# Bargaining Dynamics in Exchange Networks

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Technical Report
MSR-TR-2010-91

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**Abstract – We consider some known dynamical systems for Nash bargaining on graphs and focus on their rate of convergence. We first consider the edge-balanced dynamical system by Azar et al and fully specify its convergence for an important class of elementary graph structures that arise in Kleinberg and Tardos' procedure for computing a Nash bargaining solution on general graphs. We show that all these dynamical systems are either linear or eventually become linear and that their convergence times are quadratic in the number of matched edges. We then consider another linear system, the path bounding process of natural dynamics by Kanoria et al, and show a result that allows to improve their convergence time bound to $O(n^{4+\delta})$, for any $\delta > 0$, and for a graph of $n$ nodes that has a unique maximum-weight matching and satisfies a positive gap condition.**

## I. Introduction

Bargaining and, in particular, the concept of Nash bargaining on general graphs has been a focus of much recent research in economics, sociology and computer science [1], [2], [3], [4], [5]. In a bargaining system, players aim at making pairwise agreements to share a fixed wealth specific to a pair of players. Bargaining solutions aim at providing predictions on how the wealth will be shared and how this sharing would depend on players' positions in a network describing some notion of relationships among players.

The concept of Nash bargaining solution was introduced by Nash [6] for two players, each having an exogenous, alternative profit at its disposal were they to disagree on how to share the wealth. Recent research has focused on the concept of Nash bargaining with multiple players where each player has alternative profits determined by trading opportunities with neighbors in a graph. In computer science literature, Kleinberg and Tardos [1] were the first to establish various properties of Nash bargaining outcomes on general graphs. They also propose a polynomial-time algorithm for computing them, provided one exists. Follow up work aimed at introducing some local dynamics that are natural (so they, hopefully, have some connections with reality) and studied their convergence properties. Azar et al [3] considered the so called *edge-balanced dynamics* and established various properties about fixed points and convergence but left open the characterization of the convergence rate. In a tandem of papers [4], [5], Kanoria et al considered an alternative, *natural dynamics*, and established polynomial convergence time bounds under assumptions described later in the paper. An open research question has been to gain a better understanding of convergence properties and obtain tight bounds on the convergence time for this type of systems.

In this paper we consider dynamical systems of Nash bargaining and focus on characterizing their rate of convergence. We first consider edge-balanced dynamics of Azar et al [3] over elementary graphs that arise in the decomposition procedure of Kleinberg and Tardos [1] which include a path, a cycle, a blossom and a bicycle (see Figures 1, 2, 3 and 4 for examples). It turns out that, for all these network structures, the dynamics is either linear or eventually becomes linear. Specifically, we show that the dynamics is *linear* for a path and a cycle and is *eventually linear* for a blossom and a bicycle (and characterize the time when this takes place). This allows us to fully characterize the rate of convergence by deploying well known spectral methods for linear systems. As a result, for all these elementary structures, we find that the convergence time is *quadratic* in the number of matched edges.

We then consider a *path bounding process* introduced by Kanoria et al to study convergence properties of the so called natural dynamics introduced in [4]. This path bounding process is yet another linear system that in [4] was used to establish the convergence time upper bound $O(n^{6+\delta})$, for any $\delta > 0$, for any graph with a unique maximum-weight matching and satisfies a positive gap condition (we discuss in Section II). It turns out that an upper bound for this path bounding process asserted in Lemma 27 [4] can be improved from $1-\Theta(1/n^3)$ to $1-1/n$ which implies an improvement of the convergence time bound to $O(n^{4+\delta})$, for $\delta > 0$.

### A. Outline of the Paper

In Section II we introduce system assumptions and overview relevant concepts, including the concept of Nash bargaining outcomes, local dynamics, and the KT procedure. Section III provides the characterization of the edge-balanced dynamics and convergence times for each of the elementary graphs of the KT decomposition. In Section IV, we provide a result for a path bounding process for natural dynamics. Section V reviews related work. Finally, we conclude in Section VI.

## II. System and Assumptions

### A. Nash Bargaining Outcomes on Graphs

We consider a graph $G = (V, E)$ where $V$ is the set of nodes and $E$ is the set of edges. Each node corresponds to a distinct player that participates in the trading game defined as follows. Each edge $(i, j) \in E$ is associated with a weight $w_{i,j} \geq 0$ representing the amount that can be shared between players $i$ and $j$ should these two players decide to trade with each other. The trading game is one-exchange meaning that each player attempts to make a pairwise agreement with at most one other player, which corresponds to a matching $M \subset E$ in the graph where $(i,j) \in M$ if and only if players $i$ and $j$ reached an agreement. We denote with $x_i$ the profit of player $i$ where $x_i \geq 0$ and let $\vec{x} = (x_1, x_2, \ldots, x_n)$ denote the vector of players' profits according to an arbitrary enumeration of the $n = |V|$ players.

A balanced outcome or a Nash bargaining solution is a pair $(M, \vec{x})$ where $M$ is a matching in $G$ and $\vec{x}$ is a vector of players' profits. Such an outcome satisfies the following properties:

- **Stability**: for every edge $(i,j) \in E$,

$$x_i + x_j \geq w_{i,j}.$$

- **Balance**: for every $(i,j) \in M$, it holds

$$x_i - \max_{k \in V_i \setminus \{j\}} (w_{i,k} - x_k)_+ = x_j - \max_{k \in V_j \setminus \{i\}} (w_{j,k} - x_k)_+$$

  where, hereinafter, $V_i$ denotes the set of neighbors of a node $i$ and $(\cdot)_+ := \max(0, \cdot)$.

The stability means that there exists no player that can improve her profit by unilaterally deciding to trade with an alternative trading partner. The balance property originates from the Nash bargaining problem [6] where two players 1 and 2 aim at a pairwise agreement to share a profit $w$ having outside profit options $r_1$ and $r_2$ in case of disagreement. The Nash bargaining solution is then for players 1 and 2 to share the surplus $w - r_1 - r_2$ equally, if positive, i.e. receive profits $p_1 = r_1 + \frac{1}{2}(w - r_1 - r_2)_+$ and $p_2 = r_2 + \frac{1}{2}(w - r_1 - r_2)_+$, respectively. This allocation is balanced in the sense that $p_1 - r_1 = p_2 - r_2$, which is exactly the above asserted balance property where the outside profit options are determined by the values that players may extract through trading agreements with their neighbors.

### B. Local Nash Bargaining Dynamics

**Edge-balanced dynamics** (Azar et al [3]). First considered by Rochford [7] and Cook and Yamagishi [8], this dynamical process assumes that players already agreed on a matching $M$ and are negotiating the value of the outcome $\vec{x}$. Hence, each matched player $i$ is assigned a trading partner, which we denote with $p_i$. A version of this dynamics in discrete-time can be represented as follows. For a fixed $0 < \alpha \leq 1$ and given an initial value $\vec{x}(0)$, for $i = 1, 2, \ldots, n$ and $t = 0, 1, \ldots$, we have that

$$\begin{aligned} x_i(t+1) = x_i(t) + \alpha \Big\{ \Big[ y_i(t) + \\ + \frac{1}{2}(w_{i,p_i} - y_i(t) - y_{p_i}(t)) \Big]_0^{w_{i,p_i}} - x_i(t) \Big\} \end{aligned} \quad (1)$$

where $y_l(t)$ is the best alternate value that a matched player $l$ may get at time $t$ by trading with her other neighbors, i.e.

$$y_l(t) = \max_{k:(l,k) \in E \setminus M} (w_{l,k} - x_k(t))_+$$

and we use the notation $[\cdot]_a^b = \min(\max(\cdot, a), b)$, for $a \leq b$.

It is not difficult to observe that if players $i$ and $j$ are matched, then $x_i(t) + x_j(t) = w_{i,j}$ is time invariant, i.e. if the latter holds for a time $t$, then it still holds for time $t+1$. Note that the dynamics is not necessarily consistent with Nash bargaining solution for every time $t$ as for a matched pair $(i,j)$, the edge-surplus $w_{i,j} - y_i(t) - y_j(t)$ is allowed to be negative; the only requirement is that the allocation $y_i(t) + \frac{1}{2}(w_{i,j} - y_i(t) - y_j(t))$ is in $[0, w_{i,j}]$. However, the edge surpluses are guaranteed to be positive for $t$ large enough [3].

**Natural dynamics** (Kanoria et al [4]). For this dynamics, $x_{i,j}(t)$ is defined to be the profit that player $i$ can earn at time $t$ by partnering with one of his neighbors other than player $j$. The bargaining is assumed to evolve according the following system: given initial values $x_{i,j}(0)$, for $(i,j) \in E$, for $0 < \alpha \leq 1$ and $t = 0, 1, \ldots$,

$$x_{i,j}(t+1) = x_{i,j}(t) + \alpha \left\{ \max_{k \in V_i \setminus \{j\}} y_{k,i}(t) - x_{i,j}(t) \right\} \quad (2)$$

where $y_{i,j}(t)$ denotes the offer made by player $i$ to player $j$ at time $t$:

$$y_{i,j}(t) = (w_{i,j} - x_{i,j}(t))_+ - \frac{1}{2}(w_{i,j} - x_{i,j}(t) - x_{j,i}(t))_+.$$

Indeed, this is consistent with Nash's bargaining solution. If $x_{i,j}(t) > w_{i,j}$ then player $i$ can earn more elsewhere and makes a zero offer to player $j$. Otherwise, player $i$ offers just the right amount so that if player $j$ accepts the offer, the resulting allocation is according to a Nash bargaining solution:

$$w_{i,j} - y_{i,j}(t) = x_{i,j}(t) + \frac{1}{2}(w_{i,j} - x_{i,j}(t) - x_{j,i}(t))_+.$$

The profit of a player $i$ is equal to the current best offer made to this player, thus at time $t$ equal to

$$x_i(t) = \max_{k \in V_i} y_{k,i}(t).$$

The above dynamics was showed in [4] to converge to a Nash bargaining solution in polynomial time, provided that it exists and is unique, and the graph satisfies a positive gap condition which we define later in this section.

Both dynamical systems by Azar et al and Kanoria et al are nonlinear because of the maximum operators that appear in evaluating best profit values available to the players. Specifically, both dynamical systems are piecewise linear in $\mathbb{R}_+^{|V|}$ and $\mathbb{R}_+^{|E|}$, respectively. This fact may be leveraged in some future analysis.

### C. KT Procedure

Nash's bargaining solutions on graphs are intimately related to maximum-weight matchings. In [1] it was found that the matching $M$ of a stable outcome $\vec{x}$ is a maximum-weight matching. Furthermore, whenever a stable outcome exists, a balanced outcome exists as well [1]. The outcome vector $\vec{x}$ can be seen as a feasible solution of a dual to a fractional relaxation of a maximum-weight matching (primal):

$$\begin{array}{rl} \text{maximize} & \sum_{(i,j)\in E} w_{i,j}x_{i,j} \\ \text{over} & x_{i,j} \geq 0,\ (i,j)\in E \\ \text{subject to} & \sum_{j:(i,j)\in E} x_{i,j} \leq 1 \end{array}$$

where the dual problem is the following linear problem with two variables per inequality:

$$\begin{array}{rl} \text{minimize} & \sum_{i\in V} x_i \\ \text{over} & x_i \geq 0,\ i\in V \\ \text{subject to} & x_i + x_j \geq w_{i,j},\ (i,j)\in E. \end{array}$$

In [1], it was established that a balanced outcome $(M, \vec{x})$ can be found in polynomial time by first finding a maximum-weight matching $M$ and then solving the above dual problem to find a balanced vector $\vec{x}$. The dual problem can be solved by an iterative procedure where each iteration maximizes the smallest slack as described in the following.

A node $i \in V$ slack $s_i$ is defined by $s_i = x_i - \max_{(l,i)\in E\setminus M}(w_{i,l} - x_l)_+$ while an edge $(i,j)\in E$ slack $s_{i,j}$ is defined by $s_{i,j} = x_i + x_j - w_{i,j}$. Indeed, for a stable outcome $\vec{x}$, $s_{i,j} \geq 0$, for every $(i,j)\in E$. It is not difficult to observe that node and edge slacks satisfy $s_i = \min(x_i, \min_{(i,l)\in E\setminus M} s_{i,l})$.

The KT procedure for finding a balanced outcome proceeds by successively fixing the values $x_i$ for some nodes in $V$. This is allowed by the following key property [1]: if there exists a set $A \subset V$ and $\sigma \geq 0$ such that $s_i \leq \sigma$ for every $i\in A$ and $s_i \geq \sigma$ for $i \in V\setminus A$ and a vector $\vec{x}$ such that values $x_i$ are balanced in $A$, then there exists a vector $\vec{x}'$ such that $x_i' = x_i$ for every $i\in A$ that is a balanced outcome for $G$.

The KT algorithm is sketched as follows. Let $\sigma \geq 0$ be a variable and let $A$ be a set of nodes for which values $x_i$ have been already assigned. The set $A$ is constructed such that no matched edge crosses the cut $(A, V\setminus A)$, i.e. for every node $i\in A$ there exists no node $j \in V\setminus A$ such that $(i,j)\in M$. Initially, $\sigma = 0$ and set $A$ contains all the unmatched nodes. The algorithm then proceeds inductively with respect to the number of nodes with unassigned values as given by $|V\setminus A|$. Given $\sigma$ and $A$ the inductive step amounts to assigning values to nodes in $V\setminus A$ that maximize the minimum slack $\sigma' \geq \sigma$ which amounts to solving the following linear program

$$\begin{array}{ll} \text{maximize} & \sigma' \\ \text{subject to} & x_i' \geq \sigma',\ i\in V\setminus A \\ & x_i' + x_j' = w_{i,j},\ (i,j)\in M \\ & x_i' + x_j' \geq w_{i,j} + \sigma',\ (i,j)\in E\setminus(M\cup E(A)) \\ & x_i' = x_i, i\in A, \end{array}$$

where $E(A)$ corresponds to the set of edges of the graph $G$ linking nodes in $A$.

For a fixed $\sigma'$, this is a linear inequalities' problem with at most two variables per inequality, for which polynomial algorithms exist. In particular, by results of Aspvall and Shilach [9], for a given $\sigma'$, the system of inequalities is infeasible if there exists an infeasible simple loop in the graph construction described in [9]. A path is said to be a loop if the initial and final nodes are identical and is said to be simple if all intermediate nodes of this path are distinct. Furthermore, if a feasible solution exists than it can be constructed by finding the most constraining feasible simple loop. For the above system of inequalities, any such feasible simple loop is either a path, a cycle, a blossom or a bicycle. We refer to these as *KT elementary graphs* and define them in the following:

- **Path**. A path consists of alternating matchings with each of its end nodes anchored at either a node $i\in A$ or at a matched edge $(i,j)\in M$ such that $s_j = x_j$.[1]
- **Cycle**. A cycle consists of an even number of nodes connected by a path of alternating matchings.
- **Blossom**. A blossom is a concatenation of a cycle and a path (we refer to as a stem) as follows. The cycle consists of an odd number of nodes that are connected by a cycle of alternating matchings started from a node (we call gateway) with an unmatched edge. The stem is a path of alternating matchings such that one end node is matched to the gateway node and the other end node is anchored as for a path.

[1]Recall that if for a matched edge $(i,j)\in E$, $i\in V\setminus A$, then also $j\in V\setminus A$, and vice versa.

- **Bicycle**. A bicycle is a concatenation of two blossoms by connecting the end nodes of their respective stems such that the connected stems form alternating matchings.

The above described step is repeated until all the nodes are assigned values, i.e. until $V \setminus A = \emptyset$. Hence, the total number of such steps $k$ is at most the number of nodes $n$. At each step $l$, a KT elementary structure $C_l$ and maximum slack $\sigma_l$ are identified such that the $\sigma_l$'s form an non-decreasing sequence, $0 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_k$.

**A positive gap condition**. A balanced outcome $\vec{x}$ with slacks $\sigma_0, \sigma_1, \ldots, \sigma_k$ is said to have a gap $\sigma > 0$ if, for every $1 \leq l \leq k$,

$$\sigma_l - \sigma_{l-1} \geq \sigma$$

and, for every pair of nodes $i$ and $j$ of $C_l$ such that the edge $(i, j)$ is not part of $C_l$, we have

$$x_i + x_j - w_{i,j} \geq \sigma_l + \sigma.$$

This positive gap condition enables to study convergence of a dynamic process by partitioning into a sequence of KT elementary graphs and decoupling the dynamics over these elementary graphs, a technique introduced and used in [4].

### D. Convergence

We introduce a few elementary concepts about stability of dynamical systems in a somewhat informal manner and then define the notion of convergence time considered in this paper. We say that a dynamical system, according to which $\vec{x}(t)$ evolves over $t \geq 0$, is *asymptotically stable*, if there exists a point $\vec{x}^*$ such that for every initial value $\vec{x}(0)$, we have

$$\lim_{t \to \infty} ||\vec{x}(t) - \vec{x}^*|| = 0.$$

The system is said to be *globally asymptotically stable* if $\vec{x}^*$ is unique, i.e. does not depend on the initial condition $\vec{x}(0)$.

In particular, for a linear system $\vec{x}(t+1) = \mathbf{A}\vec{x}(t) + \vec{b}(t)$ where $\mathbf{A}$ is a given matrix and $\vec{b}(t)$ is a vector that may depend on $t$, we have that the system is globally asymptotically stable if the spectral radius of the matrix $\mathbf{A}$ is smaller than 1 (i.e. all eigenvalues are of modulo strictly smaller than 1). The concepts of asymptotic stability and global asymptotic stability are standard, see [10] for more details.

We say that the convergence to a point $\vec{x}^*$ is exponentially bounded if there exist $C > 0$ and $R > 0$ such that for every initial value $\vec{x}(0)$, we have

$$||\vec{x}(t) - \vec{x}^*|| \leq Ce^{-Rt}, \text{ for every } t \geq 0,$$

where we refer to $R$ as the rate of convergence and call $T = 1/R$ the convergence time. Moreover, If $\vec{x}(t)$ evolves according to the aforementioned linear system then the rate of convergence is given by $(i)$ $R = \log(1/\rho(\mathbf{A}))$ where $\rho(\mathbf{A})$ is the spectral radius of matrix $\mathbf{A}$ if the system is globally asymptotically stable, and $(ii)$ $R = \log(1/\lambda_2(\mathbf{A}))$ where $\lambda_2(\mathbf{A})$ is the modulus of the largest eigenvalue of matrix $\mathbf{A}$ that is smaller than 1, if the system is asymptotically stable.

## III. Dynamics for KT Elementary Graphs

In this section, we will observe that for all the elementary graphs of the KT decomposition, the values held by the nodes eventually evolve according to a *linear* discrete-time dynamical system, i.e., for given matrix $\mathbf{A}$ and vector $\vec{b}(t)$, $\vec{x}(t)$ evolves according to

$$\vec{x}(t+1) = \mathbf{A}\vec{x}(t) + \vec{b}(t). \quad (3)$$

We will find that for a path and a cycle the dynamics is linear for every time $t \geq 0$ while for a blossom and a bicycle there exists a finite time $T_0 \geq 0$ such that the dynamics is linear for every $t \geq T_0$. The asymptotic behavior is determined by spectral properties of matrix $\mathbf{A}$. Note that it suffices to consider the spectrum of matrix $\mathbf{A}$ for $\alpha = 1$. This is because, for every given $0 < \alpha \leq 1$, $\lambda' = 1 - \alpha + \alpha\lambda$ is an eigenvalue and $\vec{v}$ is an eigenvector of the matrix $\mathbf{A}$, where $\lambda$ is an eigenvalue and $\vec{v}$ is an eigenvector of the matrix $\mathbf{A}$ under $\alpha = 1$. We will see that for every KT elementary graph, the eigenvalues of matrix $\mathbf{A}$, under $\alpha = 1$, are located in the interval $[-1, 1]$ and will show that $-1$ can be an eigenvalue only for a cycle with an even number of matched edges or a bicycle with an even number of matched edges in each of its loops. In the latter two cases, for $\alpha = 1$, there is no convergence to a limit point as the asymptotic behavior is periodic because of the eigenvalue $-1$. This is ruled out by choosing the smoothing parameter $0 < \alpha < 1$, making all the eigenvalues strictly larger than $-1$, and thus ensuring convergence to a limit point.

Finally, we note that for our results in this section, we assume uniform edge weights and under this assumption, without loss of generality, we let $w_e = 1$, for every $e \in E$.

### A. Path

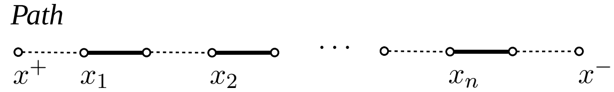


Figure 1. A path with boundary conditions.

We consider a path with boundary values $x^+(t)$ and $x^-(t)$ as illustrated in Figure 1. In this case, the evolution of the node values $\vec{x}(t)$ boils down to a discrete-time linear dynamical system (3) where $\mathbf{A}$ is the $n \times n$ *symmetric tridiagonal* matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 1/2 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1/2 \\ 0 & \cdots & 0 & 1/2 & 0 \end{pmatrix} \qquad (4)$$

and $\vec{b}(t) = (\frac{1-x^+(t)}{2}, \underbrace{0,\ldots,0}_{n-2}, \frac{x^-(t)}{2})^T$.

The eigenvalues of matrix $\mathbf{A}$ are

$$\lambda_k = \cos\left(\frac{\pi k}{n+1}\right),\ k = 1, 2, \ldots, n,$$

with the corresponding orthonormal eigenvectors

$$\vec{v}_k = \sqrt{\frac{2}{n+1}}\left(\sin\left(\frac{\pi k}{n+1}\right), \ldots, \sin\left(\frac{\pi k n}{n+1}\right)\right)^T.$$

Note that every eigenvalue is of modulo smaller than 1. This implies asymptotic stability for every $0 < \alpha \leq 1$. From the above spectrum, we have the following characterization of the convergence time whose proof is deferred to Appendix A.

*Theorem 1:* For a path of $n$ matched edges and every $0 < \alpha \leq 1$, the convergence time is

$$T = \frac{2}{\alpha\pi^2} n^2 \cdot [1 + O(1/n^2)].$$

From this theorem, we observe that the convergence time is quadratic in the number of matched edges.

### B. Cycle

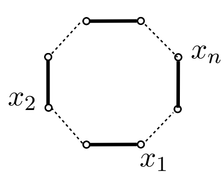


Figure 2. A cycle.

For a cycle, the dynamics of node values $\vec{x}(t)$ boils down to a linear dynamical system (3) where $\mathbf{A}$ is the following *circulant* matrix, for $n = 2$, $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and otherwise

$$\mathbf{A} = \begin{pmatrix} 0 & 1/2 & 0 & \cdots & 0 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & \ddots & \ddots & \ddots & 0 \\ 0 & 1/2 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 1/2 & 0 \\ 0 & \ddots & \ddots & \ddots & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 0 & \cdots & 0 & 1/2 & 0 \end{pmatrix} \qquad (5)$$

and where vector $\vec{b} = \vec{0}$. Note that in this case

$$\vec{x}(t) = \mathbf{A}^t \vec{x}(0),\ \text{for } t \geq 0.$$

By using similar arguments as for a path, it is not difficult to establish that the eigenvalues of matrix $\mathbf{A}$ are

$$\lambda_k = \cos\left(\frac{2\pi(k-1)}{n}\right),\ k = 1, 2, \ldots, n,$$

with the corresponding orthonormal eigenvectors

$$\vec{v}_k = \begin{cases} \frac{1}{\sqrt{n}}(1, 1, \ldots, 1, 1)^T, & \text{if } k = 1 \\ \frac{1}{\sqrt{n}}(1, -1, \ldots, 1, -1)^T, & \text{if } k = 1 + n/2 \\ \sqrt{\frac{2}{n}}(1, \cos(\phi_k), \ldots, \cos(\phi_k(n-1)))^T, & \text{o.w.} \end{cases}$$

where for easy of notation, $\phi_k = \frac{2\pi(k-1)(n-1)}{n}$.

Using the spectral decomposition of the symmetric matrix $\mathbf{A}$ (see [10] for details), we have

$$\vec{x}(t) = \sum_{k=1}^{n} \lambda_k^t \vec{v}_k \vec{v}_k^T \vec{x}(0). \qquad (6)$$

We distinguish two cases:

- **Case 1**: $n$ is even. In this case, $\lambda_k = -1$, for $k = 1 + n/2$, and $\lambda_k > -1$, otherwise. From (6), we have

  $$\vec{x}(t) = \left(\vec{v}_1 \vec{v}_1^T + (-1)^t \vec{v}_{1+n/2} \vec{v}_{1+n/2}^T\right) \vec{x}(0) + o(1).$$

  Therefore, the asymptotic behavior is periodic.
- **Case 2**: $n$ is odd. In this case, $-1 < \lambda_k \leq 1$, for every $k$, and thus we have asymptotic convergence to the limit point, $\lim_{t\to\infty} x_i(t) = \frac{1}{n}\sum_{j=1}^{n} x_j(0)$, for every $i$.

In view of the above observations, we note that for even $n$, we need to assume that $\alpha$ is strictly smaller than 1 in order to rule out asymptotically periodic behavior, while for odd $n$, we can allow for $\alpha = 1$. The following result shows that in like manner as for a path, the convergence time is quadratic in the number of matched edges, but note that it is four times smaller, asymptotically for large $n$.

*Theorem 2:* For cycle graph of $n$ matched edges and $0 \leq \alpha < 1$, if $n$ is even, and $0 < \alpha \leq 1$, if $n$ is odd, the convergence time is

$$T = \frac{1}{\alpha 2\pi^2} n^2 \cdot [1 + O(1/n^2)].$$

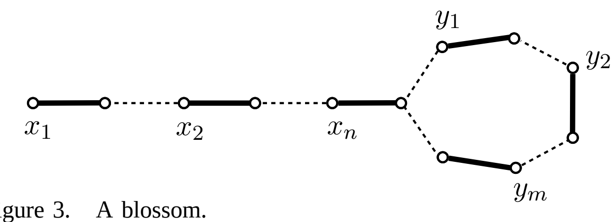


Figure 3. A blossom.

### C. Blossom

A blossom is a concatenation of a cycle and a path (we refer to as a stem); see Figure 3 for an example. We consider a blossom with $n$ matched edges in the stem and $m$ matched edges in the loop. We refer to the node that connects the stem and the loop as a *gateway* node. The matched edges of the stem are enumerated as $1, 2, \ldots, n$ along the stem towards the gateway note. We let $x_i$ denote the value of the end node of an edge $i$ of the stem that is connected to a node towards the open end of the stem. Similarly, we enumerate matched edges of the loop as $1, 2, \ldots, m$ and let $y_i$ denote the value of the node that appears first on a matched edge $i$ as we go along the loop in the clockwise direction.

It can be observed that node values $\vec{x}(t)$ and $\vec{y}(t)$ evolve according to the following non-linear dynamical system:

$$\begin{aligned} x_1(t+1) &= \frac{x_2(t)}{2} \\ x_i(t+1) &= \frac{x_{i-1}(t) + x_{i+1}(t)}{2}, 1 < i < n \\ x_n(t+1) &= \frac{1 + x_{n-1}(t) - \max[1 - y_1(t), y_m(t)]}{2} \\ y_1(t+1) &= \frac{x_n(t) + y_2(t)}{2} \\ y_i(t+1) &= \frac{y_{i-1}(t) + y_{i+1}(t)}{2}, 1 < i < m \\ y_m(t+1) &= \frac{1 + y_{m-1}(t) - x_n(t)}{2}. \end{aligned} \quad (7)$$

Note that the system is non-linear only because of the maximum operator that acts in the update for the node matched to the gateway node, which connects the stem and the loop of the blossom. The maximum operator is over the values of the nodes that are in the loop matched to the neighbors of the gateway node, $1 - y_1(t)$ and $y_m(t)$. It turns out that, eventually, one of these two values is larger or equal to the other and, hence, the system dynamics becomes linear. This is showed in the following lemma. Note that the sum $y_1(t) + y_m(t)$, if smaller or equal than 1 indicates $\max(1 - y_1(t), y_m(t)) = 1 - y_1(t)$, and otherwise, $\max(1 - y_1(t), y_m(t)) = y_m(t)$.

*Theorem 3:* For a blossom with $n$ matched edges in the stem and $m$ matched edges in the loop, for every initial value $(\vec{x}(0), \vec{y}(0))$, the sum of node values $y_1(t) + y_m(t)$ satisfies:

1) $y_1(t) + y_m(t)$, for $t \geq 0$, is autonomous of $\vec{x}(t)$, $t \geq 0$.
2) $\lim_{t\to\infty} y_1(t) + y_m(t) = 1$.
3) The asymptotic rate of convergence is $\frac{\pi^2}{2m^2}$.
4) There exists a time $T_0 \geq 0$ such that either $y_1(t) + y_m(t) \leq 1$ or $y_1(t) + y_m(t) \geq 1$ for every $t \geq T_0$.
5) $T_0 = O(m^2)$.

The theorem derives from an explicit characterization of $y_1(t) + y_m(t)$, for every $t \geq 0$, which we present in the following:

*Lemma 1:* Given initial value $\vec{y}(0)$, for every $t \geq 0$,

$$y_1(t) + y_m(t) = 1 - \frac{2}{m+1} \sum_{i=1}^{\lceil \frac{m}{2} \rceil} f_{2i-1}(\vec{y}(0)) \lambda_{2i-1}^t \quad (8)$$

where

$$f_k(\vec{y}) = 1 + \lambda_k - 2\sqrt{1 - \lambda_k^2} \sqrt{\frac{m+1}{2}} \vec{v}_k^T \vec{y}.$$

We provide proofs of Theorem 3 and Lemma 1 in Appendix B and C, respectively.

From Theorem 3 item 4, we have that the dynamics for a blossom is eventually according to the following linear system

$$\begin{pmatrix} \vec{x}(t+1) \\ \vec{y}(t+1) \end{pmatrix} = \mathbf{A} \begin{pmatrix} \vec{x}(t) \\ \vec{y}(t) \end{pmatrix} + \vec{b}$$

where matrix $\mathbf{A}$ and vector $\vec{b}$ assume one of the following two choices:

- **Case 1**: $(1 - y_1(t) \geq y_m(t))$

$$\mathbf{A} = \begin{pmatrix} \mathbf{T}_n & \mathbf{P} \\ \mathbf{Q} & \mathbf{T}_m \end{pmatrix} \quad (9)$$

with $\mathbf{T}_n$ and $\mathbf{T}_m$ tridiagonal matrices of paths of $n$ and $m$ matched edges, respectively, and

$$\mathbf{P} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \cdots & 0 \end{pmatrix}, \mathbf{Q} = \begin{pmatrix} 0 & \cdots & 0 & \frac{1}{2} \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -\frac{1}{2} \end{pmatrix}.$$

and $\vec{b} = (\underbrace{0, \ldots, 0}_{n+m-1}, 1/2)^T$.

- **Case 2**: $(1 - y_1(t) < y_m(t))$ same as under Case 1 but

$$\mathbf{P} = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & -\frac{1}{2} \end{pmatrix}$$

and $\vec{b} = (\underbrace{0, \ldots, 0}_{n-1}, 1/2, \underbrace{0, \ldots, 0}_{m-1}, 1/2)^T$.

In the following we only consider Case 1 as the spectrum of matrix $\mathbf{A}$ under Case 2 is exactly the same. We note that the eigenvalues of the matrix $\mathbf{A}$ are $(\lambda_1, \lambda_2, \ldots, \lambda_{n+\lfloor m/2 \rfloor}, \mu_1, \mu_2, \ldots, \mu_{\lceil m/2 \rceil})$ where

$$\begin{aligned} \lambda_k &= \cos\left(\frac{2\pi k}{2n+m+1}\right), \; k = 1, \ldots, n + \lfloor m/2 \rfloor \\ \mu_k &= \cos\left(\frac{\pi(2k-1)}{m+1}\right), \; k = 1, \ldots, \lceil m/2 \rceil \end{aligned}$$

with a proof provided in Appendix D.

It is noteworthy that all the eigenvalues have modulo strictly smaller and 1, and thus, the system is globally asymptotically stable. We now characterize the convergence time from an instance at which the system became linear.

*Theorem 4:* For a blossom with $n$ matched edges in the stem and $m$ matched edges in the loop, for every $0 < \alpha \le 1$, the convergence time $T$ satisfies: if $m$ is even, then

$$T = \frac{2}{\alpha\pi^2}(2n+m)^2 \cdot [1 + o(1)]$$

otherwise, for $m$ odd,

$$T = \frac{2}{\alpha\pi^2} \max\left(m^2, \frac{1}{4}(2n+m)^2\right) \cdot [1 + o(1)].$$

**Observations**. The result implies that the convergence time is $O((n+m)^2)$, i.e. quadratic in the number of matched edges. There is a significant difference with regard to whether the number of matched edges in the loop, $m$, is even or odd. The convergence is slower for $m$ even. Specifically, if the length of the stem is at least twice the length of the loop, the convergence time is larger for a factor $4$. For a fixed $n$, the convergence time is asymptotically $\frac{2}{\alpha\pi^2}m^2$ as for a path of length $m$ which is intuitive. Likewise, if $m$ is fixed and odd, the convergence time is asymptotically $\frac{2}{\alpha\pi^2}n^2$ as for a path of length $n$ and thus also in conformance to intuition.

*Proof:* For the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{n+\lfloor m/2 \rfloor}$, it is readily checked that

$$\max_k |\lambda_k| = -\lambda_{n+\lfloor m/2 \rfloor} = \cos\left(\frac{(1 + 1_{m \text{ odd}})\pi}{2n+m+1}\right)$$

while, on the other hand,

$$\max_k |\mu_k| = \mu_1 = \cos\left(\frac{\pi}{m+1}\right).$$

Therefore, the spectral radius of matrix $\mathbf{A}$, $\rho(\mathbf{A}) = \max(\max_k |\lambda_k|, \max_k |\mu_k|)$ is given by

$$\rho(\mathbf{A}) = \begin{cases} \cos\left(\frac{\pi}{2n+m+1}\right), & m \text{ even} \\ \cos\left(\frac{2\pi}{2n+m+1}\right), & m \text{ odd}, m \le 2n-1 \\ \cos\left(\frac{\pi}{m+1}\right), & m \text{ odd}, m > 2n-1. \end{cases}$$

The asserted asymptotic follows from the last identities. ■

### *D. Bicycle*

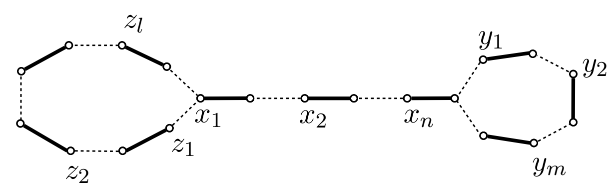


Figure 4. A bicycle.

A bicycle graph consists of two loops that are connected by a path. Without loss of generality, we refer to one of the loops as loop 1 and to other as loop 2 and refer to the path as a cross-bar; see Figure 4 for an illustration. Notice that a bicycle graph corresponds to a concatenation of two blossoms by connecting the end nodes of their respective stems so that a cross-bar is formed of alternating matchings. We let $l$ and $m$ be the number of matched edges in loop 1 and loop 2, respectively, and let $n$ be the number of matched edges of the cross-bar. The values of end nodes of matched edges are denoted by $\vec{z}(t) = (z_1(t), z_2(t), \ldots, z_l(t))^T$, $\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$ and $\vec{y} = (y_1(t), y_2(t), \ldots, y_m(t))^T$, for loop 1, cross-bar, and loop 2, respectively. See Figure 4 for positions of the corresponding nodes.

We note that for a bicycle the system evolves according to the following non-linear system:

$$\begin{aligned} z_1(t+1) &= \frac{1 + z_2(t) - x_1(t)}{2} \\ z_i(t+1) &= \frac{z_{i-1}(t) + z_{i+1}(t)}{2}, 1 < i < l \\ z_l(t+1) &= \frac{z_{l-1}(t) + x_1(t)}{2} \qquad (10) \\ x_1(t+1) &= \frac{x_2(t) + \max[1 - z_1(t), z_l(t)]}{2} \end{aligned}$$

plus other updates as for blossom (7).

In this case, the non-linearity originates because of two gateway nodes that connect the cross-bar with loops, each such gateway node having two alternative profit options with nodes in the loops. Similarly as for a blossom we have that eventually the dynamics becomes linear as stated in the following:

*Proposition 1:* For a bicycle with $l$ and $m$ matched edges in loops and $n$ matched edges in the cross-bar, there exists a time $T_0 \geq 0$ such that for every $t \geq T_0$, $(\vec{z}(t), \vec{x}(t), \vec{y}(t))$ evolves according to a linear system. Furthermore, $T_0 = O(\max(l,m)^2)$.

This observation follows from Theorem 3 applied to each loop of the bicycle. This can be done because both $y_1(t) + y_m(t)$ and $z_1(t) + z_l(t)$ evolve autonomously as given by Lemma 1 for $y_1(t) + y_m(t)$ and analogously for $z_1(t) + z_l(t)$.

We have showed that the dynamics for a bicycle is eventually according to a linear system, which is specified as follows:

$$\begin{pmatrix} \vec{z}(t+1) \\ \vec{x}(t+1) \\ \vec{y}(t+1) \end{pmatrix} = \mathbf{A} \begin{pmatrix} \vec{z}(t) \\ \vec{x}(t) \\ \vec{y}(t) \end{pmatrix} + \vec{b} \qquad (11)$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{T}_l & \mathbf{Q}' & \mathbf{0} \\ \mathbf{P}' & \mathbf{T}_n & \mathbf{P} \\ \mathbf{0} & \mathbf{Q} & \mathbf{T}_m \end{pmatrix}$$

with the given matrix blocks defined by

$$\begin{pmatrix} \mathbf{T}_l & \mathbf{Q}' \\ \mathbf{P}' & \mathbf{T}_n \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{T}_n & \mathbf{P} \\ \mathbf{Q} & \mathbf{T}_m \end{pmatrix}$$

are the matrices that correspond to two blossoms formed by loop 1 and cross-bar, and cross-bar and loop 2, respectively.

The pair $(\mathbf{A}, \vec{b})$ admits four possible values, corresponding to all possible combinations of two cases for each of the loops (Case 1 and Case 2 in Section III-C):

1) Both $\mathbf{P}'$ and $\mathbf{P}$ as in Case 1
$\vec{b} = (1/2, \underbrace{0, \ldots, 0}_{l+n+m-2}, 1/2)^T$,
2) $\mathbf{P}'$ as in Case 1, $\mathbf{P}$ as in Case 2,
$\vec{b} = (1/2, \underbrace{0, \ldots, 0}_{l-1}, 1/2, \underbrace{0, \ldots, 0}_{m+n-2}, 1/2)^T$,
3) $\mathbf{P}'$ as in Case 2, $\mathbf{P}$ as in Case 1,
$\vec{b} = (1/2, \underbrace{0, \ldots, 0}_{l+n-2}, -1/2, \underbrace{0, \ldots, 0}_{m-1}, 1/2)^T$,
4) Both $\mathbf{P}'$ and $\mathbf{P}$ as in Case 2,
$\vec{b} = (1/2, \underbrace{0, \ldots, 0}_{l-1}, 1/2, \underbrace{0, \ldots, 0}_{n-2}, -1/2, \underbrace{0, \ldots, 0}_{m-1}, 1/2)^T$

In the following, we will only consider the case under item 1 as the same end results hold for other cases. The eigenvalues of the matrix $\mathbf{A}$ are given by

$$\cos\left(\frac{\pi(2k-1)}{l+1}\right), \; k = 1, \ldots, \lceil l/2 \rceil,$$
$$\cos\left(\frac{\pi(2k-1)}{m+1}\right), \; k = 1, \ldots, \lceil m/2 \rceil,$$
$$\cos\left(\frac{2\pi k}{2n+l+m}\right), \; k = 1, \ldots, n + \lfloor l/2 \rfloor + \lfloor m/2 \rfloor$$

which we establish in Appendix E.

Remark that in any case all the eigenvalues are strictly smaller than 1. On the other hand, if both $l$ and $m$ are even, then $-1$ is an eigenvalue with eigenvector $(1, -1, 1, -1, \ldots, 1, -1)^T$, and otherwise, all the eigenvalues are strictly larger than $-1$. Therefore, if both $l$ and $m$ are even, then the asymptotic behavior of system (11) is periodic, while otherwise, it is globally asymptotically stable.

As a byproduct, similarly to Theorem 4, we can establish that from an instance at which the system became linear, the convergence time scales as follows.

*Theorem 5:* For a bicycle with $n$ matched edges in the stem and $m$ and $l$ matched edges in the loops, the convergence time $T$ satisfies the following. If $m$ or $l$ is even, then for every $0 < \alpha < 1$ (and $\alpha = 1$ if both $m$ and $l$ are even),

$$T = \frac{2}{\alpha \pi^2}(2n + m + l)^2 \cdot [1 + o(1)]$$

otherwise, if $m$ and $n$ are odd, then for every $0 < \alpha \leq 1$,

$$T = \frac{2}{\alpha \pi^2} \max\left(m^2, l^2, \frac{1}{4}(2n + m + l)^2\right) \cdot [1 + o(1)].$$

Therefore, the convergence time is $O((l + n + m)^2)$, i.e. quadratic in the number of matched edges.

## IV. The Path Bounding Process

We consider a bounding process for natural dynamics on a path, which was introduced in [4]. The bounding process, referred to as *simplified dynamics*, provides lower and upper bounds for the original dynamics by appropriately choosing initial and boundary conditions. This simplified dynamics is defined as follows. We consider a path of $n$ edges where nodes are enumerated as $0, 1, \ldots, n$ and let $M$ denote an alternating matching on this path. Let $u(t), v(t)$, $t \geq 0$, be arbitrary real-valued sequences and let $\alpha > 0$. Then, the simplified dynamics is given by

$$\begin{aligned} \hat{x}_{0,1}(t+1) &= \hat{x}_{0,1}(t) + \alpha(u(t) - \hat{x}_{0,1}(t)) \\ \hat{x}_{n,n-1}(t+1) &= \hat{x}_{n,n-1}(t) + \alpha(v(t) - \hat{x}_{n,n-1}(t)) \end{aligned}$$

while for $i = 1, 2, \dots, n-1$,

$$\begin{aligned}\hat{x}_{i,i+1}(t+1) &= \hat{x}_{i,i+1}(t) + \alpha(y_{i-1,i}(t) - \hat{x}_{i,i+1}(t)) \\ \hat{x}_{i,i-1}(t+1) &= \hat{x}_{i,i-1}(t) + \alpha(y_{i+1,i}(t) - \hat{x}_{i,i-1}(t))\end{aligned}$$

where

$$y_{i,j}(t) = \begin{cases} \frac{1}{2}(w_{i,j} - \hat{x}_{i,j}(t) + \hat{x}_{j,i}(t)), & (i,j) \in M \\ w_{i,j} - \hat{x}_{i,j}(t), & \text{otherwise.}\end{cases}$$

A path boundary process is a process $\vec{\bar{x}}(t)$ satisfying the following property: given two simplified dynamics $\vec{\hat{x}}(t)$ and $\vec{\hat{x}}'(t)$, $|\hat{x}_{i,i+1}(t) - \hat{x}'_{i,i+1}(t)| \le \bar{x}_{i,i+1}(t)$ and $|\hat{x}_{i+1,i}(t) - \hat{x}'_{i+1,i}(t)| \le \bar{x}_{i+1,i}(t)$, for every edge $(i, i+1)$ and $t \ge 0$.

It was showed in [4] that such a path bounding process can be defined as follows. Let $\bar{x}_{i,i+1}(0) = \bar{x}_{i+1,i}(0) = ||\vec{\hat{x}}(0) - \vec{\hat{x}}'(0)||_\infty$, for $i = 0, 1, \dots, n-1$. For an unmatched edge $(i, i+1)$, we let

$$\begin{aligned}&\bar{x}_{i,i+1}(t) = \bar{x}_{i,i+1}(t) + \\ &\quad + \alpha\left(\frac{\bar{x}_{i-1,i}(t) + \bar{x}_{i,i-1}(t)}{2} - \bar{x}_{i,i+1}(t)\right),\ i > 0 \\ &\bar{x}_{i+1,i}(t) = \bar{x}_{i+1,i}(t) + \\ &\quad + \alpha\left(\frac{\bar{x}_{i+2,i+1}(t) + \bar{x}_{i+1,i+2}(t)}{2} - \bar{x}_{i+1,i}(t)\right)\end{aligned}$$

while for a matched edge $(i, i+1)$,

$$\begin{aligned}\bar{x}_{i,i+1}(t) =& \bar{x}_{i,i+1}(t) + \alpha\left(\bar{x}_{i-1,i}(t) - \bar{x}_{i,i+1}(t)\right) \\ \bar{x}_{i+1,i}(t) =& \bar{x}_{i+1,i}(t) + \\ & + \alpha\left(\bar{x}_{i+2,i+1}(t) - \bar{x}_{i+1,i}(t)\right), i < n-1\end{aligned}$$

and

$$\begin{aligned}\bar{x}_{0,1}(t+1) =& \bar{x}_{0,1}(t) + \alpha(u(t) - \bar{x}_{0,1}(t)) \\ \bar{x}_{n,n-1}(t+1) =& \bar{x}_{n,n-1}(t) + \alpha(1 - u(t) - \bar{x}_{n,n-1}(t))\end{aligned}$$

where $u(t)$ is an arbitrary $\{0,1\}$-valued sequence.

The following theorem provides a stronger result than in Lemma 27 [4].

*Theorem 6:* Suppose $n$ is odd, $n > 1$, and $u(t)$ is an arbitrary $\{0,1\}$-valued sequence. Then, for initial value $\bar{x}_{i,i+1}(0) = \bar{x}_{i+1,i}(0) = 0$, for every edge $(i, i+1)$, we have that for every $t \ge 0$,

$$\bar{x}_{i,j}(t) \le 1 \tag{12}$$

and

$$\max(\bar{x}_{1,0}(t), \bar{x}_{n-1,n}(t)) \le 1 - \frac{1}{n}. \tag{13}$$

The proof of the theorem is based on analysis of a tridiagonal linear system and is provided in Appendix F.

The improvement of the theorem is in the last asserted inequality where we provide tighter bound $1 - 1/n$ in comparison with $1 - c/n^3$ for a constant $c > 0$ asserted in Lemma 27 [4]. As a consequence, the convergence time upper bound in [4] can be improved to

$$T \le C[W/\sigma + \log(\sigma/\epsilon)] \cdot n^{4+\delta}$$

where $W$ is an upper bound on the maximum edge weight, $\sigma$ is the gap parameter, and $\epsilon$, $C$ and $\delta$ are positive constants.

## V. Related Work

The concept of balanced outcomes was introduced by Nash in [6] for the case of two players with exogenous profit options. This concept follows from a set of axioms and different axioms were subsequently considered; e.g. see [11].

Kleinberg and Tardos [1] considered the concept of Nash bargaining solutions on graphs where profit options available to a player are not exogenously given but determined by her position in the graph. They established relations between stable and balanced outcomes and devised a polynomial time algorithm for computing balanced outcomes. Their work left open the question on existence and properties of local dynamics.

A local dynamics for Nash bargaining on graphs was recently considered by Azar et al [3]. This paper assumed a fixed matching of nodes and considered a local, so called edge-balanced dynamics, for outcome vector $\vec{x}$. They established that fixed points of this dynamics are balanced outcomes. The assumption that matching is fixed was removed by Kanoria et al in their natural dynamics studied in [4] and [5]. In [4], they established that provided that there exists a unique Nash bargaining solution and the graph satisfies the positive gap $\sigma > 0$ condition (Section II-C), the natural dynamics converges to this Nash bargaining solution in a polynomial time. Specifically, they showed that there exists a constant $C > 0$ such that the convergence time is upper bounded by $C[W/\sigma + \log(\sigma/\epsilon)]n^{6+\delta}$, where $W$ is an upper bound on the maximum edge weight, $\sigma > 0$ is the gap and $\epsilon, \delta > 0$. In [5], using a different approach, they established that if maximum matching is unique, then there exists $T = O(n^4/g^2)$ such that for every initial value the natural dynamics induces the maximum-weight matching, for every $t \ge T$; here $n$ is the number of the nodes and $g$ is the difference between the total weight of the maximum-weight matching and that of the second best matching, which we refer to as the *matching weight gap*.

Finally, another related work is that on maximum-weighted matchings on graphs because of a close connection between stable outcomes and maximum weight matchings and similarity of distributed algorithms considered for solving the two problems. Bayati et al [12] considered an auction-like algorithm, which is similar in spirit to the natural dynamics for solving the balanced allocation problem, and showed that for complete bipartite graphs with a unique maximum-weight matching, the convergence time is $O(Wn/g)$ where $W$ is the maximum edge weight, $g$ is the matching weight gap and $n$ is the number of nodes.

## VI. Conclusion

In this paper we showed that some known Nash bargaining dynamics on graphs can (eventually) be characterized by linear dynamical systems and this enabled us to derive tight characterizations of their convergence rates. An interesting direction for future work is to investigate tightness of the convergence time bounds derived under different assumptions such as the positive gap condition of the KT procedure or the matching weight gap.

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## Appendix

### A. Proof of Theorem 1

An eigenvalue $\lambda$ and associated eigenvector $\vec{v}$ of matrix $\mathbf{A}$ satisfy

$$\begin{aligned}\lambda v_1 &= \frac{1}{2}v_2\\ \lambda v_i &= \frac{1}{2}v_{i-1} + \frac{1}{2}v_{i+1},\ 1 < i < n\\ \lambda v_n &= \frac{1}{2}v_{n-1}.\end{aligned}$$

Using $\lambda = \cos(\phi)$ and $v_i = \sin(\phi i)$, for $\phi \geq 0$ in the above equations, along with some elementary trigonometric calculus, it readily follows that $\phi = \frac{\pi k}{n+1}$, for $k = 1, 2, \ldots, n$.

Since $-1 < \lambda_k < 1$ for every $k$ and $\lambda_1 > 0$ has the largest modulo, the convergence time is given by $T = \log(1 - \alpha + \alpha\lambda_1)$. Noting that $\lambda_1 = 1 - \frac{\pi^2}{2n^2} + O(1/n^4)$, the asserted result follows.

### B. Proof of Lemma 1

The part of the system $\vec{y}(t)$ evolves as the following non-autonomous linear system

$$\vec{y}(t+1) = \mathbf{A}\vec{y}(t) + \vec{b}(t)$$

where $\mathbf{A}$ is a tridiagonal matrix that corresponds to a path of $m$ matched edges and $\vec{b}(t) = (x_n(t)/2, \underbrace{0, \ldots, 0}_{m-2}, (1 - x_n(t))/2)^T$.

Since $\mathbf{A}$ is a symmetric matrix, we can use the spectral decomposition

$$\mathbf{A} = \sum_{k=1}^{m} \lambda_k \vec{v}_k \vec{v}_k^T$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are the orthonormal eigenvectors of matrix $\mathbf{A}$, which we identified in Section III-A.

Using the spectral decomposition, we note

$$y_i(t) = \sum_{k=1}^{m} \lambda_k^t v_{k,i} \vec{v}_k^T \vec{y}(0) + \sum_{s=0}^{t-1} \lambda_k^{t-s-1} v_{k,i} \vec{v}_k^T \vec{b}(s)$$

where $v_{k,i} = \sin\left(\frac{\pi k}{m+1} i\right)$ is the $i$-th coordinate of the eigenvector $\vec{v}_k$. Summing up $y_1(t)$ and $y_m(t)$, we obtain

$$\begin{aligned}& y_1(t) + y_m(t)\\ =& \sum_{k=1}^{m} (v_{k,1} + v_{k,m}) \left(\lambda_k^t \vec{v}_k^T \vec{y}(0) + \sum_{s=0}^{t-1} \lambda_k^{t-s-1} \vec{v}_k^T \vec{b}(s)\right)\\ =& \sum_{k \text{ odd}} 2v_{k,1} \left(\lambda_k^t \vec{v}_k^T \vec{y}(0) + \sum_{s=0}^{t-1} \lambda_k^{t-s-1} \vec{v}_k^T \vec{b}(s)\right)\end{aligned}$$

where the last inequality is because of the fact $v_{k,m} = v_{k,1}$ for $k$ odd and $v_{k,m} = -v_{k,1}$ for $k$ even. Furthermore,

$$\begin{aligned} \vec{v}_k^T \vec{b}(s) &= v_{k,1}\frac{x_n(s)}{2} + v_{k,m}\frac{1-x_n(s)}{2} \\ &= \frac{v_{k,1}}{2} \text{ for } k \text{ odd.} \end{aligned}$$

Therefore,

$$\begin{aligned} & y_1(t) + y_m(t) \\ =& \sum_{k \text{ odd}} \left( \lambda_k^t 2 v_{k,1} \vec{v}_k^T \vec{y}(0) + \sum_{s=0}^{t-1} \lambda_k^{t-s-1} v_{k,1}^2 \right) \\ =& \sum_{k \text{ odd}} \left( \lambda_k^t 2 v_{k,1} \vec{v}_k^T \vec{y}(0) + \frac{1-\lambda_k^t}{1-\lambda_k} v_{k,1}^2 \right) \\ =& \sum_{k \text{ odd}} \left( \lambda_k^t 2 \sqrt{\frac{2}{m+1}} \sqrt{1-\lambda_k^2} \vec{v}_k^T \vec{y}(0) + \right. \\ & \left. +(1-\lambda_k^t)\frac{2}{m+1}(1+\lambda_k) \right) \\ =& \frac{2}{m+1} \sum_{k \text{ odd}} (1+\lambda_k) - \frac{2}{m+1} \sum_{k \text{ odd}} \left( 1+\lambda_k \right. \\ & \left. -2\sqrt{\frac{m+1}{2}} \sqrt{1-\lambda_k^2} \vec{v}_k^T \vec{y}(0) \right) \lambda_k^t. \end{aligned}$$

It remains only to show that

$$\frac{2}{m+1} \sum_{k \text{ odd}} (1+\lambda_k) = 1$$

which follows readily by elementary trigonometric calculus.

## *C. Proof of Theorem 3*

The statements of the theorem derive from Lemma 1 as follows. Item 1 clearly holds as the function (8) depends only on the initial value $\vec{y}(0)$. Item 2 follows from (8) because all the eigenvalues $\lambda_k$ are real and with modulo strictly smaller than 1. Item 3 holds from the fact that the largest modulo eigenvalue of matrix $\mathbf{A}$ is $\lambda_1 = \cos\left(\frac{\pi}{m+1}\right) = 1 - \frac{\pi^2}{2m^2} + O(1/m^4)$ and hence $R = \log(1/\lambda_1) = \frac{\pi^2}{2m^2} + O(1/m^4)$. Item 4 holds as the sum in (8) is asymptotically dominated by the largest modulo eigenvalue $\lambda_{2i-1}$ such that $\vec{v}_{2i-1}^T \vec{y}(0) \neq 0$, i.e. the mode associated to the eigenvalue $\lambda_{2i-1}$ is excited. Let us consider the case where such an eigenvalue is $\lambda_1$ and $m$ is even; the other cases follow by similar arguments. From Lemma 1, we have $y_1(t) + y_m(t) =$

$$= 1 - \frac{2}{m+1}\lambda_1^t \left( f_1(\vec{y}(0)) + \sum_{i=2}^{\lceil \frac{m}{2} \rceil} f_{2i-1}(\vec{y}(0)) \left( \frac{\lambda_{2i-1}}{\lambda_1} \right)^t \right)$$

and, thus, since $|\lambda_{2i-1}/\lambda_1| < 1$, for every $1 < i \leq \lceil m/2 \rceil$,

$$y_1(t) + y_m(t) = 1 - \frac{2}{m+1}\lambda_1^t \left[ f_1(\vec{y}(0)) + o(1) \right].$$

Finally, item 5 holds as

$$\gamma := \max_i |\frac{\lambda_{2i-1}}{\lambda_1}| \leq \frac{\lambda_3}{\lambda_1} = 1 - \frac{4\pi^2}{m^2} + O(1/m^4).$$

For an arbitrary $\epsilon > 0$, we have $|\lambda_{2i-1}/\lambda_1|^t \leq \epsilon$, for every $i > 1$, provided that time $t$ is such that

$$t \geq \frac{\log\left(\frac{1}{\epsilon}\right)}{\log\left(\frac{1}{\gamma}\right)} = \frac{\log\left(\frac{1}{\epsilon}\right)}{4\pi^2} m^2[1+o(1)].$$

Hence, $T_0 = O(m^2)$.

## *D. Eigenvalues for a Blossom*

Remark that an eigenvalue $\lambda$ and eigenvector $\vec{v}$ of matrix $\mathbf{A}$ satisfy $\mathbf{A}\vec{v} = \lambda\vec{v}$, i.e.

$$\begin{aligned} \frac{1}{2}v_2 &= \lambda v_1 && (14) \\ \frac{1}{2}v_{i-1} + \frac{1}{2}v_{i+1} &= \lambda v_i, \ 1 < i < n+m && (15) \\ -\frac{1}{2}v_1 + \frac{1}{2}v_{n+m-1} &= \lambda v_{n+m} && (16) \end{aligned}$$

Suppose $\lambda = \cos(\phi)$ and $\vec{v} = (\sin(\phi), \sin(2\phi), \ldots, \sin((n+m)\phi))^T$. Then, by elementary trigonometric identities we note that (14) and (15) hold for every $\phi$. On the other hand, (16) is equivalent to

$$\sin((n+m-1)\phi) = \sin(n\phi) + 2\cos(\phi)\sin((n+m)\phi)$$

which by using elementary trigonometric identities is equivalent to

$$\sin\left(\frac{2n+m+1}{2}\phi\right)\cos\left(\frac{m+1}{2}\phi\right) = 0.$$

Therefore, $\phi$ is either

$$\phi_1 = \frac{2k_1}{2n+m+1}\pi \quad \text{or} \quad \phi_2 = \frac{2k_2-1}{m+1}\pi$$

where $k_1$ and $k_2$ are arbitrary integers. Since cosine is a periodic function, it can be readily checked that $\cos(\phi_1)$ attains all possible values over $k = 1, 2, \ldots, n + \lfloor m/2 \rfloor$ and similarly for $\cos(\phi_2)$ over $k = 1, 2, \ldots, \lceil m/2 \rceil$.

### E. Eigenvalues for a Bicycle

If $\lambda$ is an eigenvalue of matrix $\mathbf{A}$ with eigenvector $\vec{v}$, then we have

$$\begin{aligned}\lambda v_1 &= \frac{1}{2}v_2 - \frac{1}{2}v_{l+1}\\ \lambda v_i &= \frac{1}{2}v_{i-1} + \frac{1}{2}v_{i+1}, i = 2,\dots,n+m+l-1\\ \lambda v_{n+m+l} &= \frac{1}{2}v_{n+m+l-1} - \frac{1}{2}v_{l+n}\,.\end{aligned}$$

In the remainder, we separately consider two cases depending on whether either $l$ or $m$ is even, or otherwise.

**Case 1**: $l$ or $m$ is odd.

Without loss of generality, suppose $l$ is odd. Let us introduce the following one-to-one linear transformation $\vec{z} = \mathbf{S}\vec{v}$ where matrix $\mathbf{S}$ is defined by $z_i = v_i + v_{l-i+1}$, for $i = 1,\dots,\lfloor l/2 \rfloor$, and $z_i = 2v_i$ for $i = \lfloor l/2 \rfloor + 1,\dots,n+l+m$. It is not difficult to verify that $\mathbf{S}$ is non-singular and thus a matrix $\mathbf{B}$ such that $\mathbf{A} = \mathbf{S}^{-1}\mathbf{B}\mathbf{S}$ is similar to $\mathbf{A}$ and, therefore, has the same eigenvalues [13][Theorem 1.3.3].

Using the transformation $\vec{z} = \mathbf{S}\vec{v}$ and $\mathbf{A}\vec{v} = \lambda\vec{v}$, we have

$$\begin{aligned}\lambda z_1 &= \frac{1}{2}z_2\\ \lambda z_{\lfloor l/2 \rfloor+1} &= z_{\lfloor l/2 \rfloor}\\ \lambda z_{n+m+l} &= \frac{1}{2}z_{n+m+l-1} - \frac{1}{2}z_{n+l}\end{aligned}$$

and for $i = 2,\dots,\lfloor l/2 \rfloor$ and $i = \lfloor l/2 \rfloor + 2,\dots,n+l+m-1$,

$$\lambda z_i = \frac{1}{2}z_{i-1} + \frac{1}{2}z_{i+1}\,.$$

Notice that $\lambda\vec{z} = \mathbf{S}\mathbf{A}\mathbf{S}^{-1}\vec{z} = \mathbf{B}\vec{z}$, and from the above identities

$$\mathbf{B} = \begin{pmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{Q} & \mathbf{R} \end{pmatrix}$$

where $\mathbf{P}$ is a $\lceil l/2 \rceil \times \lceil l/2 \rceil$ tridiagonal matrix given by

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 0 & \cdots & 0 \\ 1/2 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1/2 & \ddots & 1/2 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$\mathbf{Q}$ is a $(n+m+\lfloor l/2 \rfloor) \times \lceil l/2 \rceil$ matrix with all elements equal to zero but the element in the first row and last column equal to $1/2$, and $\mathbf{R}$ is a $(n+m+\lfloor l/2 \rfloor) \times (n+m+\lfloor l/2 \rfloor)$ matrix that corresponds to a blossom with $n+\lceil l/2 \rceil$ matched stem edges and $m$ matched loop edges and is of the form (9) under Case 1 in Section III-C.

Using the properties of determinants of block matrices, we observe that eigenvalues of $\mathbf{B}$ consist of eigenvalues of matrices $\mathbf{P}$ and $\mathbf{R}$. Therefore, the eigenvalues of matrix $\mathbf{B}$, and by similarity of matrix $\mathbf{A}$, are

$$\cos\left(\frac{\pi(2k-1)}{l+1}\right),\ k = 1,\dots,\lceil l/2 \rceil, \tag{17}$$

$$\cos\left(\frac{\pi(2k-1)}{m+1}\right),\ k = 1,\dots,\lceil m/2 \rceil, \tag{18}$$

$$\cos\left(\frac{2\pi k}{2n+l+m}\right),\ k = 1,\dots,n+\lfloor l/2 \rfloor + \lfloor m/2 \rfloor \tag{19}$$

where (17) are eigenvalues of matrix $\mathbf{P}$, which is easily derived and thus omitted, and (18) and (19) are eigenvalues of $\mathbf{R}$ which we have already showed in Section III-C.

It is not difficult to see that the above eigenvalues hold whenever either $l$ or $m$ is odd.

**Case 2**: both $l$ and $m$ are even.

We use a similar but different one-to-one transformation as under Case 1: $z_i = v_i + v_{l-i+1}$, for $i = 1,\dots,l/2$, $z_i = v_i + v_{i+1}$ for $i = l/2+1,\dots,n+l+m/2$, and $z_{i+n+l} = v_{n+l+i} + v_{m+n+l-i+1}$ for $i = 0,\dots,m/2$. We have that

$$\begin{aligned}\lambda z_1 &= \frac{1}{2}z_2\\ \lambda z_{l/2} &= \frac{1}{2}z_{l/2-1} + \frac{1}{2}z_{l/2}\\ \lambda z_{n+l+m/2} &= \frac{1}{2}z_{n+l+m/2} + \frac{1}{2}z_{n+l+m/2+1}\\ \lambda z_{n+m+l} &= \frac{1}{2}z_{n+m+l-1}\end{aligned}$$

and for $i = l/2+1,\dots,n+l+m/2-1$ and $i = n+l+m/2+1,\dots,n+m+l-1$,

$$\lambda z_i = \frac{1}{2}z_{i-1} + \frac{1}{2}z_{i+1}\,.$$

Similarly as for Case 1, using the properties of determinants of block matrices, we have that the eigenvalues of $\mathbf{A}$ are

$$\begin{aligned}&\cos\left(\frac{\pi(2k-1)}{l+1}\right),\ k = 1,\dots,l/2,\\ &\cos\left(\frac{\pi(2k-1)}{m+1}\right),\ k = 1,\dots,m/2,\\ &\cos\left(\frac{\pi k}{n+l/2+m/2}\right),\ k = 1,\dots,n+l/2+m/2-1,\\ &\text{and } -1\end{aligned}$$

where $(1,-1,1,-1,\dots,1,-1)^T$ is the eigenvector of eigenvalue $-1$.

### F. Proof of Theorem 6

Let us start by noting that there are two possible alternating matchings $M$: (1) defined by letting the edge $(0,1)$ be matched and (2) otherwise. The number of unmatched edges in the two cases is $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, respectively. The analysis follows the same steps for the two cases, thus we consider only the first case.

The first assertion (12) clearly holds for every initial value such that $0 \leq \bar{x}_{i,j}(0) \leq 1$, for every edge $(i,j)$. This follows by induction because the states are updated to convex combinations of values in $[0,1]$.

In the remainder of this section, we show assertion (13) through the following steps: (1) we first identify a tridiagonal system that describes the system evolution; (2) we then show that for every $t \geq 0$, $\max(\bar{x}_{1,0}(t), \bar{x}_{n-1,n}(t))$ is maximum by taking either $u(t) = 1$ or $u(t) = 0$, for every $t \geq 0$; (3) in this step, we show that $\lim_{t\to\infty} \max(\bar{x}_{1,0}(t), \bar{x}_{n-1,n}(t)) = \mu$ where $0 < \mu \leq 1 - 1/n$; and (4) we show that from the initial value as assumed in the theorem, $\max(\bar{x}_{1,0}(t), \bar{x}_{n-1,n}(t))$ converges to $\mu$ from below. Finally, we put pieces together at the end of this section.

**Step 1: a tridiagonal-system representation**. It suffices to consider the case $\alpha = 1$ as taking $0 < \alpha \leq 1$ only affects the rate of convergence. Under this assumption, the path bounding dynamics boils down to the following:

- For an unmatched edge $(i, i+1)$:

$$\begin{aligned} \bar{x}_{i,i+1}(t+1) &= \frac{1}{2}\bar{x}_{i-1,i}(t) + \frac{1}{2}\bar{x}_{i,i-1}(t), \quad \text{for } i > 0 \\ \bar{x}_{i+1,i}(t+1) &= \frac{1}{2}\bar{x}_{i+2,i+1}(t) + \frac{1}{2}\bar{x}_{i+1,i+2}(t) \end{aligned}$$

- Otherwise, for a matched edge $(i, i+1)$:

$$\begin{aligned} \bar{x}_{i,i+1}(t+1) &= \bar{x}_{i-1,i}(t) \\ \bar{x}_{i+1,i}(t+1) &= \bar{x}_{i+2,i+1}(t), \quad \text{for } i < n-1 \end{aligned}$$

where $\bar{x}_{0,1}(t)$ and $\bar{x}_{n,n-1}(t)$ are arbitrary input sequences taking values in $\{0,1\}$ such that $\bar{x}_{0,1}(t) + \bar{x}_{n,n-1}(t) = 1$, for every $t \geq 0$.

From the above dynamics, we observe that for unmatched edges $(i, i+1)$, i.e. for $i = 1, 3, \ldots, 2m-1$ where $m = \lfloor n/2 \rfloor$ is the total number of unmatched edges, we have for $t > 0$,

$$\begin{aligned} \bar{x}_{i,i+1}(t+1) &= \frac{1}{2}\bar{x}_{i-2,i-1}(t-1) + \frac{1}{2}\bar{x}_{i+1,i}(t-1) \\ \bar{x}_{i+1,i}(t+1) &= \frac{1}{2}\bar{x}_{i+3,i+2}(t-1) + \frac{1}{2}\bar{x}_{i,i+1}(t-1) \end{aligned} \tag{20}$$

where $\bar{x}_{-1,0}(t-1) = \bar{x}_{0,1}(t)$ and $\bar{x}_{2m+2,2m+1}(t-1) = \bar{x}_{n,n-1}(t)$.

We consider the dynamics of the values $\bar{x}_{i,i+1}(t)$ and $\bar{x}_{i+1,i}(t)$ for unmatched edges at even times as the dynamics of this process boils down to a simpler linear dynamical system. Notice that knowing these state values at even times, the corresponding state values at odd times are determined by the original state updates. To this end, let us define, for $t \geq 0$ and $i = 1, 2, \ldots, m$,

$$x_i(t) := \bar{x}_{2i-1,2i}(2t) \text{ and } y_i(t) := \bar{x}_{2i,2i-1}(2t).$$

From (20), we have for $t \geq 0$ and $i = 1, 2, \ldots, m$,

$$\begin{aligned} x_i(t+1) &= \frac{1}{2}x_{i-1}(t) + \frac{1}{2}y_i(t) \\ y_i(t+1) &= \frac{1}{2}x_i(t) + \frac{1}{2}y_{i+1}(t) \end{aligned} \tag{21}$$

where $x_0(t)$ and $y_{m+1}(t)$ are arbitrary input sequences taking values in $\{0,1\}$ such that $x_0(t) + y_{m+1}(t) = 1$, for every $t \geq 0$.

Note that of our particular interest are $y_1(t)$ and $x_m(t)$ as

$$\begin{aligned} \bar{x}_{1,0}(t+1) &= \bar{x}_{2,1}(t) = y_1(t) \\ \bar{x}_{n-1,n}(t+1) &= \bar{x}_{n-2,n-1}(t) = x_m(t). \end{aligned} \tag{22}$$

From (21), it is not difficult to observe that for every $t > 0$, $y_i(t) = x_{i+1}(t)$ for $i = 1, 2, \ldots, m-1$. Therefore, we can fully describe the dynamics by the following discrete-time linear system, for $t > 0$,

$$\begin{aligned} x_1(t+1) &= \frac{1}{2}x_0(t) + \frac{1}{2}x_2(t) \\ x_2(t+1) &= \frac{1}{2}x_1(t) + \frac{1}{2}x_3(t) \\ &\vdots \\ x_{m-1}(t+1) &= \frac{1}{2}x_{m-2}(t) + \frac{1}{2}x_m(t) \\ x_m(t+1) &= \frac{1}{2}x_{m-1}(t) + \frac{1}{2}y_m(t) \\ y_m(t+1) &= \frac{1}{2}x_m(t) + \frac{1}{2}y_{m+1}(t) \end{aligned}$$

with the initial values $x_1(1) = x_0(0)/2$, $x_i(1) = 0$, for $1 < i \leq m$, and $y_m(1) = y_{m+1}(0)/2$.

In other words, $\vec{z}(t) = (x_1(t), x_2(t), \ldots, x_m(t), y_m(t))^T$ is defined by the initial point $\vec{z}(1) = (x_0(0)/2, \underbrace{0, \ldots, 0}_{m-1}, y_{m+1}(0)/2)^T$ and

$$\vec{z}(t+1) = \mathbf{A}\vec{z}(t) + \vec{b}(t), \; t > 0, \tag{23}$$

where $\mathbf{A}$ is the $(m+1) \times (m+1)$ tridiagonal matrix with elements given by (4), and $\vec{b}(t) = (\frac{1}{2}x_0(t), \underbrace{0, \ldots, 0}_{m-1}, y_{m+1}(t)/2)^T$.

In view of (22), notice that of our interest are $z_2(t)$ and $z_m(t)$ because $y_1(t) = z_2(t)$ and $x_m(t) = z_m(t)$.

**Step 2: extremal input sequence**. We will show that the input sequence $x_0(s) = 1$, for $s \geq 0$ is *extremal* in the following sense: for every given $t > 0$, it maximizes $z_{2\wedge m}(t)$ and minimizes $z_{m\vee 2}(t)$. Indeed, by symmetry, $x_0(s) = 0$, for $s \geq 0$ is extremal in the sense of minimizing $z_{2\wedge m}(t)$ and maximizing $z_{m\vee 2}(t)$, for any given $t > 0$.

Let us define $s_i(t) = z_i(t) + z_{m-i+2}(t)$ and $d_i(t) = z_i(t) - z_{m-i+2}(t)$, for $i = 1, 2, \ldots, m+1$ and $t < 0$. From these definitions, it is readily observed that

$$s_i(t) = s_{m-i+2}(t) \tag{24}$$

and

$$d_i(t) = -d_{m-i+2}(t) \tag{25}$$

and, thus, it suffices to consider only the following vectors $\vec{s}(t) = (s_1(t), s_2(t), \ldots, s_{\lceil m/2\rceil}(t))^T$ and $\vec{d}(t) = (d_1(t), d_2(t), \ldots, d_{\lceil m/2\rceil}(t))^T$.

Using (23), it is not difficult to derive that, for $i = 1, 2, \ldots, \lceil m/2\rceil$, and $t > 0$,

$$s_i(t+1) = \frac{1}{2}s_{i-1}(t) + \frac{1}{2}s_{i+1}(t) \tag{26}$$

where $s_0(t) = 1$, and

$$d_i(t+1) = \frac{1}{2}d_{i-1}(t) + \frac{1}{2}d_{i+1}(t) \tag{27}$$

where $d_0(t) = 2x_0(t) - 1$.

Remark that for $i = \lceil m/2\rceil$, $s_{i+1}(t) = s_i(t)$ while $d_{i+1}(t) = 0$ if $m$ is even and $d_{i+1}(t) = -d_i(t)$, otherwise.

It is readily observed that for $i = 1, 2, \ldots, m+1$,

$$z_i(t) = \frac{1}{2}s_i(t) + \frac{1}{2}d_i(t).$$

For $m = 1$, we can write

$$z_1(t) = \frac{1}{2}s_1(t) + \frac{1}{2}d_1(t)$$
$$z_2(t) = \frac{1}{2}s_1(t) - \frac{1}{2}d_1(t)$$

where the last equality is by using (24) and (25), while, otherwise, for $m > 1$, we can write

$$\begin{aligned} z_2(t) &= \frac{1}{2}s_2(t) + \frac{1}{2}d_2(t) \\ z_m(t) &= \frac{1}{2}s_2(t) - \frac{1}{2}d_2(t) \end{aligned} \tag{28}$$

where the last equality is by using (24) and (25).

From (26) and (27), note that that $\vec{s}(t)$ evolves according to an autonomous linear system while $\vec{d}(t)$ evolves according to a non-autonomous linear system with input sequence $x_0(t) - \frac{1}{2}$. From (27), we observe that $d_i(t)$, $1 \leq i \leq \lceil m/2\rceil$, are maximized for the input sequence $x_0(t) = 1$, for $t \geq 0$. In view of the above identities, we have that the latter sequence is extremal.

**Step 3: limit point**. Since all the eigenvalues of the system (23) are real and with modulo strictly smaller than 1, the system is globally asymptotically stable, i.e. it converges to a unique limit point from any given initial value. The rate of convergence is determined by the largest modulo eigenvalue, and the dominant asymptotic term of the rate of convergence is $\frac{\pi^2}{2m^2}$, for large $m$.

We identify the limit point of the system (23) for the input sequence $(x_0(t), y_{m+1}(t)) = (a, b)$, for $t \geq 0$, where $a$ and $b$ are positive constants. This accommodates the aforementioned extremal input sequence by choosing $a = 1$ and $b = 0$. A fixed point $\vec{z}$ is a solution of the following system of linear equations:

$$\vec{z} = \mathbf{A}\vec{z} + \vec{b}$$

where $\vec{b} = (a/2, \underbrace{0, \ldots, 0}_{m-1}, b/2)$.

It can be readily checked that there is a unique solution given by

$$z_i = a + i\frac{b-a}{m+2}, \quad \text{for } i = 1, 2, \ldots, m+1.$$

Therefore,

$$\begin{aligned} z_2 &= \left(1 - \frac{2}{m+2}\right)a + \frac{2}{m+2}b \\ z_m &= \frac{2}{m+2}a + \left(1 - \frac{2}{m+2}\right)b. \end{aligned}$$

In particular, for $a = 1$ and $b = 0$, we have

$$\max(z_2, z_m) = z_{2\wedge m} = \begin{cases} 1 - \frac{1}{m+1}, & \text{for } m = 1 \\ 1 - \frac{2}{m+2}, & \text{for } m > 1. \end{cases}$$

**Step 4: convergence to the limit point**. We consider $\max(z_2(t), z_m(t))$, for $t \geq 0$, for the input sequence $x_0(t) = 1$, for $t \geq 0$. We consider only the case $m > 1$ as the case $m = 1$ can be considered by similar steps.

From (28), we note that $\max(z_2(t), z_m(t)) = z_2(t)$, for $t \geq 0$, and thus it suffices to consider $z_2(t)$, for $t \geq 0$.

Notice that $\vec{z}(1) = \vec{b}(0)$ and thus from (23), for $t > 0$,

$$\vec{z}(t) = \sum_{s=0}^{t-1} \mathbf{A}^{t-1-s}\vec{b}(s).$$

Let us use the spectral decomposition $\mathbf{A} = \sum_{k=1}^{m+1} \lambda_k \vec{v}_k \vec{v}_k^T$ where $\lambda_k$ is an eigenvalue and $\vec{v}_k$ the corresponding eigenvector of matrix $\mathbf{A}$, which we identified in Section III-A. Noting that

$$\vec{z}(t) = \sum_{k=1}^{m+1}\sum_{s=0}^{t-1} \lambda_k^{t-1-s}\vec{v}_k\vec{v}_k^T\vec{b}(s)$$

and

$$\begin{aligned}
&\vec{v}_k^T \vec{b}(s)\sqrt{\frac{m+2}{2}} \\
= &\frac{x_0(s)}{2}\sin\left(\frac{\pi k}{m+2}\right) + \frac{1-x_0(s)}{2}\sin\left(\frac{\pi k(m+1)}{m+2}\right) \\
= &\sin\left(\frac{\pi k}{m+2}\right)\left[\frac{1}{2} - (1-x_0(s))1_{k \text{ even}}\right]
\end{aligned}$$

we obtain

$$\begin{aligned}
\vec{z}(t) &= \sqrt{\frac{2}{m+2}}\sum_{k=1}^{m+1}\vec{v}_k \sin\left(\frac{\pi k}{m+2}\right)\cdot \\
&\cdot\left[\frac{1-\lambda_k^t}{2(1-\lambda_k)} - 1_{k \text{ even}}\sum_{s=0}^{t-1}\lambda_k^{t-1-s}(1-x_0(s))\right].
\end{aligned}$$

From this, it is not difficult to derive

$$\begin{aligned}
z_2(t) &= \frac{2}{m+2}\sum_{k=1}^{m+1}\lambda_k^2 - (1+\lambda_k)\lambda_k^{t+1} \\
&-\frac{4}{m+2}\sum_{k=1}^{m+1}(1-\lambda_k^2)\lambda_k 1_{k \text{ even}}\cdot \\
&\cdot\sum_{s=0}^{t-1}\lambda_k^{t-1-s}(1-x_0(s)).
\end{aligned}$$

For the extremal input sequence $x_0(t) = 0$ for $t \geq 0$, we have

$$z_2(t) = \frac{4}{m+2}\sum_{k=1}^{\lceil m/2\rceil}\lambda_k^2 - \frac{2}{m+2}\sum_{k=1}^{m+1}(1+\lambda_k)\lambda_k^{t+1}.$$

Using the identity

$$\sum_{k=1}^{m+1}(1+\lambda_k)\lambda_k^{t+1} = \begin{cases} 2\sum_{k=1}^{\lceil m/2\rceil}\lambda_k^{t+2} & t \text{ even} \\ 2\sum_{k=1}^{\lceil m/2\rceil}\lambda_k^{t+1} & t \text{ odd}\end{cases}$$

we have

$$z_2(t) = \frac{4}{m+2}\sum_{k=1}^{\lceil m/2\rceil}\lambda_k^2 - \lambda_k(1+\lambda_k 1_{k \text{ even}})\lambda_k^t.$$

Since $0 < \lambda_k < 1$ for $1 \leq k \leq \lceil m/2\rceil$ we observe that $z_2(t)$ is increasing with $t$, i.e. it approaches its limit point from below.

**Proof of the theorem**. We showed that for every $t \geq 0$,

$$\max(\bar{x}_{1,0}(t), \bar{x}_{n-1,n}(t)) \leq \begin{cases} 1 - \frac{1}{\lfloor\frac{n}{2}\rfloor+2} & 1 < n \leq 2 \\ 1 - \frac{2}{\lfloor\frac{n}{2}\rfloor+2} & n > 2\end{cases}$$

and showed that there exists an extremal input sequence $(\bar{x}_{0,1}(t), \bar{x}_{n,n-1}(t))$, $t \geq 0$, for which the equality is achieved asymptotically as $t$ goes to infinity.

The asserted bound $1 - 1/n$ in (13) readily follows from the above displayed inequality, which completes the proof.