mathcal{C}$ and its dual $\mathcal{C}^\perp$, given G and H.

Algorithm BCJR-TBT
`Input:` The matrices G, H and a span (linear or circular) associated with every $\mathbf{g} \in G$.
`Output:` A non-mergeable linear tail-biting trellis $T = (V, E, \mathbb{F}_q)$ representing $\mathcal{C}$.

Initialization: $G_{int} = G_l$. Let $\{\mathbf{d_x}\}_{\mathbf{x}\in\mathcal{C}}$ as follows:

$$\mathbf{d_x} = \begin{cases} \sum_{j=a}^{n} x_j \mathbf{h}_j & \text{if } \mathbf{x} \in \langle \mathbf{g}_i \rangle,\ \mathbf{g}_i \in G_c \\ & \text{with circular span } (a,b] \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Step 1: Construct the BCJR labeled trellis for the subcode generated by G_l, using the matrix H instead of the parity check matrix for the code G_l. Let $V_0, V_1 \ldots V_n$ be the vertex sets created and $E_1, E_2, \ldots E_n$ be the edge sets created.

Step 2: **for** each row vector $\mathbf{g}$ of G_c
for each $\mathbf{x} \in \langle \mathbf{g} \rangle$, $\mathbf{y}$ in the rowspace of G_{int}.
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Let $\mathbf{z}$ denote the codeword $\mathbf{x} + \mathbf{y}$.
let $\mathbf{d_z} = \mathbf{d_x} + \mathbf{d_y}$.
$V_0 = V_n = V_0 \cup \{\mathbf{d_z}\}$.
$V_i = V_i \cup \left\{\mathbf{d_z} + \sum_{j=1}^{i} z_j \mathbf{h}_j\right\}, 1 \le i < n$.
There is an edge $e = (\mathbf{u}, z_i, \mathbf{v}) \in E_i$, $\mathbf{u} \in V_{i-1}$, $\mathbf{v} \in V_i$, $1 \le i \le n$ iff
$\mathbf{d_z} + \sum_{j=1}^{i-1} z_j \mathbf{h}_j = \mathbf{u}$ and $\mathbf{d_z} + \sum_{j=1}^{i} z_j \mathbf{h}_j = \mathbf{v}$.
}
$G_{int} = G_{int} + \mathbf{g}$.

Algorithm BCJR-TBT$^\perp$
`Input:` The matrices G and H.
`Output:` A non-mergeable tail-biting trellis $T^\perp = (V, E, \mathbb{F}_q)$ representing $\mathcal{C}^\perp$.

Initialization: $V_i \mid_{0\le i\le n} = E_i \mid_{1 \le i \le n} = \phi$.
for each $\mathbf{y} = \langle y_1, y_2, \ldots, y_n \rangle \in \mathcal{C}^\perp$.
{
let $\mathbf{d}^T = \langle d_1 d_2 \ldots d_k \rangle$ **s.t.**

$$d_i = \begin{cases} 0 & \text{if } 1 \le i \le l \\ \sum_{j=a}^{n} y_j g_{i,j} & \text{otherwise} \end{cases}$$

where $\mathbf{g}_i \in G$ has circular span $(a,b]$.
$V_0 = V_n = V_0 \cup \{\mathbf{d}\}$.
$V_i = V_i \cup \left\{\mathbf{d} + \sum_{j=1}^{i} y_j \langle g_{j,1} g_{j,2} \ldots g_{j,k} \rangle^T\right\}$.
There is an edge $e = (\mathbf{u}, z_i, \mathbf{v}) \in E_i$, $\mathbf{u} \in V_{i-1}$, $\mathbf{v} \in V_i, 1 \le i \le n$, iff
$\mathbf{d} + \sum_{j=1}^{i} y_j \langle g_{j,1}, g_{j,2}, \ldots, g_{j,k} \rangle^T = \mathbf{u}$, and
$\mathbf{d} + \sum_{j=1}^{i} y_j \langle g_{j,1}, g_{j,2}, \ldots, g_{j,k} \rangle^T = \mathbf{v}$.
}

The preceding algorithms lead us to our main result.

Theorem 1 *Let T be a non-mergeable linear trellis, either conventional or tail-biting, for a linear code $\mathcal{C}$. Then there exists a non-mergeable linear dual trellis $T^\perp$ for $C^\perp$ such that the state-complexity profile of $T^\perp$ is identical to the state-complexity profile of T.*

Finally, as we know that for tail-biting trellises there are several measures of minimality [2], if any of these definitions requires the trellis to be non-mergeable, it immediately follows that there exist under that definition of minimality, minimal trellises for a code and its dual with identical state-complexity profiles.

References

[1] R. Koetter and A. Vardy, “On the theory of linear trellises,” *Information, Coding and Mathematics* (M. Blaum, Editor), Boston:Kluwer, May 2002.

[2] R. Koetter and A. Vardy, “The Structure of Tail-Biting Trellises: Minimality and Basic Principles,” *IEEE Trans. on Inform. Theory*, submitted for publication, 2002.

[3] F.R. Kschischang, “The trellis structure of maximal fixed-cost codes,” *IEEE Trans. Inform. Theory*, vol. 42, pp. 1828-1838, 1996.