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HEAT-PULSE PROPAGATION IN TOKAMAKS AND THE RÔLE OF DENSITY PERTURBATIONS*

C. M. BISHOP and J. W. CONNOR
Culham Laboratory, Abingdon, Oxfordshire OX14 3DB, U.K.
(EURATOM/UKAEA Fusion Association)

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Abstract—Studies of the propagation of heat pulses, launched by a sawtooth collapse or by modulated auxiliary heating, yield information on transport properties in the Tokamak. Corresponding values of the thermal conductivity are often found to be significantly larger than values obtained from power balance calculations. A full treatment of the heat pulse must, however, include possible coupling to an associated density pulse. Such effects are discussed in the context both of neoclassical transport theory and of a model of anomalous transport due to drift waves. It is found that the discrepancies between heat pulse and power balance measurements could arise from coupling between density and temperature perturbations due to the presence of off-diagonal terms in the transport matrix.

1. INTRODUCTION

THE SAWTOOTH COLLAPSE in the central region of a Tokamak plasma can launch a heat pulse which propagates radially outwards. By fitting solutions of heat conduction equations to the observed shape of the pulse, information about the thermal conductivity can be obtained (CALLEN and JAHNS, 1977). It is often found that the thermal conductivity deduced from heat-pulse measurements (χ^{HP}) is about a factor 2 greater than that obtained from power balance calculations (χ^{PB}). Similar experiments can also be performed using modulated ECRH (ASHRAF *et al.*, 1988). The sawtooth also launches a density pulse which can yield information on the particle diffusion coefficient (KIM *et al.*, 1988). Again, the density-pulse value is found to be about a factor 2–5 larger than that obtained from gas-puff measurements, which are indicative of the equilibrium value. GOEDHEER (1986) has shown that neglect of perturbed source and sink terms can lead to an overestimate of transport coefficients, though not all of the discrepancy can be explained in this way.

In this paper we consider the effects on heat-pulse (and density-pulse) measurements of possible "off-diagonal" terms in the transport equations; that is, terms in which a density gradient drives a heat flux, or a temperature gradient drives a particle flux. A pure temperature perturbation is then no longer a normal mode of the system (HOSSAIN *et al.*, 1987), and it is necessary instead to consider the eigenvectors of the transport matrix. These involve some combination of density and temperature perturbations, so that even a source of pure temperature perturbations would generate density perturbations at some distance from the source. If the propagation of a heat pulse in such a system were to be modelled by a simple heat conduction equation, the value χ^{HP} of the thermal conductivity so obtained could differ substantially from the value χ^{PB} corresponding to the equilibrium temperature gradient. The presence of off-diagonal terms therefore provides a possible explanation for the discrepancy between equilibrium and perturbation transport coefficients.

In Section 2 we consider a model transport matrix with constant coefficients in order to demonstrate the basic mechanisms. We then illustrate these ideas with two examples: in Section 3 we consider neoclassical transport theory for which off-diagonal terms are known to exist and have been calculated, and in Section 4 we consider a model of anomalous transport due to the dissipative trapped electron mode for which off-diagonal terms are also present. Some concluding remarks are presented in Section 5.

2. MODEL SYSTEM

The evolution of temperature and density profiles is governed by the coupled transport equations

$$\frac{\partial n}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\Gamma) = S_n, \tag{1}$$

$$\frac{3}{2}\frac{\partial}{\partial t}(nT)+\frac{1}{r}\frac{\partial}{\partial r}(rQ) = S_h. \tag{2}$$

Here n and T denote density and temperature, Γ and Q are the particle and heat fluxes, and S_n and S_h are the corresponding source terms. The heat flux Q can be decomposed into diffusive and convective terms:

$$Q = q+\tfrac{5}{2}\Gamma T. \tag{3}$$

The particle and diffusive heat fluxes can be expressed in terms of transport coefficients as follows:

$$\Gamma = -D\frac{\partial n}{\partial r}-M_{12}\frac{n}{T}\frac{\partial T}{\partial r}, \tag{4}$$

$$q = -M_{21}T\frac{\partial n}{\partial r}-\chi_0 n\frac{\partial T}{\partial r}, \tag{5}$$

and we shall neglect thermal and particle pinch terms. In steady state the time derivative terms in equations (1) and (2) vanish, and thus the diffusion terms balance the source terms. Suppose the resulting temperature profile were to be modelled by a simple heat equation of the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left(-n\chi^{\mathrm{PB}}\frac{\partial T}{\partial r}+\tfrac{5}{2}\Gamma T\right)\right\} = S_h, \tag{6}$$

i.e. one in which off-diagonal terms have been neglected. The value of χ^{PB} so obtained depends on the density profile; if we assume

$$\eta \equiv \frac{n}{T}\frac{\mathrm{d}T}{\mathrm{d}n} = 1, \tag{7}$$

then we obtain

$$\chi^{\mathrm{PB}} = \chi_0 + M_{21}, \tag{8}$$

and we see that the presence of off-diagonal terms implies $\chi^{\mathrm{PB}} \neq \chi_0$.

Information about transport coefficients can also be obtained by studying the propagation of temperature and density pulses. When the time-dependent source terms are localized in space, we can study the propagation in the region where the source terms are small, in which case the time derivative terms balance the diffusion terms. This has the considerable advantage that models for the source terms, which are often complex and uncertain, are not needed (TANG, 1988, private communication). We begin by writing the density and temperature as the sum of steady state and perturbed quantities:

$$n(r,t) = n_0(r) + \tilde{n}(r,t), \tag{9}$$

$$T(r,t) = T_0(r) + \tilde{T}(r,t). \tag{10}$$

We now linearize the transport equations (1)–(5) with the assumption

$$n_0 \frac{\partial \tilde{n}}{\partial r} \gg \tilde{n}\frac{\partial n_0}{\partial r}, \tag{11}$$

$$T_0 \frac{\partial \tilde{T}}{\partial r} \gg \tilde{T}\frac{\partial T_0}{\partial r}. \tag{12}$$

This gives a set of coupled transport equations which can be written in matrix notation as

$$\frac{\partial}{\partial t}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\mathbf{P}\cdot\frac{\partial}{\partial r}\mathbf{u}\right), \tag{13}$$

where

$$\mathbf{u} = \begin{pmatrix} \tilde{n}/n_0 \\ \tilde{T}/T_0 \end{pmatrix}, \tag{14}$$

$$\mathbf{P} = \begin{pmatrix} D & M_{12} \\ \frac{2}{3}(D+M_{21}) & \frac{2}{3}(\chi_0+M_{12}) \end{pmatrix}. \tag{15}$$

For simplicity we take the elements of $\mathbf{P}$ to be constants; more realistic examples will be given in the next two sections. The general solution of equation (13) is given by

$$\mathbf{u} = \alpha_1(r,t)\mathbf{u}_1 + \alpha_2(r,t)\mathbf{u}_2, \tag{16}$$

where the $\mathbf{u}_j$ are the two solutions of the eigenvalue equation

$$\mathbf{P}\cdot\mathbf{u}_j = \lambda_j\mathbf{u}_j, \tag{17}$$

and α_j are given by

$$\alpha_j(r,t) = A(z_j)\exp(\mathrm{i}\omega t), \tag{18}$$

where $A(z_j)$ satisfies the Bessel equation

$$\frac{\mathrm{d}^2A}{\mathrm{d}z_j^2} + \frac{1}{z_j}\frac{\mathrm{d}A}{\mathrm{d}z_j} + A = 0 \tag{19}$$

with

$$z_j = r\left(\frac{\omega}{\lambda_j}\right)^{1/2}\frac{1+\mathrm{i}}{2}. \tag{20}$$

Away from the source, the term with the smaller eigenvalue λ_1 decays more rapidly, leaving a pure eigenmode $\mathbf{u}_2$. If the resulting heat pulse were to be fitted by a solution of a simple heat equation of the form

$$\frac{3}{2}\frac{\partial\tilde{T}}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\chi^{\mathrm{HP}}\frac{\partial\tilde{T}}{\partial r}\right), \tag{21}$$

the value of χ^{HP} would be given from the eigenvalue λ_2 by $\chi^{\mathrm{HP}} = 3\lambda_2/2$ and thus

$$\chi^{\mathrm{HP}} = \tfrac{1}{2}\chi_0 + \tfrac{1}{2}M_{12} + \tfrac{3}{4}D + \tfrac{3}{4}\{(\tfrac{2}{3}\chi_0 + \tfrac{2}{3}M_{12} - D)^2 + \tfrac{8}{3}M_{12}(D + M_{21})\}^{1/2}. \tag{22}$$

The presence of off-diagonal terms therefore implies $\chi^{\mathrm{HP}} \neq \chi^{\mathrm{PB}}$. If $M_{12} = M_{21} = 0$, however, we obtain $\chi^{\mathrm{HP}} = \chi^{\mathrm{PB}} = \chi_0$. Thus the existence of off-diagonal terms can provide an explanation for the apparent discrepancy between χ^{PB} and χ^{HP}.

3. NEOCLASSSICAL TRANSPORT

We next consider neoclassical transport theory as an illustration of the results of the previous section. The fluxes Γ and Q can be written as (HINTON and HAZELTINE, 1976)

$$\Gamma = -f(r)n[1.04A' + 1.20T'/T], \tag{23}$$

$$Q = -f(r)nT[1.20A' + 2.55T'/T], \tag{24}$$

where primes denote derivatives with respect to r, and we have set $T_i = T_e = T$ and neglected ion gradients and electric field terms. Here

$$f(r) = \text{const}\cdot r^{-3/2}q^2 n T^{-1/2}, \tag{25}$$

$$A' = \frac{n'}{n} - \frac{3}{2}\frac{T'}{T}. \tag{26}$$

In these canonical variables (A, T) the Onsager Symmetry of the fluxes (23) and (24) is explicit. For simplicity we shall approximate $f(r)$ by C/r^2 where C is a constant.

If the equilibrium temperature profile were to be fitted by a simple heat equation of the form given in equation (6), and we again assume $\eta = 1$, we would obtain

$$\chi^{\text{PB}} = 0.25\left(\frac{C}{r^2}\right). \tag{27}$$

To study heat-pulse propagation we again linearize the transport equations with the assumptions (11) and (12). The resulting coupled equations can be written as

$$\frac{\partial}{\partial t}\mathbf{u} = C\mathbf{M}\cdot\frac{\partial^2}{\partial x^2}\mathbf{u}, \tag{28}$$

where $x = r^2/2$, and

$$\mathbf{u} = \begin{pmatrix} \tilde{n}/n_0 \\ \tilde{T}/T_0 \end{pmatrix}, \tag{29}$$

$$\mathbf{M} = \begin{pmatrix} 1.04 & -0.36 \\ -0.24 & 0.86 \end{pmatrix}. \tag{30}$$

These equations can again be written in terms of the eigenvectors of $\mathbf{M}$, giving

$$\mathbf{u} = \alpha_1(x, t)\mathbf{u}_1 + \alpha_2(x, t)\mathbf{u}_2, \tag{31}$$

where

$$\mathbf{M}\cdot\mathbf{u}_j = \lambda_j\mathbf{u}_j, \tag{32}$$

$$\lambda_1 = 0.64, \quad \mathbf{u}_1 = \begin{pmatrix} 0.67 \\ 0.74 \end{pmatrix}, \tag{33}$$

$$\lambda_2 = 1.26, \quad \mathbf{u}_2 = \begin{pmatrix} 0.86 \\ -0.51 \end{pmatrix}. \tag{34}$$

The solution with frequency ω and bounded as $x \to \infty$ can be written as

$$\alpha_j = \exp(-k_j x)\{a_j \cos(\omega t - k_j x) + b_j \sin(\omega t - k_j x)\}, \tag{35}$$

$$k_j = (\omega/2C\lambda_j)^{1/2}, \tag{36}$$

where the coefficients $\{a_j, b_j\}$ are determined by the boundary conditions. We consider the case of a δ-function heat source at $r = 0$ varying like $\cos(\omega t)$, and no source of particles. This gives $b_j = a_j$, and $a_1/a_2 = -1.80$. Thus close to the source (the "near field") there are density fluctuations even though there is no (fluctuating) particle source. This is a consequence of the presence of off-diagonal terms in the transport matrix. From equation (31) and (35) we have

$$\frac{\tilde{n}}{n_0} \simeq 0.19 \frac{\tilde{T}}{T_0} \quad \text{(near field).} \tag{37}$$

Further away from the origin the two eigenmodes decay at different rates so that at large distances from the source (the "far field") the solution is dominated by the eigenmode $\mathbf{u}_2$ since this has the largest eigenvalue. From equation (34) we then have

$$\frac{\tilde{n}}{n_0} \simeq -1.69 \frac{\tilde{T}}{T_0} \quad \text{(far field).} \tag{38}$$

Note that the far field result is independent of the boundary conditions. If the far field temperature profile were to be fitted by a solution of a simple heat equation of the form (21) the value of χ^{HP} would be determined by the eigenvalue λ_2 giving

$$\chi^{\text{HP}} = 1.89\left(\frac{C}{r^2}\right). \tag{39}$$

Comparing equations (27) and (39) we see that the presence of off-diagonal terms in this case leads to a large discrepancy between χ^{HP} and χ^{PB}. This calculation can be generalized to allow for a modulated heat source localized away from the axis. The far field results remain unchanged, since they are independent of the boundary conditions.

4. ANOMALOUS TRANSPORT

As a second example of a transport matrix with off-diagonal terms we consider the anomalous transport due to the dissipative trapped electron mode, as calculated by HORTON (1976). Using the results derived in the Appendix we can write the fluxes in the form

$$\Gamma = -n\hat{D}\left\{2\frac{1}{n}\frac{\partial n}{\partial r} + 6\frac{1}{T}\frac{\partial T}{\partial r}\right\}, \tag{40}$$

$$q = -nT\hat{D}\left\{\frac{1}{n}\frac{\partial n}{\partial r} + 15\frac{1}{T}\frac{\partial T}{\partial r}\right\}, \tag{41}$$

where

$$\hat{D} = \left(\frac{cT_e}{eB}\right)^2 \frac{\varepsilon L_s}{8\nu_e}\left(\frac{\pi}{2\varepsilon}\right)^{1/2}\left(\frac{1}{n}\frac{\partial n}{\partial r}\right)^3, \tag{42}$$

and L_s is the shear scale length, ε is the inverse aspect ratio and the other symbols have their usual meaning. Unlike the cases considered so far, the fluxes are non-linear functions of the gradients. This can lead to discrepancies between χ^{HP} and χ^{PB}, in addition to those due to the off-diagonal terms (GENTLE, 1988). If we define χ^{PB} by equation (6), and take $\eta = 1$, we obtain

$$\chi^{PB} = 16\hat{D}T. \tag{43}$$

Linearizing the transport equations, again using (11) and (12), we pick up additional terms arising from the ∇n terms in $\hat{D}$. The resulting equations take the form

$$\frac{\partial}{\partial t}\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\hat{D}T\mathbf{R}\cdot\frac{\partial}{\partial r}\mathbf{u}\right), \tag{44}$$

where

$$\mathbf{u} = \begin{pmatrix}\tilde{n}/n_0\\ \tilde{T}/T_0\end{pmatrix}, \tag{45}$$

$$\mathbf{R} = \begin{pmatrix}26 & 6\\ 50 & 14\end{pmatrix}. \tag{46}$$

The eigenvalue equation for **R** is given by

$$\mathbf{R}\cdot\mathbf{u}_j = \lambda_j\mathbf{u}_j, \tag{47}$$

with solutions

$$\lambda_1 = 1.67, \quad \mathbf{u}_1 = \begin{pmatrix}0.24\\ -0.97\end{pmatrix}, \tag{48}$$

$$\lambda_2 = 38.33, \quad \mathbf{u}_2 = \begin{pmatrix}0.44\\ 0.90\end{pmatrix}. \tag{49}$$

In the far field region we then have

$$\chi^{HP} = \tfrac{3}{2}\lambda_2\hat{D}T = 57.5\hat{D}T, \tag{50}$$

and thus

$$\chi^{HP} \simeq 3.6\chi^{PB}. \tag{51}$$

In this model the discrepancy between χ^{HP} and χ^{PB} arises partly from the existence of off-diagonal terms, and partly from the non-linearity of the fluxes.

5. CONCLUSIONS

Measurements of transport coefficients in a Tokamak by perturbative (heat-pulse and density-pulse) methods often give values which are significantly different from those obtained from equilibrium profile measurements. We have shown that such apparent anomalies can arise from the presence of off-diagonal terms in the transport matrix. Another consequence of off-diagonal terms is that a pure temperature perturbation (or a pure density perturbation) is no longer a normal mode of the system. Away from the source region, the ratio of density to temperature perturbations is determined by the eigenvector of the transport matrix, and can be compared with the experimentally determined ratio (BISHOP *et al.*, 1989). We have illustrated these results with the examples of neoclassical transport, and anomalous transport due to the dissipative trapped electron mode.

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APPENDIX: EVALUATION OF FLUXES FOR THE DTE MODE

HORTON (1976) has given the following expressions for the quasilinear fluxes due to the dissipative trapped electron mode:

$$\Gamma = -n\bar{D}\left\{(\mathrm{Im}\,G^0)\Delta\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}r}+(\mathrm{Im}\,G^1)\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right\}, \tag{52}$$

$$K = -nT\bar{D}\left\{(\mathrm{Im}\,G^1)\Delta\frac{1}{n}\frac{\mathrm{d}n}{\mathrm{d}r}+(\mathrm{Im}\,G^2)\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right\}, \tag{53}$$

where

$$\bar{D} = \left(\frac{cT_e}{eB}\right)\frac{\rho L_s}{8}\left(\frac{1}{n}\frac{\partial n}{\partial r}\right)^3, \tag{54}$$

$$G^n \simeq -\left(\frac{2\varepsilon}{\pi}\right)^{1/2}\int_0^\infty \frac{\mathrm{d}t\, t^{1/2}\,e^{-t}\,(t-\frac{3}{2})^n}{\frac{3}{2}-\varepsilon_n t+\mathrm{i}(t_3/t)^{3/2}}, \tag{55}$$

$$t_3 = \left(\frac{\nu_e}{\varepsilon\omega_*}\right)^{2/3}, \tag{56}$$

$$\omega_* = \left(\frac{k_\perp c T_e}{eBr_n}\right), \tag{57}$$

and, in Horton's notation,

$$Q = K + \tfrac{3}{2}\Gamma T. \tag{58}$$

The distinction between $\frac{3}{2}\Gamma T$ and $\frac{5}{2}\Gamma T$ is largely a matter of convention, and is discussed in Ross (1988); we shall use the $\frac{5}{2}\Gamma T$ convention. Only the total heat flux Q has physical significance. Note that, if $\Delta = 1$ in equations (52) and (53), the fluxes would satisfy an Onsager symmetry. In fact we shall take $\Delta = 1/2$ since this is a more typical value. In the $t_3 > 1$ regime the dominant contribution to Im G^n comes from small t, and this allows the following approximate analytic expression to be obtained:

$$\left(\frac{\pi}{2\varepsilon}\right)^{1/2} t_3^{3/2}\, \mathrm{im}\, G^n = 2 \ldots (n = 0) \tag{59}$$

$$= 3 \ldots (n = 1) \tag{60}$$

$$= 21/2 \ldots (n = 2). \tag{61}$$

Equations (40) and (41) then follow.