Deriving Probability Density Functions from Probabilistic Functional Programs

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EAPLS award winner

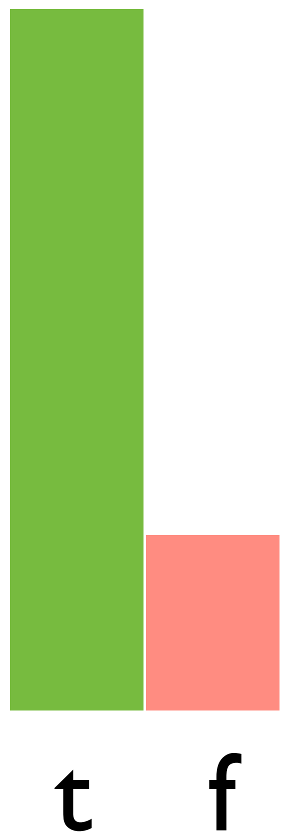
Probabilistic Programs

with discrete return type

flip(0.8)

if flip(0.8)
then flip(0.9)
else flip(0.4)

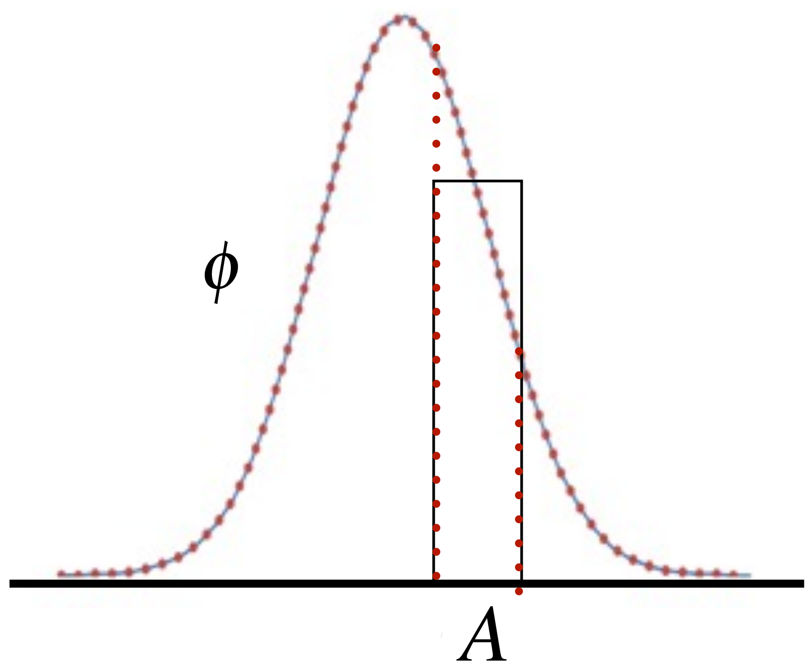
random(Bernoulli(0.8))



The probability mass of a value V
is the proportion of successful program runs
that return V

Density Functions I

random(Normal(0.0))

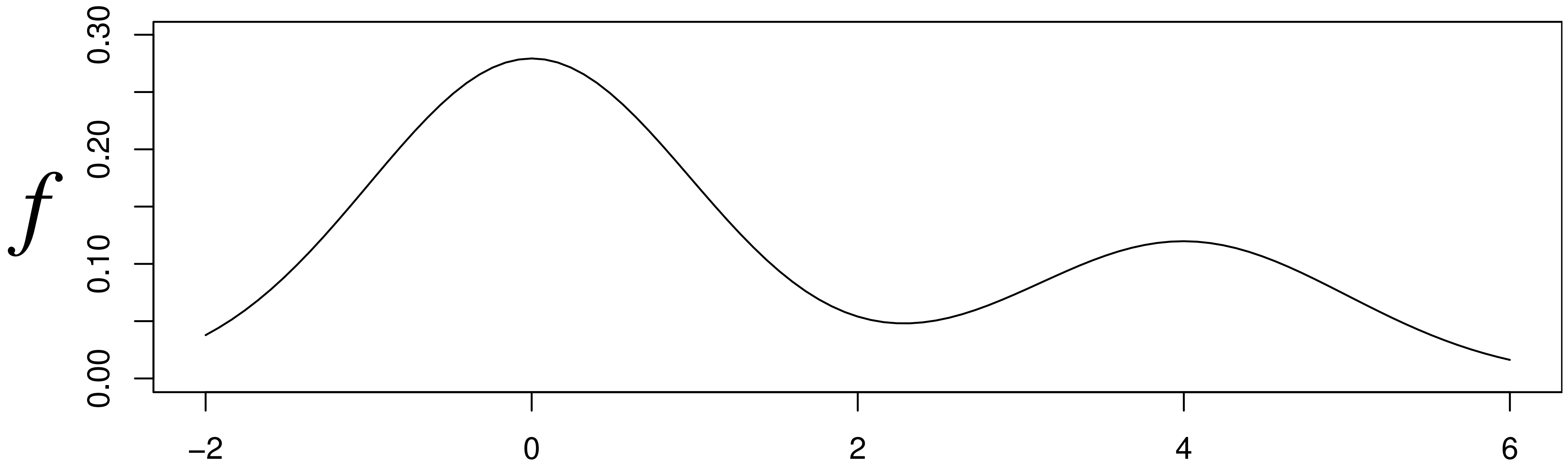


$$\int_A \phi$$

Integrating the density function over an interval A yields the proportion of successful program runs that return a value in A

Density Functions 2

if flip(0.7)
then random(Normal(0.0))
else random(Normal(4.0))



$$f(x) = 0.7 \cdot \phi(x) + 0.3 \cdot \phi(x-4)$$

Densities by compilation

- Given: Program M with result type t
- Sought: Function F from t to double
  that gives the density of the result of M

- Generalisation to programs with free variables:
  - Such parameters are treated as constants
  - The density F depends on a parameter valuation

```
if flip(p)
then random(Normal(m))
else random(Normal(n))
```

$$f(x) = \mathsf{p} \cdot \phi(x - \mathsf{m}) + (1 - \mathsf{p}) \cdot \phi(x - \mathsf{n})$$

Outline

- Motivation
- Density compiler
- Experimental results

Motivation

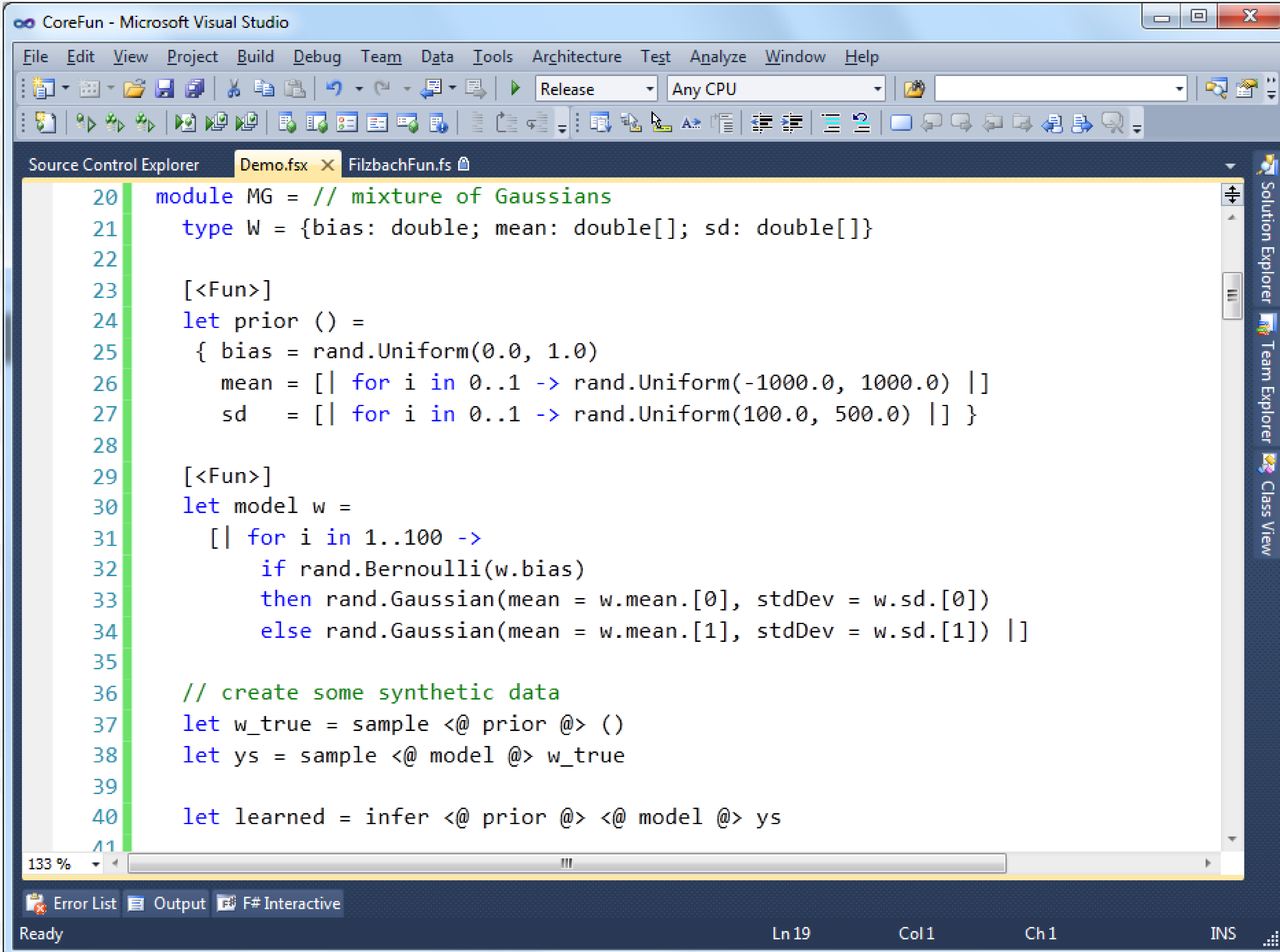
- Bayesian ML is based on probabilistic models
  - Conveniently written in a programming language
- Density functions are widely used in ML
  - Here: Markov chain Monte Carlo (MCMC)
- Current practice: code up both model *and* density
  - Do the model and the density agree?
  - What if you want to change one or the other?
  - (Non)Existence of an efficient density function limits the class of models used in practice. (Mostly because of "hidden variables")

Source and Target

- Base: language of finite computations (no recursion or while-loops)
- Source (Fun): base + $\textbf{random}(\mathrm{Dist}(M))$
  $\textbf{fail}$

Based on "Stochastic lambda-calculus", by Ramsey & Pfeffer, POPL'02.

- Target: base + $\lambda(x_1,\ldots,x_n).\ E$
  $E\ F$
  $\int E$


CoreFun - Microsoft Visual Studio
File Edit View Project Build Debug Team Data Tools Architecture Test Analyze Window Help
Release
Any CPU
Source Control Explorer
Demo.fsx
FilzbachFun.fs
module MG = // mixture of Gaussians
type W = {bias: double; mean: double[]; sd: double[]}
[<Fun>]
let prior () =
{ bias = rand.Uniform(0.0, 1.0)
mean = [| for i in 0..1 -> rand.Uniform(-1000.0, 1000.0) |]
sd = [| for i in 0..1 -> rand.Uniform(100.0, 500.0) |] }
[<Fun>]
let model w =
[| for i in 1..100 ->
if rand.Bernoulli(w.bias)
then rand.Gaussian(mean = w.mean.[0], stdDev = w.sd.[0])
else rand.Gaussian(mean = w.mean.[1], stdDev = w.sd.[1]) |]
// create some synthetic data
let w_true = sample <@ prior @> ()
let ys = sample <@ model @> w_true
let learned = infer <@ prior @> <@ model @> ys
133 %
Solution Explorer
Team Explorer
Class View
Error List
Output
F# Interactive
Ready
Ln 19
Col 1
Ch 1
INS

TrueSkill

Player ranking model, used in XBox Live.
Computes a skill distribution for each player.

```
let skillprior() = random (Gaussian(10.0,20.0))
let Alice,Bob = skillprior(),skillprior()
let performance player = random (Gaussian(player,1.0))
observe (performance Alice > performance Bob)
Alice
```

assume

To compute the marginal probability density of Alice, we need to integrate over all values of Bob.

observe x := **if** x **then** () **else fail**

Density compiler

Environment, compilation rules and correctness

Compilation

- Given a list Υ of random variables
  (and deterministic variables and their definitions)
  and an expression E for
  the joint density of the random variables
  and of being in the current branch in the program
- Returns the density function F

Inductively Defined Judgments of the Compiler:

$\Upsilon; E \vdash \mathrm{dens}(M) \Rightarrow F$	in $\Upsilon; E$ expression F gives the PDF of M
$\Upsilon; E \vdash \mathrm{marg}(x_1, \ldots, x_k) \Rightarrow F$	in $\Upsilon; E$ expression F gives the PDF of $(x_1, \ldots, x_k)$

Example rules, I

(VAR RND)

$$\frac{x \in \mathrm{rands}(\Upsilon) \quad \Upsilon; E \vdash \mathrm{marg}(x) \Rightarrow F}{\Upsilon; E \vdash \mathrm{dens}(x) \Rightarrow F}$$

Marginal Density: $\Upsilon; E \vdash \mathrm{marg}(x_1, \ldots, x_k) \Rightarrow F$

(MARGINAL)

$$\frac{\{x_1, \ldots, x_k\} \cup \{y_1, \ldots, y_n\} = \mathrm{rands}(\Upsilon) \qquad x_1, \ldots, x_k, y_1, \ldots, y_n \text{ distinct}}{\Upsilon; E \vdash \mathrm{marg}(x_1, \ldots, x_k) \Rightarrow \lambda(x_1, \ldots, x_k).\ \int \lambda(y_1, \ldots, y_n).\ E\sigma_\Upsilon}$$

(LET RND)

$$\frac{\neg(M\ \mathrm{det}) \qquad \varepsilon; 1 \vdash \mathrm{dens}(M) \Rightarrow F_1 \qquad \Upsilon, x; E \cdot (F_1\ x) \vdash \mathrm{dens}(N) \Rightarrow F_2}{\Upsilon; E \vdash \mathrm{dens}(\mathbf{let}\ x = M\ \mathbf{in}\ N) \Rightarrow F_2}$$

Example rules, 2

(Tuple Proj L)

$$\frac{\Upsilon; E \vdash \mathrm{dens}(M) \Rightarrow F}{\Upsilon; E \vdash \mathrm{dens}(\mathbf{fst}\ M) \Rightarrow \lambda z.\ \int \lambda w.\ F\ (z, w)}$$

↑ projection from a pair

(If Det)

$$\frac{M\ \mathrm{det} \qquad \Upsilon; E \cdot [M\sigma_\Upsilon = \mathbf{true}] \vdash \mathrm{dens}(N_1) \Rightarrow F_1 \qquad \Upsilon; E \cdot [M\sigma_\Upsilon = \mathbf{false}] \vdash \mathrm{dens}(N_2) \Rightarrow F_2}{\Upsilon; E \vdash \mathrm{dens}(\mathbf{if}\ M\ \mathbf{then}\ N_1\ \mathbf{else}\ N_2) \Rightarrow \lambda z.\ (F_1\ z) + (F_2\ z)}$$

arguments M are a deterministic function of the parameters

prob. of being in the current branch

(Random Const)

$$\frac{M\ \mathrm{det} \qquad \mathrm{rands}(\Upsilon) \sharp (M\sigma_\Upsilon) \qquad \Upsilon; E \vdash \mathrm{marg}(\varepsilon) \Rightarrow F}{\Upsilon; E \vdash \mathrm{dens}(\mathbf{random}(\mathrm{Dist}(M))) \Rightarrow \lambda z.\ (\mathrm{pdf}_{\mathrm{Dist}(M\sigma_\Upsilon)}\ z) \cdot (F\ ())}$$

Standard distribution and its probability density function

Correctness

Types of variables in Υ

Lemma 1 (Derived Judgments).
If $\Gamma, \Gamma_\Upsilon \vdash \Upsilon$ wf *and* $\mathrm{dom}(\Gamma_\Upsilon) = \mathrm{rands}(\Upsilon) \cup \mathrm{dom}(\sigma_\Upsilon)$ *and* $\Gamma, \Gamma_\Upsilon \vdash E :$ **double** *then*

If $\Upsilon; E \vdash \mathrm{dens}(M) \Rightarrow F$ *and* $\Gamma, \Gamma_\Upsilon \vdash M : t$ *then* $\Gamma \vdash F : t \rightarrow$ **double**.

Theorem 1 (Soundness). *If* $\varepsilon; 1 \vdash \mathrm{dens}(M) \Rightarrow F$ *and* $\varepsilon \vdash M : t$ *then*

$$(\mathscr{P}[\![M]\!]\ \varepsilon)\ A = \int_A F$$

The probabilistic semantics of M
(Ramsey & Pfeffer '02, Gordon et al. '13)

Inductively Defined Judgments of the Compiler:

$\Upsilon; E \vdash \mathrm{dens}(M) \Rightarrow F$	in $\Upsilon; E$ expression F gives the PDF of M
$\Upsilon; E \vdash \mathrm{marg}(x_1, \ldots, x_k) \Rightarrow F$	in $\Upsilon; E$ expression F gives the PDF of $(x_1, \ldots, x_k)$

Example

$$\varepsilon; 1 \vdash \mathrm{dens}\left(\begin{array}{l}\textbf{let } \mathsf{b} = \textbf{random}(\mathsf{Bernoulli}(\mathsf{p})) \textbf{ in} \\ \textbf{if } \mathsf{b} \\ \textbf{then random}(\mathsf{Normal}(\mathsf{m})) \\ \textbf{else random}(\mathsf{Normal}(\mathsf{n}))\end{array}\right) \Rightarrow$$

(modulo beta-eq)

$$\lambda x.\Sigma_{b\in\{\mathbf{t},\mathbf{f}\}}\mathbf{P}_{\mathsf{Bernoulli(p)}}(b)\cdot[b=\mathbf{t}]\cdot\phi(x-\mathsf{m})+$$
$$\Sigma_{b\in\{\mathbf{t},\mathbf{f}\}}\mathbf{P}_{\mathsf{Bernoulli(p)}}(b)\cdot[b=\mathbf{f}]\cdot\phi(x-\mathsf{n})$$

$$\lambda x.\mathbf{P}_{\mathsf{Bernoulli(p)}}(\mathbf{t})\cdot\phi(x-\mathsf{m})+$$
$$\mathbf{P}_{\mathsf{Bernoulli(p)}}(\mathbf{f})\cdot\phi(x-\mathsf{n})$$

Implementation

- Direct implementation of the compilation rules
  - In F#, operating on a subset of (quoted) F#
  - Operates on log-probabilities
  - Uses let-expansion in the MARGINAL rule
  - Parametric in integration function (currently a simple Riemann sum)

Evaluation

- Synthetic models, and ecological systems models from Computational Sciences, MSR Cambridge

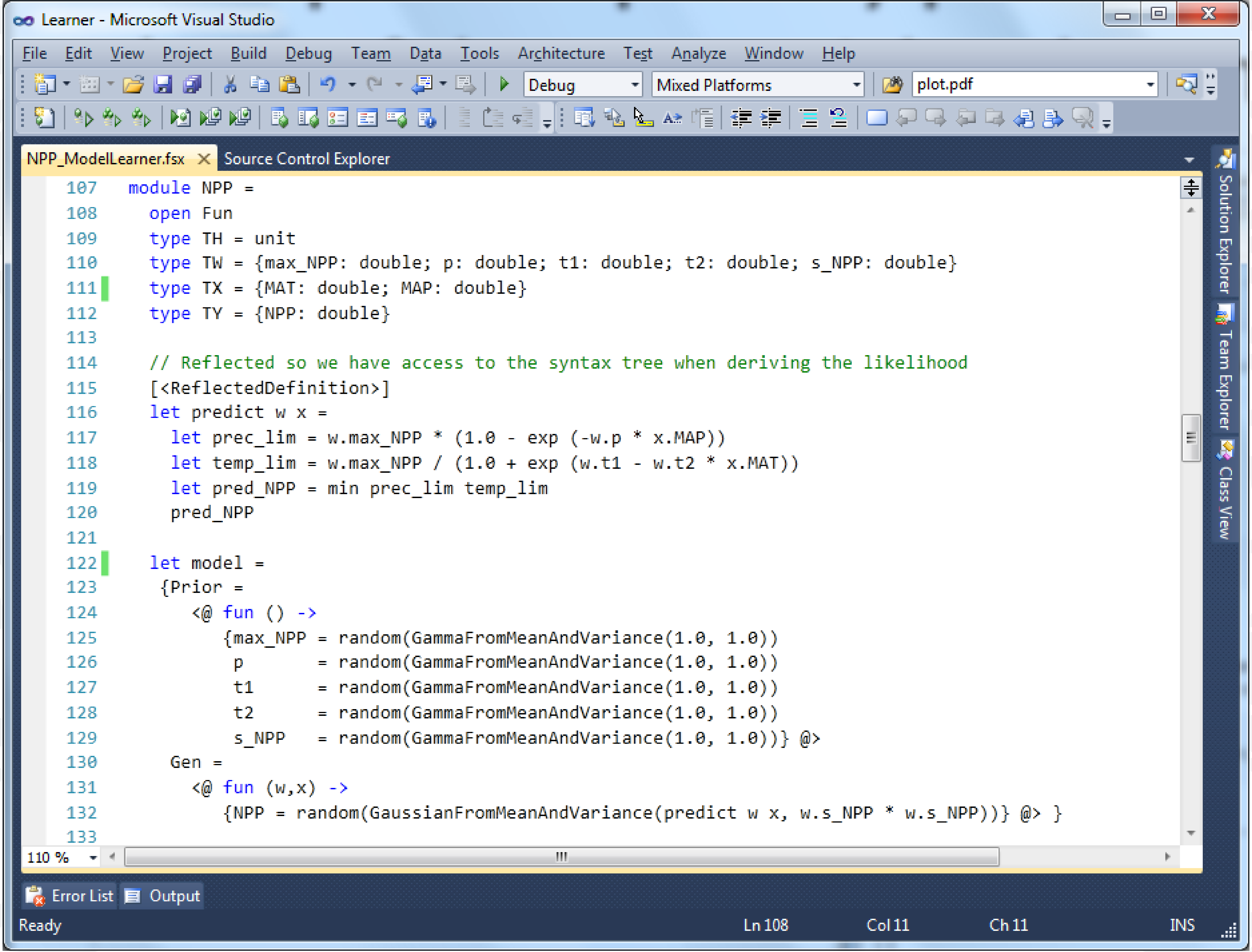
Orig. model+density

Fun model

Get a 2-3x slowdown

Example	orig	LOC, orig	LOC, Fun		time (s), orig	time (s), Fun	
mixture of Gaussians	F#	32	20	0.63x	1.77	4.78	2.7x
linear regression	F#	27	18	0.67x	0.63	2.08	3.3x
species distribution	C#	173	37	0.21x	79	189	2.4x
net primary productivity	C#	82	39	0.48x	11	23	2.1x
global carbon cycle	C#	1532	402	0.26x	n/a	764	n/a

Write a quarter as much code



Learner - Microsoft Visual Studio
File Edit View Project Build Debug Team Data Tools Architecture Test Analyze Window Help
Debug
Mixed Platforms
plot.pdf
NPP_ModelLearner.fsx
Source Control Explorer
module NPP =
open Fun
type TH = unit
type TW = {max_NPP: double; p: double; t1: double; t2: double; s_NPP: double}
type TX = {MAT: double; MAP: double}
type TY = {NPP: double}
// Reflected so we have access to the syntax tree when deriving the likelihood
[<ReflectedDefinition>]
let predict w x =
let prec_lim = w.max_NPP * (1.0 - exp (-w.p * x.MAP))
let temp_lim = w.max_NPP / (1.0 + exp (w.t1 - w.t2 * x.MAT))
let pred_NPP = min prec_lim temp_lim
pred_NPP
let model =
{Prior =
<@ fun () ->
{max_NPP = random(GammaFromMeanAndVariance(1.0, 1.0))
p = random(GammaFromMeanAndVariance(1.0, 1.0))
t1 = random(GammaFromMeanAndVariance(1.0, 1.0))
t2 = random(GammaFromMeanAndVariance(1.0, 1.0))
s_NPP = random(GammaFromMeanAndVariance(1.0, 1.0))} @>
Gen =
<@ fun (w,x) ->
{NPP = random(GaussianFromMeanAndVariance(predict w x, w.s_NPP * w.s_NPP))} @> }
110 %
Solution Explorer
Team Explorer
Class View
Error List
Output
Ready
Ln 108
Col 11
Ch 11
INS

Related Work

- Naive prototype (interpreter) reported at POPL'13.
- Builds on work by Bhat *et al.*, POPL'12.
- We have a soundness proof
- We have a simpler algorithm (and fewer judgments)
- We implement our algorithm, and study real models
- We use a more expressive language:
  integer operations, **`fail`**, general **`if`** and **`match`**, deterministic **`let`**
- We are less complete (admit fewer joint densities)

Conclusion

- We compile probabilistic programs to their density functions
  - The algorithm is sound.
- We validate the approach by compiling existing ecology models
  - The implementation is reasonably efficient
- Future work:
  - optimisation, improve completeness, clean up `match`
  - more complex real-life models or variations
  - different ways of treating hidden variables