APPENDIX

Theorem A.1: $\mathbb{V}(\cdot)$ is the variance. If $L_A = \{a_i\}_{i=1}^n$ and $L_B = \{b_i\}_{i=1}^m$ are two disjoint set of positive real numbers. Let $L = \{a_i\} \cup \{b_i\}$, then

$$\triangle\mathbb{V}(L) = \mathbb{V}(L) - \frac{|L_A|\mathbb{V}(L_A) + |L_B|\mathbb{V}(L_B)}{|L|} \geq 0.$$

If $a_1 \leq a_2 \ldots a_n \leq b_1 \ldots \leq b_m$, then the equality holds only if $a_1 = a_2 = \ldots = a_n = b_1 \ldots = b_m$

Proof: Let

$$\frac{\sum a_i}{n} = a, \sum a_i^2 = A, \frac{\sum b_i}{m} = b, \sum b_i^2 = B$$

We have

$$\begin{aligned}\triangle\mathbb{V}(L) &= \left[\frac{A+B}{m+n} - (\frac{an+bm}{m+n})^2\right] - \\ &\quad \left[\frac{A-na^2+B-mb^2}{m+n}\right] \\ &= \frac{mn}{(m+n)^2}\cdot(a-b)^2 \qquad (14)\end{aligned}$$

the statement follows. □

Theorem A.2 (Theorem 3.6): $L = \{x_i\}_{i=1}^N$ is sorted, denote $L_A^{(i)} = \{x_j\}_{j=1}^i$ and $L_B^{(i)} = \{x_j\}_{j=i+1}^N$, let

$$\triangle\mathbb{V}^{(i)}(L) = \mathbb{V}(L) - \frac{|L_B^{(i)}|\mathbb{V}(L_B^{(i)}) + |L_B^{(i)}|\mathbb{V}(L_B^{(i)})}{|L|}.$$

If $\triangle\mathbb{V}(L) = \max_i\{\triangle\mathbb{V}^{(i)}(L)\}$, then $\triangle\mathbb{V}(L)|L| \geq \triangle\mathbb{V}(L_A^{(i)})|L_A^{(i)}|$ and $\triangle\mathbb{V}(L)|L| \geq \triangle\mathbb{V}(L_B^{(i)})|L_B^{(i)}|$ for $\forall i = 1, 2, \ldots, N$, the equality holds only if $\triangle\mathbb{V}(L_A^{(i)}) = 0$ and $\triangle\mathbb{V}(L_B^{(i)}) = 0$ respectively.

Proof: Without loss of generality, let L be ascending. We prove $\triangle\mathbb{V}(L)|L| \geq \triangle\mathbb{V}(L_A^{(i)})|L_A^{(i)}|$, the other inequality can be similarly proved. Suppose

$$\underset{i}{\text{argmax}}\{\triangle\mathbb{V}^{(i)}(L)\} = n.$$

let $m = N - n$ then according to the proof of Theorem A.1,

$$\triangle\mathbb{V}(L)|L| = \frac{mn}{(m+n)}\cdot(a-b)^2. \qquad (15)$$

Denote

$$\triangle\mathbb{V}(L_A^{(i)})|L_A^{(i)}| = \frac{m'n'}{(m'+n')}\cdot(a'-b')^2.$$

$$\because \triangle\mathbb{V}(L) = \max_i\{\triangle\mathbb{V}^{(i)}(L)\} = \triangle\mathbb{V}^{(n)}(L)$$

$$\therefore \frac{m'(m+n-m')}{(m'+(m+n-m'))^2}\cdot(a'-c')^2 \leq \frac{mn}{(m+n)^2}\cdot(a-b)^2$$

$$\therefore \frac{m'(m+n-m')}{(m'+(m+n-m'))}\cdot(a'-c')^2 \leq \frac{mn}{(m+n)}\cdot(a-b)^2 \qquad (16)$$

where

$$c' = \frac{ma+nb-m'a'}{m+n-m'} \geq b' \text{ (L is ascending)}.$$

We now show

$$\frac{m'n'}{(m'+n')}\cdot(a'-b')^2 \leq \frac{m'(m+n-m')}{(m'+(m+n-m'))}\cdot(a'-c')^2. \qquad (17)$$

Note the function $(x-a')^2$ is increasing w.r.t x when $x > a'$, then

$$(b'-a')^2 \leq (c'-a')^2 \qquad (18)$$

And function

$$\frac{m'x}{(m'+x)} = \frac{m'}{\frac{m'}{x}+1}$$

is increasing w.r.t x. Since $n' < m+n-m'$, we have

$$\frac{m'n'}{(m'+n')} < \frac{m'(m+n-m')}{(m'+(m+n-m'))} \qquad (19)$$

Combined with Eq. 18, Eq. 17 holds. According to Eq. 16, then $\triangle\mathbb{V}(L)|L| \geq \triangle\mathbb{V}(L_A^{(i)})|L_A^{(i)}|$, the equality holds only if $a' = b'$ thus $\triangle\mathbb{V}(L_A^{(i)}) = 0$.

□

Definition A.3 (FIFO property): Given a network $G = (V, E)$, where the travel time of each edge in G is time-dependent, we say G is FIFO if for any arc (i,j) in E, given A leaves node i starting at time t_1 and B leaves node i at time $t_2 \geq t_1$, then B cannot arrive at j before A .

Theorem A.4: Let $c_{ij}(t)$ be a strictly positive function defined for a time interval $[0, T]$, which specifies how much time it takes to travel from i to j if departing i at time t. The graph is FIFO $\Leftrightarrow$ $t+c_{ij}(t)$ is non-decreasing for any $(i,j) \in E$, $t \in [0,T]$.

Proof: Note that $t + c_{ij}(t)$ is the earliest arriving time at node j for one leaving node i at time t. Then the correctness of this theorem is a direct consequence of the FIFO's definition. □

Theorem A.5: If $c_{ij}(t)$ is piecewise linear, then G is FIFO if and only if the right derivative $c'_{ij+}(t) \geq -1$, $\forall t \in [0,T]$.

Proof: Since $c_{ij}(t)$ is piecewise linear, then according to Theorem A.4, we have

$$\begin{aligned}&G \text{ is FIFO if and only if} \quad (t+c_{ij}(t))'_+ \geq 0, \forall t \in [0,T] \\ &G \text{ is FIFO if and only if} \quad c'_{ij+}(t) \geq -1, \forall t \in [0,T] \qquad (20)\end{aligned}$$

Then the theorem follows. □

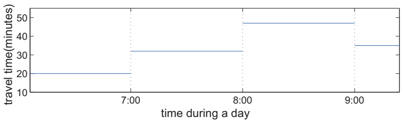
Theorem A.6: If the range of a single time slot $\triangle t$ satisfies:

$$\triangle t \geq t_{max} \qquad (21)$$

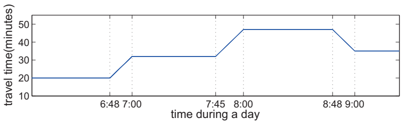
, we can reconstruct an continuous travel time function from a step travel time function to obey the FIFO rule (given the user's custom factor).

Proof: For each time interval $(t_1, t_2) \subseteq [0,T]$ containing a discontinuity point t_M, e.g.

$$s_{ij}(t) = \begin{cases} f_1 & \text{if } t_1 < t \leq t_M \\ f_2 & \text{if } t_M < t < t_2. \end{cases} \qquad (22)$$



(a) A step travel time function



(b) A refined travel time function

Fig. 18. Travel Time Function

denote $t_{M'} = t_M - |f_1 - f_2|$. The refined function $c_{ij}(t)$ can be defined as:

$$c_{ij}(t) = \begin{cases} f_1 & \text{if } t_1 < t \leq t_{M'} \\ f_1 + \frac{f_2 - f_1}{|f_2 - f_1|}(t - t'_M) & \text{if } t_{M'} < t \leq t_M \\ f_2 & \text{if } t_M < t < t_2. \end{cases} \quad (23)$$

Note that $|s_{ij}(t)| \leq t_{max}$ so that $|f_1 - f_2| \leq t_{max}$, then according to inequality (21), $t_{M'}$ is in the same time interval with t_M. Fig. 18(b) is the refined travel time function of Fig. 18(a). Since the gradient of the piecewise linear function $c_{ij}(t)$ can only be -1,0,1, it is clear that the refined travel time function satisfies the inequity (20). □