Crowdsourcing and All-Pay Auctions [1]

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[1]The work by Dominic DiPalantino was performed in part while an intern with Microsoft Research.

Abstract – In this paper we present and analyze a model in which users select among, and subsequently compete in, a collection of contests offering various rewards. The objective is to capture the essential features of a *crowdsourcing* system, an environment in which diverse tasks are presented to a large community. We aim to demonstrate the precise relationship between incentives and participation in such systems.

We model contests as *all-pay auctions* with incomplete information; as a consequence of revenue equivalence, our model may also be interpreted more broadly as one in which users select among auctions of heterogeneous goods. We present two regimes in which we find an explicit correspondence in equilibrium between the offered rewards and the users' participation levels. The regimes respectively model situations in which different contests require similar or unrelated skills. Principally, we find that rewards yield logarithmically diminishing returns with respect to participation levels. We compare these results to empirical data from the crowdsourcing site Taskcn.com; we find that as we condition the data on more experienced users, the model more closely conforms to the empirical data.

1. INTRODUCTION

Methods of soliciting solutions to tasks via open calls to large-scale communities have proliferated since the advent of the Internet; the term *crowdsourcing* was recently coined to refer to these approaches. Examples of tasks found on crowdsourcing sites are the graphical design of logos, the creation of a marketing plan, the identification and labeling of an image, and the answering of an individual's question. Many crowdsourcing sites exhibit a similar structure – a task is described, a reward and time period are stated, and during the period users compete to provide the best submission. At the conclusion of the period, a subset of submissions are selected, and the corresponding users are granted the reward.

The rewards offered may be monetary or non-monetary; non-monetary rewards can take the form of reputation points in community Q&A sites, and confer a measure of social status within these communities. In either case, the rewards directly influence user participation, and so it is important that they are designed to induce the appropriate levels of participation and quality of submissions. It appears, however, that most crowdsourcing sites rely on ad-hoc design choices or trial and error when determining these incentives, and current research seems limited to a few empirical studies [14, 4].

The goal of this paper is to provide a better understanding of the relationship between rewards and participation in the context of crowdsourcing sites. Toward this end, we model crowdsourcing as a two-stage game in which strategic users (i) select among contests offering different rewards and (ii) upon joining a contest, those who selected it compete amongst themselves for the reward. In this model, the users are endowed with a private *skill* at producing a submission for each type of contest; skills for different users are drawn independently at random according to a commonly known distribution. In the second stage of the game, we model the contest as an *all-pay auction*, a well-studied mechanism that is frequently employed in the contest literature [2]. We summarize our main contributions in the following.

In Section 3, we develop and rigorously analyze a model in which users strategically choose a contest and then compete in their chosen contest. We show the existence of an equilibrium under general assumptions.

In Section 4, we explore a special case in which each user has the same skill for all contests; this corresponds to a situation in which users' skills are principally determined by their opportunity cost, or all tasks are similar in nature. We fully characterize the equilibrium assignment of users to contests. We find that there is a threshold reward below which users will choose not to participate in contests, and that users are partitioned according to their skill; users with higher skills choose to restrict their attention to contests with higher rewards, while users with lower skills participate broadly. Additionally, we find a closed-form correspondence between rewards and the average number of users, and show that it exhibits decreasing returns when rewards are used to incent users to participate.

In Section 5, another special case is explored, in which a user's skill for each contest is drawn independently; this corresponds to a situation in which contests are dissimilar and skills are uncorrelated. We analyze the asymptotic properties of equilibria in this setting, and establish that the same closed-form correspondence between rewards and participation holds approximately for large systems.

Section 6 concerns relaxations of some of our assumptions. In particular, we examine the consequences of allowing asymmetry in distributions of different skills, and imposing minimum levels of quality on submissions.

In Section 7, we pose a design problem in which a system operator must select the rewards in order to induce participation and maximize a well-defined welfare objective. In those situations in which the the offered rewards are centrally controlled, or in which those offering the rewards cooperate, a social welfare optimization problem may be posed and solved within our framework.

Finally, in Section 8, we complement our analysis with an interpretation of empirical data from the "witkey" site Taskcn, which presents tasks to users in a manner similar to that in our model. We discuss other relevant design factors, which we suggest as extensions for future models.

Proofs of our results are presented in the appendix.

2. RELATED WORK

Some recent studies of empirical data have analyzed the effects of incentives in crowdsourcing sites. Yang, Adamić, and Ackerman in [14] explore users of the site Taskcn. This Chinese site is an intermediary where users submit solutions for various tasks (such as designing a logo for a business); the solution chosen as the winner earns a monetary reward (fixed in advance) for its creator. The authors find that while users who are new to the site fare poorly by unrealistically choosing tasks with high rewards, those that repeatedly use the site pursue a more profitable strategy by better balancing the magnitude of the rewards with the likelihood of success. The results suggest that users who remain on the site learn to behave more strategically. Another investigation of users' responses to incentives is provided by Chen, Ho, and Kim in [4]. They study Google Answers, a now-defunct site in which users answered questions in exchange for monetary rewards. They found that questions offering more money received longer answers. This and [14] motivate the search for a better understanding of how strategic users should behave in these environments, and how a designer might select the appropriate incentives.

Our model owes a debt to auction theory – this is due to our choice to model contests as *all-pay auctions*. The all-pay auction mechanism is well-studied in the contest literature; the authors in [2] characterize equilibria under the assumption that skills are public knowledge, and cite numerous examples in which all-pay auctions have been used to model contests such as political races, R&D, and job promotion. We assume, however, that skills are private knowledge, in line with auction theory in the vein of [8]; the sites we study involve a higher degree of uncertainty than one might find in a simple contest with a small number of players. Our assumptions satisfy the requirements of the Revenue Equivalence Theorem [6]; consequently the all-pay nature of our auction may be elided for much of our analysis (with some noted exceptions).

Our work differs from single-item auction theory in that several rewards are simultaneously offered on crowdsourcing sites. In [10], Moldovanu and Sela explore the design of contests in which multiple rewards are available to users. Different from their setting, in our model users must *select* which reward they are pursuing; the crowdsourcing model we consider is more akin to a collection of separate contests than to a single contest with multiple rewards.

There are several examples in the literature of models in which users must choose among auctions. In [7], McAfee explores a dynamic environment in which many sellers of identical items choose mechanisms to present to buyers. He finds an equilibrium in which sellers conduct an auction with an efficient reserve price, and buyers randomize over sellers. In [13], Wolinsky likewise examines a dynamic model in which identical sellers conduct first-price auctions with reserve for a large number of bidders that are randomly selecting sellers; bidders learn their valuation after meeting the seller. The line of inquiry of these two models is continued by [12], where the number of bidders is endogenized, by [11], which examines a common value environment, and by [5]. Our work differs from these settings in that we assume that 'sellers' have heterogeneous items. Additionally, the mechanism design question is different, as revenue maximization is not necessarily the principal objective in contest design – for instance, we may be more concerned with the number of submissions or the quality of the best submission than with aggregate quality.

Ashlagi, Monderer, and Tennenholtz in [1] explore the revenue properties of competing advertisement auctions; among other situations, they examine in detail a setting in which two auctions with different click-through rates (and hence proportionally different user valuations) are conducted, and users choose between the two auctions. This may be viewed as a special case of the setting examined in Section 4.

3. THE MODEL

At a broad level, many crowdsourcing websites may be viewed as systems in which tasks are associated with rewards and presented to users; each user selects a task, and effortfully creates a submission. For each task, the submissions are judged, and a subset of users (often a singleton) are granted the associated reward. For instance, on TopCoder.com, users may select between several contests asking for submission of a Quality Assurance plan for a piece of software, each offering different rewards. On Taskcn, a business may solicit logo designs, offering money to the creator of the best logo; users have many such projects to choose from.

It is natural to view the competitive nature of the submission phase as a contest; indeed, sites such as TopCoder.com label them as such. In light of evidence that users of these sites become more strategic in their decisions over time [14], game-theoretic tools may be of use.

We consider a one-shot game in which players select a contest, exert effort (at a cost that depends on their skill), and in each contest the player with the best effort wins a prize. Specifically, consider a game in which N players choose among J contests. Let R_j denote the reward offered in contest $j \in \{1, \cdots, J\}$. Associated with each player i is a vector of *skills* $\vec{v_i} = (v_{i1}, \cdots, v_{iJ})$, where v_{ij} represents player i's skill at contest j. We suppose that the skill vector for each player is drawn from a continuous joint probability distribution over $[0, m]^J$, that skill vectors for different players are drawn independently from each other, and that the distribution is known to all players but that skill vector $\vec{v_i}$ is known only to player i. In subsequent sections we will make stronger assumptions about this distribution. The parameter m represents a maximum possible skill, corresponding to an upper limit on the amount of effort a player can obtain from unit cost.

The game consists of two stages. In the first stage, each player i selects a contest j and a bid b_{ij}. In the second stage, in each contest j, the prize is awarded to the player with the highest bid among those who selected the contest. Since bids represent effort (which cannot be unspent), all bids are collected. The payoff to player i is $v_{ij}R_j - b_{ij}$ if he submitted the highest bid, and $-b_{ij}$ otherwise. In the event of a tie, a winner is selected uniformly at random among the high bidders.

These payoffs reflect our decision to model the contests as *all-pay auctions* – these are auctions in which the high bidder receives the object, but all bidders pay their bid to the auctioneer. All-pay auctions capture the essential properties of contests and have been used to model rent-seeking, R&D races, and political contests among other applications [2]. To see the connection between contests and all-pay auctions, suppose we were to model the skill of player i at contest j by a unit cost of effort c_{ij}. If he exerts effort b_{ij} and wins, his payoff is $R_j - c_{ij}b_{ij}$; if he loses, he still pays the cost $c_{ij}b_{ij}$. Scaling his payoffs by dividing by c_{ij}, we recover the game above when $v_{ij} = \frac{1}{c_{ij}}$. Thus, in our setting a player's skill v_{ij} may be interpreted as the amount of effort he is able to exert per unit cost.

While a given player does not know the skills of the other players, he is aware of the underlying distribution. Additionally, all other information is public – all players are aware of the number of players N, the number of contests J, and the reward offered in each contest. In other words, ours is a model of *incomplete information* – the assumption that players have only distributional knowledge of others' skills directly influences the resulting equilibria, and is in contrast to cases of *complete information* contests that have also been studied [2]. The presumption of incomplete information is motivated by the fact that players in large systems typically have only limited statistics about the performance of other players, and there may be large uncertainty about which players will choose to participate in a given task.

A mixed strategy for player i with skills $\vec{v_i}$ consists of a probability distribution $\vec{\pi_i} = (\pi_{i1}, \cdots, \pi_{iJ})$ over the contests together with a bid b_{ij} for each contest j. (The players could choose a random bid as well – this does not arise in the equilibria we study.) Player i's payoff is the expected payoff in the all-pay auction, with the expectation taken over his own mixed strategy and i's beliefs about other players' types and strategies. His mixed strategy is a *best response* if it yields him at least as high a payoff as any other strategy. We are interested in a symmetric *Bayes-Nash equilibrium*, which specifies a mixed strategy for each possible skill vector $\vec{v}$ such that each mixed strategy is a best response, assuming other players also follow these strategies [9]. In such a setting, $\vec{\pi_i}$ is independent of the player i and we write $\pi_j(\vec{v})$ to denote the probability that a player with skills $\vec{v}$ joins contest j.

PROPOSITION 3.1. *There exists a symmetric equilibrium to this game.*

The proof of the proposition makes several uses of the Revenue Equivalence Theorem; one may note that neither the precise form of the auction nor the disclosure of the number of players has any impact on players' surplus or contest selection. This is because participants are risk neutral and in each contest face symmetric, independent bidders with private values.

We will focus on the symmetric equilibrium. In this setting, let p_j be the probability that a player selects contest j, $\hat{F}_j(v)$ be the cumulative distribution over skill for a player in contest j given that he selects this contest, and $\hat{F}^c_j(v) = 1 - \hat{F}_j(v)$. Then a player with skill v would win contest j with probability $(1 - p_j\hat{F}^c_j(v))^{N-1}$, as this is the probability that none of the other $N-1$ players joins contest j with a higher skill.

Let $g_j(v)$ denote the expected profit of a player with skill v for contest j were he to select that contest. From the Revenue Equivalence Theorem [6], a player's surplus is the integral of his probability of winning, and so we have

$$g_j(v) = R_j \int_0^v (1 - p_j\hat{F}^c_j(x))^{N-1} dx. \tag{1}$$

In equilibrium, a player i will only participate in those contests j that maximize $g_j(v_{ij})$.

The symmetric equilibrium has the property that the number of players in contest j is a binomial distribution with success probability p_j. Additionally, we would like to work with a symmetric case such that $p_j = p_k$ whenever $R_j = R_k$. This generally requires a corresponding symmetry in the joint probability distribution. Such a symmetry yields that the class of all contests having a particular reward has a common p_j and $\hat{F}_j(v)$. This is formalized in the following proposition.

PROPOSITION 3.2. *If the distribution of skills is symmetric with respect to contests that have the same rewards, then there is an equilibrium that is also symmetric with respect to contests that have the same rewards.*

We focus on the equilibrium in Proposition 3.2. We do so despite the existence of equilibria that violate the symmetries we are imposing. Consider, for instance, a setting with two players and two identical contests – suppose moreover that each player's skills for the two contests are perfectly correlated (though the skills between players are uncorrelated). It would clearly benefit the players if player 1 joined contest 1 and player 2 joined contest 2, regardless of their skills, for they could then receive the reward without expending any effort. We presume that coordination of this variety does not occur, as the analogous situation in a large system would either require the assistance of the system operator or an exceptional amount of communication between the players.

The following corollary refers to the equilibrium in Proposition 3.2 and may be observed from this symmetry and the property that surplus is increasing in skill.

COROLLARY 3.1. *Suppose there are J_k contests having reward equal to R_k, and a player has skills $v_1, v_2, \cdots, v_{J_k}$ for these contests. Then if the player participates in any of these, he participates in those having maximum skill; and if more than one skill obtains this maximum, he selects over such contests uniformly at random.*

The Large-System Limit. We are interested in properties of the player decisions as the system becomes large. Popular crowdsourcing sites are by definition well-populated, and often dramatically so; Yahoo! Answers has posed millions of questions to its community, and Taskcn has seen hundreds of thousands of submissions.

Accordingly, we define what it means for this system to scale. Consider a sequence of these games. We allow the number of players N and the number of contests J to increase, supposing asymptotically that $J \sim \frac{1}{\lambda}N$, so that the number of players per contest remains roughly constant at $\lambda > 0$. (For sequences a_n and b_n, we write $a_n \sim b_n$ when $\frac{a_n}{b_n} \to 1$ as $n \to \infty$.) A site for which this proportion cannot be bounded becomes too crowded or sparse; a similar scaling concept is utilized in [5, 11, 12]. Implicitly, as J grows we have a sequence of joint probability distributions over $[0, m]^J$ – in later sections where specific cases are examined, the evolution of this sequence will be made more precise.

Additionally, we assume that as the system grows there remain only finitely many different values of R_j – that is, that for some constant K, we have $R_1 > R_2 > \ldots > R_K$, with the reward for each contest taking on one of these values. Thus the contests are naturally partitioned into K *classes*. Let J_j be the number of contests of class j . Then we suppose that asymptotically, $J_j \sim \nu_j J$ for some $\nu_j \geq 0$, so that the proportion of contests that are in a given class is roughly constant. Since $\sum_{j=1}^K J_j = J$, we must also have $\sum_{j=1}^K \nu_j = 1$.

In the large limit, due to this symmetry, the binomial distribution over players will approach a common Poisson limit for each contest in a given class. Let λ_j be the mean number of players in a given contest in class j. Since λ is the average number of players per contest, we have $\sum_{j=1}^K \nu_j\lambda_j \sim \lambda$; hence, for a large enough number of contests, $\sum_{j=1}^K \nu_j\lambda_j \leq \lambda + \epsilon$, for $\epsilon > 0$. Note that the sequence of vectors of λ_j is thus confined to a compact space – hence the sequence has a limit vector which corresponds to a Poisson distribution of players per contest. In the cases examined in later sections, we show that this limit is unique, and that all equilibria converge to it.

4. PLAYER-SPECIFIC SKILLS

In this section we assume that each player is endowed with a skill that applies across all contests. More concretely, for each player i the skill vector $\vec{v}_i$ is equal to $(v, v, \ldots, v)$ where v is drawn from the distribution $F(v)$ independently of the skills of other players.

This assumption is justified for systems where the ability to successfully perform in individual contests predominantly depends on a player's skill in a way that is independent of the particular contests – for example, when the underlying tasks are closely related or require a similar kind of talent. Another example is when all players would require the same amount of time to put forth effort, but different players face different hourly opportunity costs.

On the other hand, this assumption may not apply to systems that are comprised of tasks that require diverse specialized skills. That case would be better accommodated by the underlying assumption in Section 5, as its primary feature is a lack of correlation between a player's skill at performing different tasks.

In Section 3 we established that a symmetric equilibrium exists under general assumptions. In the following, we show show uniqueness of equilibrium under the additional assumption of player-specific skills.

PROPOSITION 4.1. *Under the assumption of player-specific skills, there is a unique symmetric equilibrium.*

We fully characterize the symmetric equilibrium for any given number of players $N > 1$. This characterization is presented in the two main theorems of this section. The theorems are stated under the assumption that $F(v)$ is an atomless distribution over $[0, m]$, where m is the maximum skill and lies in the support of $F(v)$.

Recall that we have K classes of contests with rewards $R_1 > R_2 > \ldots > R_K$. In the sequel, we will use the notation $\vec{R} = (R_1, \ldots, R_K)$ and, for any subset $A \subseteq \{1, \ldots, K\}$, let

$$H_A(\vec{R}) = \left(\sum_{k\in A} \frac{J_k}{J_A} R_k^{-\frac{1}{N-1}}\right)^{-1} \quad (2)$$

$$J_A = \sum_{k\in A} J_k. \quad (3)$$

Additionally, for $A = \{1, \ldots, \ell\}$ we write $A = [1, \ell]$.

THEOREM 4.1. *Under the player-specific skills, the symmetric equilibrium satisfies the following two properties.*
1. Threshold reward. *A contest is selected by a player with strictly positive probability only if the reward offered by this contest is one of the $\tilde{K}$ highest rewards, where*

$$\tilde{K} = \max\left\{i : R_i^{\frac{1}{N-1}} > \left(1 - \frac{1}{J_{[1,i]}}\right) H_{[1,i]}(\vec{R})\right\}. \quad (4)$$

2. Participation rates. *A player selects a particular contest of class j with probability p_j given by*

$$p_j = \begin{cases} 1 - \left(1 - \frac{1}{J_{[1,\tilde{K}]}}\right) \frac{H_{[1,\tilde{K}]}(\vec{R})}{R_j^{\frac{1}{N-1}}}, & \text{if } j \le \tilde{K} \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

Item 1 tells us that there exists a *threshold reward* below which, with probability 1, a contest attracts no players. In other words, all players select from the set of contests that offer one of $\tilde{K}$ highest rewards where $\tilde{K}$ is explicitly determined by (4). An explicit characterization of the correspondence between the equilibrium mean number of players per contest Np_j and the offered rewards is readily seen from (5).

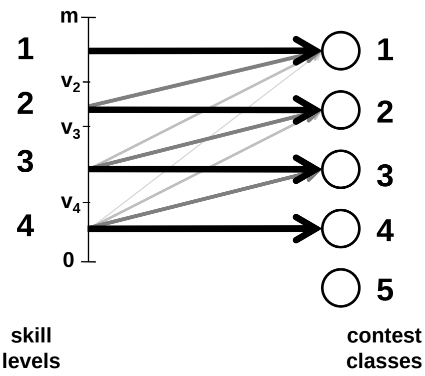


Figure 1: Selection of contests in equilibrium under player-specific skills. Players are partitioned over skill levels. A player of skill level ℓ selects from the set of top ℓ reward contests.

In the following theorem, we establish how in equilibrium, contests are selected by players of given skill.

THEOREM 4.2. *In the equlibrium, players select contests as given in the following.*
1. Skill levels. *Players are partitioned over $\tilde{K}$ skill levels such that a skill level ℓ corresponds to the interval of skill values $[v_{\ell+1}, v_\ell)$, where*

$$F(v_\ell) = 1 - J_{[1,\ell]}\left(1 - \frac{R_\ell^{\frac{1}{N-1}}}{H_{[1,\ell]}(\vec{R})}\right), \quad (6)$$

for $\ell = 1, \ldots, \tilde{K}$, and $v_\ell = 0$ for $\ell = \tilde{K}+1, \ldots, K$.
2. Contest selections vs. skill. *A player of skill v selects a particular contest of class j with probability $\pi_j(v)$ given by*

$$\pi_j(v) = \begin{cases} \frac{R_j^{-\frac{1}{N-1}}}{\sum_{k=1}^{\ell} J_k R_k^{-\frac{1}{N-1}}}, & \text{for } j = 1, \ldots, \ell \\ 0, & \text{for } j = \ell+1, \ldots, K, \end{cases} \quad (7)$$

for $v \in [v_{\ell+1}, v_\ell)$. Thus, a player of skill level ℓ selects a contest that offers one of ℓ highest rewards.

Item 1 says that in equilibrium, players are partitioned over a finite set of *skill levels*. Item 2 tells us that a player of skill level ℓ randomly selects a contest among those that offer one of ℓ highest rewards. Note that a smaller value of ℓ indicates a higher skill. The players of skill level ℓ select contests that offer the ℓ-th highest reward with the largest probability and those that offer larger reward are selected with smaller probability. Indeed, (7) establishes that a player of skill level ℓ selects a contest that offers the j-th highest reward where $j = 1, \ldots, \ell$, with probability inversely proportional to $R_j^{1/(N-1)}$.

In Fig. 1, we provide an illustration of the equilibrium selection of contests for $K = 5$ and $\tilde{K} = 4$. The thickness of the arrows indicate the likelihood that a player joins that particular contest. Here, contest 1 has the highest reward, and players with skills in level 1 are in the highest segment of $[0, m]$.

4.1 The Large-System Limit

In this subsection, we consider the properties of the equilibrium under the large-system scaling, which was introduced in Section 3.

THEOREM 4.3. *In the large-system limit, the number of players that participate in a given contest of class j is a Poisson random variable with mean λ_j given by*

$$\lambda_j = \begin{cases} \frac{\lambda}{\nu_{[1,\tilde{K}]}} + \log \frac{R_j}{\prod_{k=1}^{\tilde{K}} R_k^{\nu_k/\nu_{[1,\tilde{K}]}}}, & j = 1, \ldots, \tilde{K}, \\ 0, & j = \tilde{K}+1, \ldots, K, \end{cases}$$

where

$$\tilde{K} = \max\left\{ i : \; R_i > \left(\prod_{k=1}^{i} R_k^{\nu_k/\nu_{[1,i]}} \right) e^{-\frac{\lambda}{\nu_{[1,i]}}} \right\}. \quad (8)$$

The result reveals that for large systems, the participation rate λ_j for contest j is logarithmic in the offered reward R_j, provided that the reward is larger or equal to the threshold reward $R_{\tilde{K}}$. Hence, the participation rates exhibit diminishing returns as the rewards increase.

Additionally, we also note the following refined asymptotic characterization:

$$\lambda_j \sim \frac{\lambda}{\nu_{[1,\tilde{K}]}} + \log \frac{R_j}{\prod_{k=1}^{\tilde{K}} R_k^{\nu_k/\nu_{[1,\tilde{K}]}}} + \left(\log \frac{\prod_{k=1}^{\tilde{K}} R_k^{\nu_k/\nu_{[1,\tilde{K}]}}}{R_j} \right) \frac{1}{N}, \quad \text{large } N,$$

for $j = 1, \ldots, \tilde{K}$. This tells us that as N goes to infinity the mean participation over contest classes become more *imbalanced* as the term that contains $1/N$ is positive (negative) for contests that offer smaller (larger) rewards.

Moreover, we can invert Theorem 4.3 to obtain rewards as a function of participation rates.

COROLLARY 4.1. *Given $(\lambda_1, \ldots, \lambda_K)$ with $\lambda_j > 0$ for $j = 1, \ldots, \tilde{K}$ for some $1 \leq \tilde{K} \leq K$ and $\lambda_j = 0$ otherwise, the rewards $R_1, \ldots, R_{\tilde{K}}$ are uniquely determined up to a multiplicative constant. For any $c > 0$, we have*

$$R_j = ce^{\lambda_j}, \; \textit{for } j = 1, \ldots, \tilde{K},$$

and

$$R_j < ce^{\frac{-\lambda + \sum_{k=1}^{\tilde{K}} \nu_k \lambda_k}{\nu_{[1,\tilde{K}]}}}, \; \textit{for } j = \tilde{K}+1, \ldots, K.$$

The threshold in (8) admits the following interpretation. Suppose there are two contest classes – the first offers a guaranteed reward of R_i, while the other offers the reward $\prod_{k=1}^{i} R_k^{\nu_k/\nu_{[1,i]}}$; we would bid zero in both, but win the latter only if we are lucky enough to be the only player. The threshold is the largest i for which we prefer R_i in such a situation.

We have the following analogue of Theorem 4.2.

COROLLARY 4.2. *In the large-system limit, we have the following.*

1. Skill levels. *Each skill level ℓ corresponds to the interval of skill values $(v_{\ell+1}, v_\ell]$ where v_ℓ is given by*

$$F(v_\ell) = 1 - \frac{\nu_{[1,\ell]}}{\lambda} \log \frac{\prod_{k=1}^{\ell} R_k^{\nu_k/\nu_{[1,\ell]}}}{R_\ell},$$

for $\ell = 1, \ldots, \tilde{K}$, and $v_\ell = 0$, otherwise.

2. Contest selections vs. skill.

$$\pi_j(v) = \begin{cases} \frac{1}{J_{[1,\ell]}}, & j = 1, \ldots, \ell \\ 0, & j = \ell+1, \ldots, K, \end{cases}$$

for $v \in [v_{\ell+1}, v_\ell)$.

Item 2 reveals the following *insensitivity property* that holds in the large-system limit. A player of skill level ℓ selects a contest *uniformly at random* from those that offer any of ℓ highest rewards. This may be of interest to system designers – the result suggests that for large systems, the mapping of the skill to the set of contests to choose from may be more important than the precise probabilities at which the contests are selected within this set. For example, this may inform the design of recommendations that would match the player skill to the contest rewards.

Interestingly, Theorems 4.1 and 4.3 reveal that the participation rates are independent of the underlying probability distribution. Indeed, only Item 1 of Theorem 4.2 (and respectively, Corollary 4.2) is dependent on $F(v)$ – the distribution influences which players join which contests, but the resulting participation rates are invariant.

5. CONTEST-SPECIFIC SKILLS

In contrast with the previous section, it is also natural to examine situations in which the talents required for contests are very diverse. In such situations, a player's intrinsic skills for different contests would exhibit little correlation. Toward this end, we now suppose that skills are drawn independently across not only players but also across contests for each player.

Let $F_j(v)$ be an atomless distribution on $[0, m]$ from which the skills for contests of class j are drawn. We assume that m is in the support of this distribution.

An implication of Corollary 3.1 is that each player needs only pay attention to his highest skill in each class of contests (since the distribution is atomless, a tie is a zero-measure event). If there are J_k contests in class k, then the highest skill among these contests will have distribution $F_k^{J_k}(v)$ and each contest in this class is equally likely to have the highest skill (and hence be the only candidate for participation).

We explore the large-scale limit of this environment. Let $(\lambda_1, \cdots, \lambda_K)$ be a limit of the vector of the mean number of players per contest of each type, and consider a sequence of equilibria that lead to this limit. Uniqueness of the limit will follow from Theorem 4.3; in the interim we may abuse notation and refer to this limit by the definite article.

In this sequence, $\lim_{N\to\infty} Np_j = \lambda_j$, where p_j is the probability of a player selecting one particular contest of type j. Recall that $\hat{F}_j(v)$ is the distribution of a player's skill contingent on his selecting a contest of class j.

LEMMA 5.1. *Suppose $\lambda_j > 0$. Then we have*

$$\lim_{N\to\infty} \hat{F}_j(v) = \begin{cases} 0, & \textit{if } v < m \\ 1, & \textit{if } v = m. \end{cases}$$

For the purpose of developing intuition, we can again consider (via revenue equivalence) a situation in which the number of players is revealed after players join the contest. As the distribution approaches one in which all participants have maximal skill, the player's surplus becomes increasingly dependent on the outcome in which no other players compete (a contest between two equally skilled players will give both zero surplus). Since the number of players approaches a Poisson limit, the probability of this event for a contest of class j is $e^{-\lambda_j}$.

For notational convenience, define ρ_j for each contest type j by $\rho_j = R_j e^{-\lambda_j}$.

LEMMA 5.2. *For each j and $v \in [0, m]$, we have*

$$g_j(v) \sim \rho_j v, \quad \text{for large } N.$$

Moreover, for any $\epsilon > 0$, we have for all sufficiently large N

$$|g_j(v) - \rho_j v| \leq \epsilon \quad \text{uniformly for all } v \in [0, m].$$

PROPOSITION 5.1. *In the large-scale limit, whenever $\lambda_j > 0$, we have $\rho_j \geq \rho_k$ for all contest classes k.*

Note that this implies a single value of ρ_j for all contest classes attended by a positive fraction of players in the limit.

Let $\tilde{K}$ be such that $\lambda_j > 0$, for $j = 1, \ldots, \tilde{K}$, and $\lambda_j = 0$, for $j = \tilde{K}+1, \ldots, K$. For contests i and j such that $\lambda_i, \lambda_j > 0$, we have from Proposition 5.1,

$$\lambda_j = \lambda_{j+1} + \log\left(\frac{R_j}{R_{j+1}}\right), \; j = 1, \ldots, \tilde{K} - 1.$$

Hence,

$$\lambda_j = \lambda_{\tilde{K}} + \log\left(\frac{R_j}{R_{\tilde{K}}}\right), \; j = 1, \ldots, \tilde{K} - 1. \qquad (9)$$

These relationships suffice to determine λ_j for all classes j as in the following corollary. Asymptotic rates of convergence are explored in Appendix C.4.

COROLLARY 5.1. *For the large-system limit, precisely the same limit holds as is stated in Theorem 4.3. Moreover, the reverse mapping in Corollary 4.1 also holds.*

In the proof of Lemma 5.1, we see that as the system grows, a player's skill at each contest class becomes closer to m, and hence his skills at any two contest classes become more alike. Consequently in a large system we are very nearly in a situation in which a player has a common skill for all contests – thus in the limit we observe the same player participation as in Section 4.

6. EXTENSIONS

In this section we explore situations in which some of our previous assumptions are relaxed. In the following subsection we explore the consequences of allowing distributions for different skills to have different supports. Subsection 6.2 examines the impact of imposing a minimum effort requirement on participating players.

6.1 Asymmetric Skills

Recall that our maximum skill m enforced a lower bound on the unit cost of effort; we may be interested in cases where this bound varies by contest class. Thus we observe that in Section 5 the use of the same maximum skill level m for each contest class may be generalized. For instance, suppose we wish to use distributions $F_j(v)$ with maximum skill level m_j for each class j (i.e. m_j is the supremum of the support of $F_j(v)$). If we scale to distribution $F_j^*(v) = F_j(m_j v)$ for $v \in [0, 1]$ and reward $R_j^* = m_j R_j$, then the players face precisely the same decisions in the two games. For instance, in this setting, observe the following generalization of Theorem 4.3.

COROLLARY 6.1. *Suppose the maximum skill in contest class j is m_j. In the large-system limit, the number of players that participate in a given contest of class j is a Poisson random variable with mean λ_j given by*

$$\lambda_j = \begin{cases} \frac{\lambda}{\nu_{[1,\tilde{K}]}} + \log \frac{m_j R_j}{\prod_{k=1}^{\tilde{K}} (m_k R_k)^{\nu_k/\nu_{[1,\tilde{K}]}}}, & j = 1, \ldots, \tilde{K}, \\ 0, & j = \tilde{K}+1, \ldots, K, \end{cases}$$

where

$$\tilde{K} = \max\left\{ i: \; m_i R_i > \left(\prod_{k=1}^{i} (m_k R_k)^{\nu_k/\nu_{[1,i]}} \right) e^{-\frac{\lambda}{\nu_{[1,i]}}} \right\}.$$

The relaxation of this assumption raises interesting modeling and informational requirements. As distributional knowledge is common to players in our game, the set of values m_j must also be known by the players. Additionally, it raises the question of how these values might be inferred from observing the participation levels of a system. An interesting future direction would be to model these parameters endogenously, since it is reasonable to assume that a player's surplus at the maximum skill would bear some relationship to his outside option.

6.2 Minimum Effort

Some contests may require entrants to put forth a minimum level of effort. For instance, a software design contest may disqualify all programs that are unable to process a particular set of inputs, the remaining entrants to compete on the basis of elegance of code, general performance, or less rigid criteria. In joining such a contest, a player bears the cost of this effort – in the auction setting, this corresponds to a minimum bid, or reserve price. In all-pay auctions, a reserve price is particularly restrictive for the bidders, and is roughly equivalent to an entry cost (presuming a player joined the auction intending to bid, and that bids are declared before the number of opponents is revealed).

Suppose that players may exit the system if they anticipate negative surplus upon seeing their skills (alternatively, we may suppose there is an additional contest offering negligible reward and requiring zero minimum effort), and that contests of class j enforce a minimum effort requirement of e_j. We assume that $R_j \geq \frac{e_j}{m}$, for otherwise we are assured that no players would participate in such a contest class. Then for each contest class j, there is a minimum type $\underline{v}_j$ who puts forth minimal effort and receives zero surplus, according to the equation

$$R_j \underline{v}_j (1 - p_j)^{N-1} = e_j.$$

Note that the left-hand side is the expected effort that would be excerted from a player with skill $\underline{v}_j$ if this player selected to participate in a contest of class j; in the corresponding all-pay auction, $R_j \underline{v}_j$ is the value and $(1 - p_j)^{N-1}$ is the probability of winning for the given player. We have that no type below $\underline{v}_j$ would willingly join this contest. Note that the imposition of minimum effort influences both p_j and the posterior distribution $\hat{F}_j(v)$ – that is, its effect must be fully integrated into the equilibrium.

The techniques leading to closed-form solutions of equilibria and participation do not carry over to this setting. Qualitatively, however, it may be shown that analogous results hold – participation is roughly logarithmically increasing in reward and logarithmically decreasing in required minimum effort. The interested reader may refer to Appendix D.1 for related discussion.

7. SYSTEM OBJECTIVES

In those situations in which the the offered rewards are centrally controlled, or in which those offering the rewards cooperate, it is natural to inquire as to the optimal specification of rewards for contests. Toward this end, this section explores possible social welfare objectives that may be desired by such a system. We make use of the characterization results, established in the preceding sections, on the relationship between reward and participation.

Suppose that each contest j is associated with a utility $U_j(\lambda_j)$ for the mean number of participants in this contest $\lambda_j \geq 0$. Suppose also that each contest j is associated with a cost $C_j(\vec{R})$ for a vector of given nonnegative rewards $\vec{R} = (R_1, \cdots, R_K)$. We assume $\vec{R}$ takes values from a given set $\mathcal{R}$ that is a subset of $[0, \infty)^K$.

We consider the system welfare problem defined as follows:

SYSTEM

$$\begin{aligned} \text{maximize} \quad & \sum_{k=1}^{K} \nu_k [U_k(\lambda_k(\vec{R})) - C_k(\vec{R})] \\ \text{over} \quad & \vec{R} \in \mathcal{R} \\ \text{subject to} \quad & \sum_{k=1}^{K} \nu_k \lambda_k = \lambda. \end{aligned}$$

For the mapping between rewards $\vec{R}$ and the mean number of contest participants, we may use the participation rates derived in Theorem 4.1, or alternatively the formula specified by Theorem 4.3 and Corollary 5.1. In the former case, we inspect a finite system under the player-specific skill assumption; in the latter case, we use the large system limit, which yields the same populations for both the player-specific and contest-specific skill assumptions.

In the following sections, we consider specific instances of the SYSTEM problem. First, we consider the special case where for each contest there is zero cost of offering any reward, i.e. $C_k(\cdot) \equiv 0$ for each contest k. Such zero-cost contests are of interest for systems where rewards are non-monetary, e.g. reputation points in a Q&A system. Second, we consider cases where the utility for a contest is increasing with the expected revenue.

7.1 Zero-Cost Contests

We consider the SYSTEM problem with $C_k(\cdot) \equiv 0$ for each contest k and rewards taking values on $\mathcal{R} = [0, \infty)^K$.

Suppose that for each contest class k, $U_k(\lambda_k)$ is an increasing, strictly concave function of $\lambda_k \geq 0$. Let U_k' denote the marginal utility and $U_k'^{-1}$ its inverse.

PROPOSITION 7.1. *Under player-specific skills, optimal rewards are unique up to a multiplicative constant. Moreover, for any $c > 0$,*

$$R_j = c\left(1 - \frac{U_j'^{-1}(\mu)}{N}\right)^{-(N-1)}, \ j = 1, \ldots, K, \tag{10}$$

where μ is a unique solution of

$$\sum_{k=1}^{K} \nu_k U_k'^{-1}(\mu) = \lambda.$$

For the large-system limit under both player- and contest-specific skill assumptions, we have the following optimal allocation of rewards. The proposition may be demonstrated by an analogous argument, employing Theorem 4.3 in lieu of Theorem 4.1.

PROPOSITION 7.2. *In the large-system limit, optimal rewards are unique up to a multiplicative constant. Moreover, for any $c > 0$,*

$$R_j = c e^{U_j'^{-1}(\mu)}, \ j = 1, \ldots, K, \tag{11}$$

where μ is a unique solution of

$$\sum_{k=1}^{K} \nu_k U_k'^{-1}(\mu) = \lambda.$$

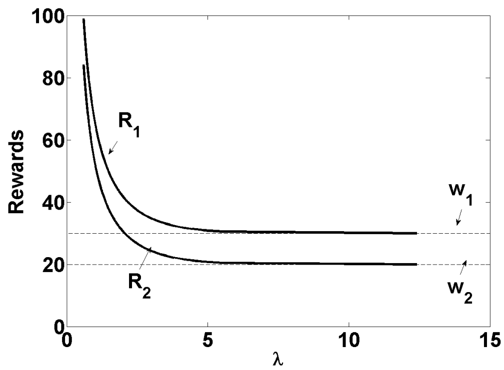


Figure 2: Optimal rewards versus the mean number of players per contest λ with willingness to pay parameters $w_1 = 30$ and $w_2 = 20$. With fewer players per contest, rewards are higher.

7.2 Revenue Optimal Rewards

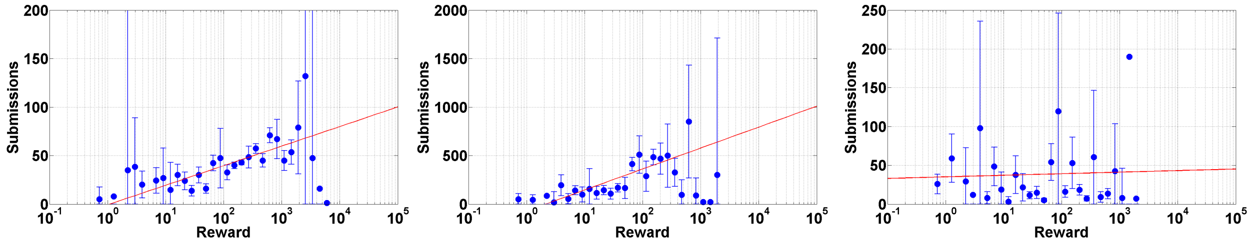
In the large-system limit under the assumption of contest-specific skills, consider the revenue for a contest of class j given by

$$\Pi_j(\lambda_j) = R_j m_j \left(1 - (1 + \lambda_j) e^{-\lambda_j}\right) \tag{12}$$

where R_j is the offered reward, m_j is the maximum skill and λ_j is the expected number of participants for contest j in equilibrium. This revenue corresponds to the total amount of effort put forth by players in this contest, and may be derived from the Revenue Equivalence Theorem. (It corresponds to a revenue of m_j when two or more players are present, and 0 otherwise.) We note that this revenue is not relevant in all circumstances; in many contests, only

Category	Number of tasks	Submissions per task	Median submission period (days)	Median reward (RMB)	Median total submissions per user (observed on Feb '09)
Graphics	2,392	44.66	16.97	200	n/a
Logos	571	50.36	16.96	240	24
2-D	1,431	49.14	17.68	200	23
Characters	613	323.75	17.15	80	n/a
Miscellaneous	565	38.47	12.29	79.85	n/a

Table 1: Summary of dataset basic properties.



(i) Graphics (ii) Characters (iii) Miscellaneous

Figure 3: The mean number of submissions versus the reward. The bars indicate 95% confidence intervals. The lines are linear regressions.

the effort put forth by the strongest player is important. Nonetheless, in contests where the players' effort may be usefully aggregated, this quantity warrants inspection.

We consider the SYSTEM problem with

$$U_j(\lambda_j) = V_j(\Pi_j(\lambda_j)),$$

where $V_j(\Pi_j)$ is the utility from contest j when the revenue in that contest is Π_j. We also suppose that the cost is $D_j(R_j)$ for reward R_j if the contest is attended by at least one player; this corresponds to $C_j(\vec{R}) = (1-e^{-\lambda_j(\vec{R})})D_j(R_j)$.

We can solve SYSTEM by a two-step procedure as follows. By Corollary 4.1 we have that for some $r > 0$, $R_j = re^{\lambda_j}$, whenever $\lambda_j > 0$. The first step amounts to solving, for fixed $r > 0$, and $j = 1, \ldots, K$,

$$\begin{aligned} \text{maximize} \quad & V_j\left(re^{\lambda_j} m_j \left(1 - (1+\lambda_j)\, e^{-\lambda_j}\right)\right) - \\ & (1-e^{-\lambda_j})D_j(re^{\lambda_j}) \\ \text{over} \quad & \lambda_j \geq 0. \end{aligned} \tag{13}$$

This yields a solution $\lambda_j(r)$. The second step amounts to finding $r \geq 0$ such that $\sum_{k=1}^{K} \nu_k \lambda_k(r) = \lambda$.

Example. Let us consider two contest classes with the following utility and costs. The utility functions are given by $V_k(x) = w_k \log(x)$, $x \geq 0$, $k = 1, 2$, where $w_k > 0$ is a willingness to pay parameter. Suppose also that cost functions are linear: $D_k(x) = x$, $x \geq 0$, $k = 1, 2$. We assume that the contest classes contain the same number of contests, i.e. $\nu_1 = \nu_2 = 1/2$.

We inspect the solution near the limit when there are a large mean number of participants in each contest (i.e. weak competition). In this case we have that the objective function in (13) is approximately $w_j\lambda_j - re^{\lambda_j}$. It follows that the optimal values of λ_j are given by $\lambda_j = \log w_j - \log r$. Combining this with $\sum_{k=1}^{K} \nu_k \lambda_k = \lambda$, we obtain $\log r = \log\sqrt{w_1 w_2} - \lambda$. Hence,

$$\lambda_j = \begin{cases} \lambda + \frac{1}{2}\log\frac{w_1}{w_2}, & j = 1, \\ \lambda - \frac{1}{2}\log\frac{w_1}{w_2}, & j = 2. \end{cases}$$

It follows that, in the prevailing limit, $R_j = w_j$, $j = 1, 2$, i.e. the rewards offered by contest classes are equal to the respective willingness to pay parameters.

The interaction between these two contests may be explored by examining how the rewards change as the number of players per contest changes. Intuitively, for smaller λ the contests would offer higher rewards since they must more intensely court a smaller number of players. This effect is illustrated by Fig. 2, where a numerical solution is presented.

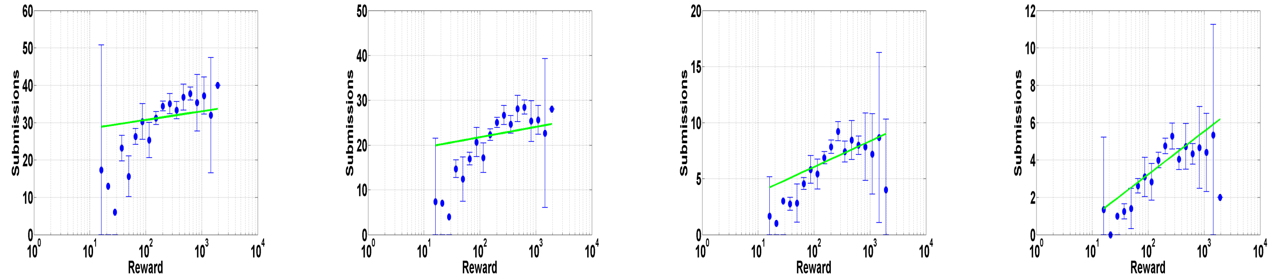
8. EMPIRICAL RESULTS

In this section we compare the results of our empirical analysis with the predictions of our analytical model. To this end, we use data collected from the crowdsourcing site Taskcn, covering a year-long period. We first provide basic information about the data set and then present our results.

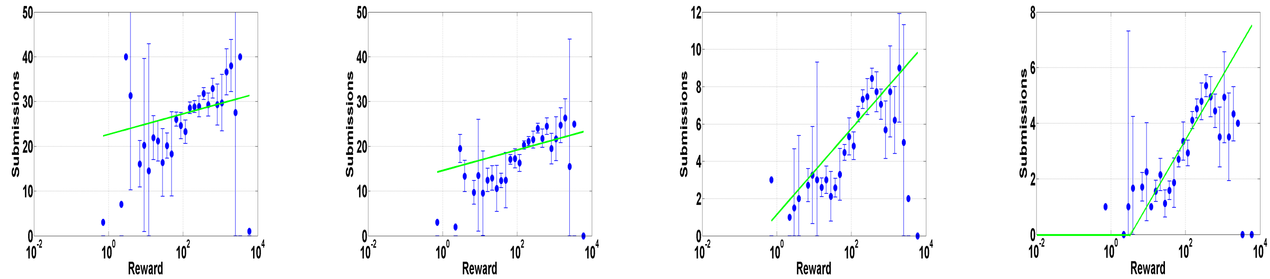
8.1 Data

Taskcn groups posted tasks into a few broad categories, together with subcategories. The categories are Graphics, Characters, Miscellaneous, and Super Challenge. Graphics and Characters respectively concern graphic design and writing-oriented tasks. Miscellaneous is a diverse category, containing programming tasks, odd jobs, and the catch-all "other." Super Challenge is a small collection of miscellaneous high-reward tasks.

In Table 1, we summarize the basic properties of the tasks that we consider. We consider only the tasks posted in year 2008 for which a single winner was chosen. We observed that for all categories (and all but one subcategory) the majority of the tasks had a single winner. Furthermore, for each



(i) Subcategory: Logos



(ii) Subcategory: 2-D

Figure 4: The mean number of submissions versus the reward for the subcategories (i) Logos and (ii) 2-D. From left to right, the conditioning is on users who submitted at least X solutions with $X = 1, 10, 100, 200$. The bars indicate 95% confidence intervals. The thick lines indicate the result of Theorem 4.3. The more experienced the users are (i.e. the larger the threshold X), the better the fit to the model.

subcategory of the category Graphics, more than 80% of the tasks awarded a single winner. In some of our analysis we focus particularly on the subcategories Logos and 2-D Design, which account for the bulk of the Graphics category. We do so under the conjecture that the tasks in these subcategories are roughly homogeneous in terms of the required skills, thus avoiding the modeling difficulty mentioned in Section 6.1 and lending some support to the player-specific skill assumption.

As previously noted, our model is independent of the underlying distribution of players' skills for large systems; it does not appear to be simple to infer this distribution from the data. We note incidentally that, in agreement with [14], we found that there is small group of users that accounts for a surprisingly large number of successful submissions. For instance, in the subcategory 2-D, there were 14,197 distinct users (individuals who participated at least once) responsible for a bit over 70,000 submissions. Of the 1431 tasks, there were 774 distinct winners. However, a group of only 122 users accounted for 50% of all winning submissions.

8.2 Participation vs. Rewards

In Figure 3 we show the mean number of submissions per task versus the task reward for the categories (i) Graphics, (ii) Characters, and (iii) Miscellaneous. The results for (i) and (ii) indicate an increasing trend for participation with the reward over a wide interval of rewards. There is some noticeable drop in participation at high rewards. For (iii), there is a lack of apparent trend which may be due to the heterogeneity of the tasks in the given category. The empirical results for (i) and (ii) support the hypothesis that the level of participation depends on the reward. The relationship between the rewards and the mean participation established by our model in Theorem 4.3 predicts a linear increase of the mean participation with the logarithm of the reward. The linear regression lines in Figure 3-(i) and (ii) suggest a faster increase than predicted by our model (which predicts a slope of $\frac{1}{\log_{10} e} \approx 2.3$). In the following, we discuss this discrepancy in more detail.

To more closely examine the relationship between reward and participation, we turn our attention to the experience of the users. We consider the same statistics as in the preceding paragraph but restrict our attention to those users who submit to several tasks in their lifetime. This enables us to filter out submissions from users who submitted only a few times and perhaps never learned an effective strategy. Intuitively, one would expect that a user would become more proficient at strategically selecting tasks with the acquired experience; this was confirmed in [14].

Consider a threshold X such that we restrict our attention to those users who submit at least X solutions in total (as observed early February 2009). We examine thresholds X taking values $1, 10, 100$, and 200. For a user who was active the entire year, these respectively correspond to submitting approximately (i) any number of tasks, (ii) once a month, (iii) every fourth day, and (iv) every second day, over this period. In Figure 4 we provide the same plots as in Figure 3 but for subcategories Logos and 2-D, and we condition on these thresholds. The data suggests that as we condition on more experienced users, we more closely obtain the relationship predicted by the model.

8.3 Other Considerations

While we focus on the influence of rewards on participation, this is not the only factor capable of influencing user participation. Users may avoid a task that is poorly worded or posted by someone with a poor reputation. Additionally, we have examined the influence of the duration of the submission period on participation, and found that is a significant factor in some cases; we omit further discussion due to space constraints. While a thorough factorial analysis of user participation is of interest, it is beyond the scope of the present paper.

9. CONCLUSION

In this paper we have presented and analyzed a model of crowdsourcing in which strategic players select among, and subsequently compete in, contests. Our focus is on the relationship between participation and incentives; in the regimes we have studied, we find that participation rates are logarithmically increasing as a function of the offered reward. Despite the simplicity of our model, it appears to be consistent with data from Taskcn, when the data is restricted to include only those who repeatedly use the site.

Nonetheless, we have noted that reward is not the only factor that can influence participation levels. Future work should consider the impact of the duration for which the task is posted, and how this might affect users in a dynamic setting. This introduces another dimension of empirical analysis as well.

These characterizations and the game-theoretic perspective lead naturally to the questions of how best to structure the incentives, and how to aid users in their selection of tasks. We have considered how the former might be posed in our framework; the latter question invites inquiry into how users' skills might be ascertained, and in what manner users respond to offered suggestions. Currently, the tasks on Taskcn are chiefly presented to users in order of recency; given even rudimentary information about users' skills and interests, there is room for improvement.

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APPENDIX

A. APPENDIX FOR SECTION 3

A.1 Proof of Proposition 3.1

Consider a simpler game in which players select contests, but do not submit bids. After the assignment of players to contests is realized, the players' skills are revealed and each player receives his outcome under a second-price auction. (The auction is not actually conducted – these outcomes are simply used as a payoff function.) Note that the action spaces are finite and the types of the players are independent – these together with the boundedness of the payoffs satisfy the technical conditions for existence of mixed-strategy equilibria in [9]. Since the game is symmetric, we may select a symmetric equilibrium.

Next, consider the two-stage game in which players first select contests, and then the number of players is revealed and an all-pay auction is carried out. By the Revenue Equivalence Theorem [6], players' expected surplus in the second stage (under symmetric all-pay auction equilibrium bidding) is equal to their surplus in the previous game. Another application of the Revenue Equivalence Theorem establishes that they would also receive the same surplus if they were to bid before the number of players is revealed. This is our original game.

A.2 Proof of Proposition 3.2

In the simplified context of Proposition 3.1, we examine a similar game in which this equilibrium symmetry is apparent, and then observe that the resulting equilibrium is an equilibrium in the original game. This yields an equilibrium in the simplified setting with the desired properties, which corresponds to an equilibrium in our model via the argument in Proposition 3.1.

In the modified game, after players select a contest, for each player Nature selects a random permutation of similar contests, and modifies the player's skills and selection accordingly. Effectively, Nature's move enforces symmetry of the actions of each player – even if a player selects different actions for different contests with the same rewards, such a strategy is payoff-equivalent to one that randomizes uniformly over these actions. It is clear that since the distribution of skills is symmetric with respect to similar contests, examining a player's symmetric post-Nature equilibrium strategy yields a symmetric strategy that is also an equilibrium in our original game.

B. APPENDIX FOR SECTION 4

In this section we first establish some properties that hold for any symmetric equilibria of the game under the player-specific skill assumption. We then provide proofs for Theorem 4.1, Theorem 4.2, and Proposition 4.1.

The equilibrium expected payoff $g_j(v)$ to a user with skill v in a contest of class j is given by (1). For each j, the marginal equilibrium expected payoff $g'_j(v) = (d/dv)g_j(v)$ satisfies

$$g'_j(v) = R_j(1 - p_j\hat{F}^c_j(v))^{N-1},\ v \in [0, m]. \tag{14}$$

By the definition of conditional probability, we have

$$p_j\hat{F}^c_j(v) = \int_v^m \pi_j(x)dF(x),\ v \in [0, m]. \tag{15}$$

Hence,

$$p_j = \int_0^m \pi_j(x)dF(x). \tag{16}$$

The expected payoff g_j function, for any j, satisfies the following properties; these properties hold generally and do not depend on the assumption of player-specific skills.

(G1) g_j is a non-decreasing function;
(G2) g_j is a continuous convex function;
(G3) g'_j is a continuous function;
(G4) $g_j(0) = 0$;
(G5) $g'_j(m) = R_j$.

Note that (G1) follows from (1) as the integrand therein is positive. (G2) follows from (1) as the integrand is non-decreasing with v for $v \in [0, m]$. (G3) follows from (14) and (15) and non-atomicity. (G4) follows from (1). Finally, (G5) derives from (14) and the fact that $\hat{F}_j$ is a distribution function with support in $[0, m]$, hence $\hat{F}^c_j(m) = 0$.

The following holds for any symmetric equilibrium with the probability distribution $\vec{\pi}$. If for some $v \in [0, m]$,
(E1) $\pi_j(v) > 0$, then $g_j(v) \geq \max_k g_k(v)$;
(E2) If $g_j(v) < g_k(v)$ for some $k \neq j$, then $\pi_j(v) = 0$.

The following lemma presents a key property of any symmetric equilibrium.

Lemma B.1. *Suppose that for some $v_0 \in [0, m]$, and $i, j \in \{1, \dots, K\}$ such that $g_i(v_0) \geq \max_k g_k(v_0)$, we have*

$$g'_j(v_0) < g'_i(v_0).$$

Then,

$$g_j(v) < \max_k g_k(v), \quad \text{for all } v \in (v_0, m].$$

Proof. To contradict, suppose that $g_j(v) = \max_k g_k(v)$ for some $v \in (v_0, m]$ and let v_1 denote the infimum of such v. Hence

$$g_j(v_1) = \max_k g_k(v_1). \tag{17}$$

Fix $\epsilon > 0$ such that $g'_i(v_0) - g'_j(v_0) > \epsilon$. By property (E2), $\pi_j(v) = 0$, for all $v \in (v_0, v_1)$ and hence by (14)–(15), $g'_j(v) = g'_j(v_0)$ for all $v \in [v_0, v_1)$. By (G2), $g'_i(v) \geq g'_i(v_0)$, for all $v \in [v_0, v_1)$. Thus

$$g'_i(v) - g'_j(v) > \epsilon,\ \text{for all } v \in [v_0, v_1).$$

From (G3), it follows that $g'_i(v) - g'_j(v) > \epsilon$, for all $v \in [v_0, v_1]$. This along with the assumption $g_i(v_0) \geq g_j(v_0)$ yields

$$\begin{aligned} g_i(v_1) - g_j(v_1) &= g_i(v_0) - g_j(v_0) + \int_{v_0}^{v_1}(g'_i(v) - g'_j(v))dv \\ &\geq \int_{v_0}^{v_1}(g'_i(v) - g'_j(v))dv \\ &> 0, \end{aligned}$$

which is a contradiction. □

The lemma entails the following corollary.

Corollary B.1. *For any j such that $p_j > 0$,*

$$g'_j(0) \geq g'_k(0), \quad \text{for all } k.$$

PROOF. To contradict, suppose that $p_j > 0$ and $g'_j(0) < \max_k g'_k(0)$. Apply Lemma B.1 with $v_0 = 0$ to conclude that $g_j(v) < \max_k g_k(v)$, for all $v \in (0, m]$. By (E2) we have $\pi_j(v) = 0$ for all $v \in (0, m]$, and hence by (16), $p_j = 0$, a contradiction. □

Futhermore, combining Lemma B.1 with (E2) and (15) we have

COROLLARY B.2. *Under the assumptions of Lemma B.1,*

(i) $\pi_j(v) = 0$, *for all* $v \in (v_0, m]$*;*

(ii) $\hat{F}^c_j(v_0) = 0$, *if* $p_j > 0$.

LEMMA B.2. *If* $p_j > 0$ *and* $p_i = 0$ *then* $R_j > R_i$.

PROOF. To contradict, suppose that $R_j \leq R_i$. By Corollary B.1, $g'_j(0) \geq g'_i(0)$, i.e. $R_j(1-p_j)^{N-1} \geq R_i$. Combining with $p_j > 0$, it follows $R_j > R_i$, which is a contradiction. □

For each contest class j, we define $v_j \in [0, m]$ by

$$p_j \hat{F}^c_j(v) > 0 \quad \text{for all } v \in [0, v_j)$$
$$p_j \hat{F}^c_j(v) = 0 \quad \text{for all } v \in [v_j, m].$$

Note that if $p_j = 0$ then $v_j = 0$. We define $v_{K+1} = 0$.

LEMMA B.3. *For any symmetric equilibrium, we have*

(i) $v_j \leq v_i$ *for* $R_j < R_i$*;*

(ii) $g'_j(v) = g'_1(v)$ *for* $j = 1, \ldots, \ell$ *and* $v \in [v_{\ell+1}, v_\ell)$*;*

(iii) $g'_j(v) < g'_1(v)$ *for* $j = \ell + 1, \ldots, K$ *and* $v \in [v_{\ell+1}, v_\ell)$.

PROOF. **Item (i)**. In view of Lemma B.2 and the fact $v_j = 0$ if $p_j = 0$, item (i) is true if $p_j = 0$ or $p_i = 0$. It thus suffices to consider only contests for which $p_j > 0$.

We proceed progressively over $[0, m]$, starting at $v = 0$. Let $A = \{k : \ p_k > 0\}$. By Corollary B.1, $g'_j(0) = g'_i(0)$ for all $i, j \in A$. Let v' be smallest value in $[0, m]$ such that $g'_j(v') < \max_{k \in A} g'_k(v')$ for some $j \in A$. Then, it must hold that $R_j = \min_{k \in A} R_k$. To contradict, suppose that there exist $i \in S$ such that $R_i < R_j$. By Corollary B.2-(ii) and (G3) we have $g'_j(v') = R_j$. Combining the last two relations with $g'_i(v') \geq g'_j(v')$ we conclude $g'_i(v') > R_i$, which cannot hold as by (14), $g'_k(v) \leq R_k$, for all k. This establishes item (i) for j^* such that $R_{j^*} = \min_{k \in A} R_k$.

The result follows by continuing the argument over $[v', m]$ with $A = \{k : \ p_k > 0\} \setminus \{j^*\}$.

Items (ii) and (iii). These items can be shown via arguments similar to those in the proof of item (i). □

B.1 Proof of Theorem 4.1

Let A be defined as follows

$$A = \{j \in \{1, \ldots, K\} : p_j > 0\}. \quad (18)$$

By Corollary B.1, we have that for some $C > 0$,

$$R_j(1 - p_j)^{N-1} = C, \quad \text{for all } j \in A. \quad (19)$$

Now, using the fact $\sum_{k \in A} J_k p_k = 1$, we obtain

$$C = \left(1 - \frac{1}{J_A}\right)^{N-1} H_A(R)^{N-1}$$

where $H_A(R)$ is defined in (2). From (19), it follows

$$p_j = 1 - \left(1 - \frac{1}{J_A}\right) \frac{H_A(R)}{R_j^{\frac{1}{N-1}}}, \text{ for } j \in A. \quad (20)$$

It remains to show that $A = \{1, \ldots, \tilde{K}\}$ with $\tilde{K}$ as asserted in (4). From Lemma B.2, we have that $A = \{1, \ldots, n\}$ for some $n \in \{1, \ldots, K\}$. From (20), it is readily checked that $p_j > 0$ for all $j = 1, \ldots, n$ is equivalent to

$$\phi(n) < 1 \quad (21)$$

where

$$\phi(n) = J_{[1,n]} - \sum_{k=1}^{n} J_k \left(\frac{R_n}{R_k}\right)^{\frac{1}{N-1}}.$$

In addition, n must satisfy $p_{n+1} = 0$, by the definition (18). Combining $p_n > 0$ and $p_{n+1} = 0$ with Corollary B.1, we have that it must hold $R_{n+1} \leq R_n(1-p_n)^{N-1}$. Using (20), it can be readily checked that the latter condition is equivalent to

$$\phi(n+1) \geq 1. \quad (22)$$

It is easy to check that $\phi(n)$ is non-decreasing with n and thus (21) and (22) imply $n = \tilde{K}$.

B.2 Proof of Theorem 4.2

In Lemma B.3 we found that for any symmetric equilibrium there exists a sequence $0 = v_{K+1} \leq v_K \leq \ldots \leq v_2 \leq v_1 = m$ such that the expected payoffs satisfy (ii) and (iii) in Lemma B.3. In the remainder we refer to the conditions (ii) and (iii) of Lemma B.3.

From (1) and (ii), we have

$$R_j \left(1 - p_j \hat{F}^c_j(v)\right)^{N-1} = R_1 \left(1 - p_1 \hat{F}^c_1(v)\right)^{N-1}$$

for all $j = 1, \ldots, \ell$ and $v \in [v_{\ell+1}, v_\ell)$. It follows that

$$p_j \hat{F}^c_j(v) = 1 - a_j + a_j p_1 \hat{F}^c_1(v) \quad (23)$$

with

$$a_j = \left(\frac{R_1}{R_j}\right)^{\frac{1}{N-1}},$$

for $j = 1, \ldots, \ell$ and $v \in [v_{\ell+1}, v_\ell)$.

From Corollary B.2 and (iii), we have

$$\hat{F}^c_j(v) = 0, \quad (24)$$

for all $j = \ell, \ldots, K$ and $v \geq v_\ell$.

By the property of conditional probability,

$$\sum_{k=1}^{K} J_k p_k \hat{F}^c_j(v) = F^c(v), \ v \in [0, m]. \quad (25)$$

Plugging (23) and (24) in the last relation, we obtain

$$p_1 \hat{F}^c_1(v) = 1 - \frac{J_{[1,\ell]}}{\sum_{k=1}^{\ell} J_k a_k} + \frac{1}{\sum_{k=1}^{\ell} J_k a_k} F^c(v)$$

for $v \in [v_{\ell+1}, v_\ell)$. Combining with (23), we have

$$p_j \hat{F}^c_j(v) = 1 - R_j^{-\frac{1}{N-1}} H_{[1,\ell]}(R) \left(1 - \frac{1}{J_{[1,\ell]}} F^c(v)\right) \quad (26)$$

for $j = 1, \ldots, \ell$ and $v \in [v_{\ell+1}, v_\ell)$ where $H_{[1,\ell]}(\cdot)$ is defined in (2).

From $p_\ell \hat{F}^c_\ell(v_\ell) = 0$ and (26), we obtain Eq. (6). This establishes item (i).

Item (ii) is derived upon differentiating both sides in (15) and using (26).

B.3 Proof of Proposition 4.1

The asserted uniqueness follows by the uniqueness of the equilibrium constructed in the proof of Theorem 4.2.

B.4 Proof of Theorem 4.3

In the proof, we repeatedly use the asymptote, for any $a > 0$, $a^x \sim 1 + \log(a)x$, for small x.

Consider (5) in Theorem 4.1. We have

$$H_{[1,\tilde{K}]}(\vec{R}) \sim 1 + \log\left(\prod_{k=1}^{\tilde{K}} R_k^{\nu_k/\nu_{[1,\tilde{K}]}}\right)\frac{1}{N}, \text{ large } N. \quad (27)$$

Furthermore, for large N,

$$\begin{aligned}\frac{H_{[1,\tilde{K}]}(\vec{R})}{R_j^{\frac{1}{N-1}}} &\sim H_{[1,\tilde{K}]}(\vec{R})\left(1 - \frac{\log R_j}{N}\right) \\ &\sim 1 + \log\left(\frac{\prod_{k=1}^{\tilde{K}} R_k^{\nu_k/\nu_{[1,\tilde{K}]}}}{R_j}\right)\frac{1}{N}\end{aligned}$$

where the last asymptote follows from (27). Using (28) in (5), we obtain

$$p_j \sim \frac{1}{J_{[1,\tilde{K}]}} + \log\left(\frac{R_j}{\prod_{k=1}^{\tilde{K}} R_k^{\nu_k/\nu_{[1,\tilde{K}]}}}\right)\frac{1}{N}, \text{ large } N,$$

for $j = 1, \ldots, \tilde{K}$. Similarly, (8) derives from (4).

We have showed that for each j,

$$\lim_{N\to\infty} Np_j = \lambda_j,$$

where λ_j is as asserted in the corollary. The asserted Poisson limit then follows by recalling that in equilibrium, the number of players that select a contest of class j is a binomial random variable with parameters N and p_j.

C. APPENDIX FOR SECTION 5

C.1 Proof of Lemma 5.1

Suppose $v < m$. Since m is in the support of $F_j(\cdot)$ we have $F_j(v) < 1$. Also, $\hat{F}_j(v) \le \frac{F_j^{J_j}(v)}{J_j p_j} \sim \frac{\lambda F_j^{J_j}(v)}{\nu_j N p_j}$. The inequality follows from conditional probability, and the asymptote from $\nu_j N \sim \lambda J_j$. Then $Np_j \to \lambda_j$ and $F_j^{J_j}(v)$ decreases exponentially in N, and we have $\hat{F}_j(v) \to 0$.

C.2 Proof of Lemma 5.2

In Eq. (1), taking the large scale limit and using dominated convergence, we have for large N,

$$g_j(v) \sim R_j \int_0^v \left[\lim_{N\to\infty}\left(1 - \frac{Np_j \hat{F}_j^c(x)}{N}\right)^{N-1}\right] dx.$$

Since $\hat{F}_j^c(x) \to 1$ and $Np_j \to \lambda_j$, the integrand is simply $e^{-\lambda_j}$. The second part of the lemma follows from the observation that $g_j(m) - \frac{\epsilon}{2} \le g_j(m - \frac{\epsilon}{2R_j}) \le g_j(m)$; hence we may take this convergence at a point strictly bounded away from m. This ensures uniformity, since $\hat{F}_j^c(x)$ is decreasing in x.

C.3 Proof of Propositon 5.1

Suppose for the sake of contradiction that $\lambda_j > 0$ but $\rho_j(1+\epsilon) = \rho_k$ for some $\epsilon > 0$. The probability of a player joining a contest of class j is asymptotically $J_j p_j \sim \frac{\lambda_j \nu_j}{\lambda} > 0$. Let the random variables v_j and v_k denote a player's highest skill in classes j and k respectively. Then since participation in class j requires his surplus to be higher there than in class k, we must have $J_j p_j \le \mathbf{P}[g_j(v_j) \ge g_k(v_k)]$. By Lemma 5.2, for all N large enough we have $\mathbf{P}[g_j(v_j) \ge g_k(v_k)] \le \mathbf{P}[\rho_j v_j + \rho_j \frac{\epsilon}{2} \ge \rho_k v_k]$. But the right hand side is $\mathbf{P}[v_j \ge v_k + \frac{\epsilon}{2}] \le F_k(m - \frac{\epsilon}{2})^{J_k} \to 0$, a contradiction.

C.4 Convergence to the Limit

We briefly discuss the convergence to the limit established in Corollary 5.1. Our objective is to examine, as the system grows, how quickly player populations approach the limit populations. Let us consider the case of two contest classes. Let $\rho(N)$ be a sequence such that for $v \in [0, m]$, $\rho(N) \sim g_2(v)/g_1(v)$, for large N. In view of Lemma 5.2, we have $\lim_{N\to\infty} \rho(N) = \rho_2/\rho_1$. It is readily checked that $\rho(N)$ satisfies the following asymptote

$$\frac{R_1}{R_2}\rho \sim e^{\frac{1}{\nu_1}(\lambda - Np_2)}, \text{ large } N. \quad (28)$$

Note further that

$$Np_2 \sim \frac{\lambda}{\nu_2}\int_0^m F(\rho v)^{\frac{\nu_1}{\lambda}N} dF(v)^{\frac{\nu_2}{\lambda}N}, \text{ large } N. \quad (29)$$

Example. Suppose $F(v) = (v/m)^a$, $v \in [0, m]$, for any $a > 0$. From (29), we have

$$Np_2 \sim \lambda \rho^{\frac{\nu_1}{\lambda}aN}, \text{ large } N. \quad (30)$$

Thus (28) reads as

$$\frac{R_1}{R_2}\rho \sim e^{\frac{\lambda}{\nu_1}\left(1 - \rho^{\frac{\nu_1}{\lambda}aN}\right)}, \text{ large } N. \quad (31)$$

Case 1: $R_2 < R_1 e^{-\frac{\lambda}{\nu_1}}$. By Theorem 4.3, $\lim_{N\to\infty} Np_2 = 0$ and hence from (31), $\lim_{N\to\infty}\rho(N) = \frac{R_2}{R_1}e^{\frac{\lambda}{\nu_1}}$. From (30), it then follows

$$Np_2 \sim e^{-[\frac{\nu_1}{\lambda}\log\frac{R_1}{R_2} - 1]aN}, \text{ large } N.$$

This establishes the exponential convergence of Np_2 to 0 as N tends to infinity.

Case 2: $R_2 \ge R_1 e^{-\frac{\lambda}{\nu_1}}$. From (28),

$$Np_2 \sim \lambda + \nu_1 \log\frac{R_2}{R_1} - \nu_1 \log\rho, \text{ large } N.$$

From (30),

$$\rho \sim \left(\frac{Np_2}{\lambda}\right)^{\frac{\lambda}{\nu_1}\frac{1}{aN}}, \text{ large } N.$$

Combining with the fact $\lim_{N\to\infty} Np_2 = \lambda_2 > 0$ (from Theorem 4.3), it follows

$$Np_2 \sim \lambda_2 + \lambda \log\left(\frac{\lambda}{\lambda_2}\right)\frac{1}{aN}, \text{ large } N.$$

This establishes the $1/N$ convergence of Np_2 to λ_2. Hence, in this example, both participation levels quickly converge to their respective limits.

D. APPENDIX FOR SECTION 6

D.1 Discussion of Minimum Effort

In this section, we consider special cases under the player-specific and contest-specific skills settings. We present the following to demonstrate where the complexities arise and to show qualitative impacts of minimum effort requirements on participation.

D.1.1 Player-Specific Skills

Obtaining a closed-form relation for the mean participation levels and rewards appears difficult. This is already demonstrated for the case of two contest classes that we consider in the following. Consider the case of two contest classes with respective rewards R_1 and R_2. Let e_1 and e_2 denote the minimum efforts for the contest classes 1 and 2, respectively. Without loss of generality, let $e_1 \geq 0$ and $e_2 = 0$.

If the minimum effort e_1 is larger than the maximum possible user payoff $R_1 m$, i.e. $e_1 > R_1 m$, then in equilibrium, no user selects a contest of class 1 and in this case we have $p_1 = 0$. In the sequel, we consider the case $e_1 \leq R_1 m$.

PROPOSITION D.1. *Suppose* $e_1 \in (0, R_1 m]$. *If*

$$R_1 \leq R_2 \left(1 - \frac{1}{J_2} + \frac{1}{J_2} F(e_1/R_1)\right)^{N-1} \tag{32}$$

then $p_1 = 0$. *Otherwise,* $p_1 = 0$ *if and only if*

$$R_1 \leq \frac{e_1}{v^*} + R_2 \frac{1}{v^*} \int_0^{v*} \left(1 - \frac{1}{J_2} + \frac{1}{J_2} F(x)\right)^{N-1} dx \tag{33}$$

where $v^* = m$ *if* $R_1 \geq R_2$ *and*

$$F(v^*) = 1 - J_2 \left(1 - \left(\frac{R_1}{R_2}\right)^{\frac{1}{N-1}}\right), \tag{34}$$

otherwise.

PROOF. $p_1 = 0$ is equivalent to

$$g_1(v) < g_2(v) \text{ almost everywhere on } [0, m]. \tag{35}$$

Under $p_1 = 0$, we have

$$g_1(v) = \begin{cases} 0, & v \in [0, e_1/R_1] \\ R_1 v - e_1, & v \in [e_1/R_1, m] \end{cases}$$

$$g_2(v) = R_2 \int_0^v \left(1 - \frac{1}{J_2} + \frac{1}{J_2} F(x)\right)^{N-1} dx, \; v \in [0, m].$$

It follows that condition (35) is equivalent to

$$\phi(v) > 0 \text{ almost everywhere on } [e_1/R_1, m] \tag{36}$$

where for $v \in [e_1/R_1, m]$,

$$\phi(v) = R_2 \int_0^v \left(1 - \frac{1}{J_2} + \frac{1}{J_2} F(x)\right)^{N-1} dx - R_1 v + e_1.$$

Note that

$$\phi'(v) = R_2 \left(1 - \frac{1}{J_2} + \frac{1}{J_2} F(v)\right)^{N-1} - R_1$$

is increasing for each $v \in [e_1/R_1, m]$.

Suppose that (32) holds. Note that this is equivalent to $\phi'(e_1/R_1) \geq 0$. By the fact that $\phi'(v)$ is increasing with v, we have $\phi'(v) \geq 0$ for all $v \in [e_1/R_1, m]$. Combined with $\phi(e_1/R_1) \geq 0$, we have that (36) holds.

In the remainder, we consider the cases when (32) does not hold. If $R_1 \geq R_2$ note that $\phi'(v) \leq 0$ for all $v \in [e_1/R_1, m]$ and hence (36) is equivalent to $\phi(m) > 0$. This is the same as (33). If $R_1 \leq R_2$, then (36) is equivalent to $\phi(v^*) > 0$ where v^* is the minimizer of $\phi(v)$ over $[e_1/R_1, m]$. The minimizer satisfies $\phi'(v^*) = 0$, which is the same as (34). □

In the sequel, we discuss the equilibrium under the assumption that $p_1 > 0$. We argue that the equilibrium is such that for some $0 \leq \underline{v} \leq \overline{v} \leq m$, the probability of selecting the contest class 1, $\pi_1(v)$, satisfies the following. First, $\pi_1(v) = 0$, for $v \in [0, \underline{v}]$. Second, $0 < \pi_1(v) < 1$ for $v \in (\underline{v}, \overline{v})$. Finally, if $R_1 \geq R_2$, $\pi_1(v) = 1$, for $v \in [\overline{v}, m]$, and otherwise, $\pi_1(v) = 0$ for $v \in [\overline{v}, m]$. We next characterize the threshold skills $\underline{v}$ and $\overline{v}$. It is readily seen that these thresholds need to satisfy the following properties.

$$g_1(v) < g_2(v), \text{ for } v \in (0, \underline{v}) \tag{37}$$

$$g_1(\underline{v}) = g_2(\underline{v}) \tag{38}$$

$$g_1'(v) = g_2'(v), \text{ for } v \in [\underline{v}, \overline{v}] \tag{39}$$

Case 1: $R_1 \geq R_2$. First note that for $v \in [0, \underline{v}]$,

$$g_1(v) = \max(R_1 v (1 - p_1)^{N-1} - e_1, 0)$$

$$g_2(v) = R_2 \int_0^v (1 - p_2 \hat{F}_2^c(x))^{N-1} dx.$$

The threshold $\underline{v}$ is a solution of Eq. (38) that we can rewrite as

$$R_1 \underline{v} (1 - p_1)^{N-1} - e_1 = R_2 \int_0^{\underline{v}} (1 - p_2 \hat{F}_2^c(x))^{N-1} dx. \tag{40}$$

The latter is equivalent to

$$\begin{aligned} & \underline{v} \left(J - 1 + F(\underline{v})\right)^{N-1} - \tfrac{e_1}{R_2} \left(J_1 r_N + J_2\right)^{N-1} \\ = & \int_0^{\underline{v}} \left(J - 1 - \tfrac{J_1}{J_2} r_N F(\underline{v}) + \left(1 + \tfrac{J_1}{J_2} r_N\right) F(x)\right)^{N-1} dx \end{aligned} \tag{41}$$

where $r_N = \left(\frac{R_2}{R_1}\right)^{\frac{1}{N-1}}$. In the sequel, we show equivalence of (41) and (40).

From (15) and $\pi_2(v) = 1$ for $v \in [0, \underline{v}]$, we have

$$J_2 p_2 \hat{F}_2^c(v) = F(\underline{v}) - F(v) + p_2 \hat{F}_2^c(\underline{v}), \text{ for } v \in [0, \underline{v}]. \tag{42}$$

Eq. (39) can be written as

$$R_1 (1 - p_1 \hat{F}_1^c(v))^{N-1} = R_2 (1 - p_2 \hat{F}_2^c(v))^{N-1}, v \in [\underline{v}, \overline{v}].$$

Using (25), it follows that

$$p_2 \hat{F}_2^c(v) = \frac{J_1 \left(R_1^{-\frac{1}{N-1}} - R_2^{-\frac{1}{N-1}}\right) + R_2^{-\frac{1}{N-1}} F^c(v)}{J_1 R_1^{-\frac{1}{N-1}} + J_2 R_2^{-\frac{1}{N-1}}}, \tag{43}$$

for each $v \in [\underline{v}, \overline{v}]$. Combined with (15), we have

$$\pi_2(v) = \frac{R_2^{-\frac{1}{N-1}}}{J_1 R_1^{-\frac{1}{N-1}} + J_2 R_2^{-\frac{1}{N-1}}}, \; v \in [\underline{v}, \overline{v}]. \tag{44}$$

Note

$$p_1 = \int_{\underline{v}}^{\overline{v}} \pi_1(x) dF(x) + \frac{1}{J_1} F^c(\overline{v}).$$

Using (44) and $J_1\pi_1(v) = 1 - J_2\pi_2(v)$, we obtain

$$J_1 p_1 = 1 - \frac{J_1 R_1^{-\frac{1}{N-1}}}{J_1 R_1^{-\frac{1}{N-1}} + J_2 R_2^{-\frac{1}{N-1}}} F(\underline{v}) - \frac{J_2 R_2^{-\frac{1}{N-1}}}{J_1 R_1^{-\frac{1}{N-1}} + J_2 R_2^{-\frac{1}{N-1}}} F(\overline{v}). \quad (45)$$

The threshold $\overline{v}$ is determined by the condition $\hat{F}_2^c(\overline{v}) = 0$. From (43) we have

$$F(\overline{v}) = 1 - J_1 \left(1 - \left(\frac{R_2}{R_1}\right)^{\frac{1}{N-1}}\right). \quad (46)$$

The equivalence of (41) and (40) follows by using (45) with $F(\overline{v})$ replaced by the right-hand side in (46), using (42) with $p_2 \hat{F}_2^c(\underline{v})$ replaced by the right-hand side in (43) evaluated at $\underline{v}$, and simple calculus.

Case 2: $R_1 < R_2$. By the same arguments as in Case 1, one can check that the threshold skill $\underline{v}$ satisfies (41). The threshold skill $\overline{v}$ is now obtained by the condition $p_1 \hat{F}_1^c(\overline{v}) = 0$ which yields

$$F(\overline{v}) = 1 - J_2 \left(1 - \left(\frac{R_1}{R_2}\right)^{\frac{1}{N-1}}\right).$$

D.1.2 Contest-Specific Skills

Consider the contest-specific skill assumption in the presence of minimum effort requirements. Suppose that the minimum effort for a contest of class j is $e_j \geq 0$.

Let $\underline{v}_j$ be such that

$$R_j \underline{v}_j (1 - p_j)^{N-1} - e_j = 0.$$

Then a player's surplus in this regime becomes

$$g_j(v) = R_j \int_{\underline{v}_j}^{v} (1 - p_j \hat{F}_j^c(x))^{N-1} dx.$$

The equivalent of Proposition 5.1 in this setting is

$$g_j(v) = \rho_j v - e_j = \rho_j (v - \underline{v}_j),\ v \in [\underline{v}_j, m],$$

which yields the following.

PROPOSITION D.2. *In the large-scale limit, whenever $\lambda_j > 0$, we have $\rho_j m - e_j \geq \rho_k m - e_k$ for all contest classes k.*

Let $A = \{j :\ \lambda_j > 0\}$. By Proposition D.2, there exists a constant C such that $\rho_j - e_j/m = C$, for all $j \in A$. It follows that

$$\lambda_j = \begin{cases} \log R_j - \log\left(C + \frac{e_j}{m}\right), & j \in A \\ 0, & \text{otherwise} \end{cases}$$

where C is given by the implicit function

$$\prod_{j \in A} \left(C + \frac{e_j}{m}\right)^{\nu_j} = \left(\prod_{j \in A} R_j^{\nu_j}\right) e^{-\lambda}.$$

The latter follows from $\sum_{j \in A} \nu_j \lambda_j = \lambda$.

Unlike the case without minimum effort requirements, it does not seem tractable to obtain a closed-form correspondence, in general. An explicit characterization can be obtained in some special cases.

Example. Suppose we have two contest classes with $R_1 > R_2$ and $e_1 > e_2 = 0$. Suppose further that $\nu_1 = \nu_2 = \frac{1}{2}$, and that $\lambda_1 > 0$ and $\lambda_2 > 0$ in the limit. Let $C = \rho_1 - e_1/m = \rho_2$. Then from $\nu_1\lambda_1 + \nu_2\lambda_2 = \lambda$, we find that C must satisfy

$$C^2 + \frac{e_1}{m} C - e^{-2\lambda} R_1 R_2 = 0.$$

We can solve this to find

$$\lambda_1 = \log R_1 - \log\left(\frac{1}{2}\sqrt{\left(\frac{e_1}{m}\right)^2 + 4R_1R_2e^{-2\lambda}} - \frac{1}{2}\frac{e_1}{m}\right),$$

$$\lambda_2 = \log R_2 - \log\left(\frac{1}{2}\sqrt{\left(\frac{e_1}{m}\right)^2 + 4R_1R_2e^{-2\lambda}} + \frac{1}{2}\frac{e_1}{m}\right).$$

The restrictions $\lambda_1 > 0$ and $\lambda_2 > 0$ yield conditions necessary to ensure that e_1 is not too high nor R_2 too low to ensure positive participation in both contest classes.

Although the presence of more contest classes results in a higher-order polynomial in C, it is possible to demonstrate that when ν_j is sufficiently small, slightly changing R_j or e_j does not greatly perturb C. This in turn may be used to show in general that λ_j increases logarithmically in R_j and decreases logarithmically in $C + e_j$ (and hence in e_j).

E. APPENDIX FOR SECTION 7

E.1 Proof of Proposition 7.1

From (5) we have

$$\lambda_j = N\left[1 - \left(1 - \frac{1}{J}\right)\frac{H_{[1,\tilde{K}]}(\vec{R})}{R_j^{\frac{1}{N-1}}}\right]$$

for $j = 1, \ldots, \tilde{K}$, and $\lambda_j = 0$, otherwise. We perform the optimization in SYSTEM by first fixing $H_{[1:\tilde{K}]}(\vec{R}) = r$, for given $r > 0$. We can then consider λ_j as a function of R_j only. Note that λ_j is a strictly concave function of R_j. By composition [3], $U_j(\lambda_j(\vec{R}))$ is also a strictly concave function of R_j. It thus follows that for any given $r > 0$, the concave program SYSTEM has a unique solution.

Rewards $\vec{R}$ are optimal if and only if the following holds:

$$U_k'(\lambda_k) = \mu + \delta_k,\ k = 1, \ldots, K,$$

where μ is the Lagrange multiplier associated with the constraint, $r = H_{[1:\tilde{K}]}(\vec{R})$, and $\delta_k = 0$ for $k = 1, \ldots, \tilde{K}$ and $\delta_k > 0$, otherwise. The Lagrange multiplier μ is given by $\sum_{k=1}^{K} \nu_k \lambda_k = \lambda$, which is equivalent to

$$\sum_{k=1}^{\tilde{K}} \nu_k U_k'^{-1}(\mu) = \lambda.$$

Now, noting that for each k, $\lambda_k = U_k'^{-1}(\mu + \delta_k)$ and observing that $U_k'^{-1}(x) > 0$ for any $x \geq 0$, we have that $\tilde{K} = K$. The result follows from (5) by substituting $p_k = \lambda_k/N = U_k'^{-1}(\mu)/N$.