# The Two-Variable Guarded Fragment with Transitive Relations

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**Abstract**

*We consider the restriction of the guarded fragment to the two-variable case where, in addition, binary relations may be specified as transitive. We show that (i) this very restricted form of the guarded fragment without equality is undecidable and that (ii) when allowing non-unary relations to occur only in guards, the logic becomes decidable. The latter subclass of the guarded fragment is the one that occurs naturally when translating multi-modal logics of the type K4, S4 or S5 into first-order logic. We also show that the loosely guarded fragment without equality and with a single transitive relation is undecidable.*

## 1 Introduction

We consider first-order logic without non-constant function symbols, but with equality and with relation symbols of arbitrary arities. The class of all closed formulas containing at most two variables is called the *two-variable fragment* of first-order logic and is denoted by $FO^2$. The decidability of $FO^2$ without equality was first noted by Scott [1962] by a reduction to formulas with quantifier prefix $\forall\forall\exists^*$, a fragment that was proved decidable by Gödel [1932]. Gödel claimed without proof that this fragment remains decidable also with equality, which was later refuted by Goldfarb [1984]. The decidability and finite model property for the full class $FO^2$ was first established by Mortimer [1975]. From Mortimer's [1975] proof follows also that (the satisfiability problem for) $FO^2$ is decidable in *nondeterministic doubly exponential* time. This upper bound was recently improved by Grädel, Kolaitis & Vardi [1997] to *nondeterministic exponential* time. The NEXPTIME-hardness of $FO^2$ even without equality follows from results by Fürer [1981].

**Why the two-variable fragment?** Since (propositional) modal logic can be embedded into $FO^2$, that was already shown by Gabbay [1971], the decidability of $FO^2$provides some understanding of the tractability of (propositional) modal logics. However, while several extensions of modal logic, like *computational tree logic* or CTL [Clarke & Emerson 1981], remain decidable (for validity), corresponding extensions of $FO^2$ lead to undecidability. In particular, Vardi [1997] shows that CTL can be embedded into $FO^2$ fragment of fixed-point logic. The validity problem of the latter was recently shown to be undecidable by Grädel, Otto & Rosen [1998], whereas Fischer & Ladner [1979] have shown that the validity problem for CTL is EXPTIME-complete. Similarly, Immerman & Vardi [1997] show that, CTL can be viewed as a $FO^2$ fragment of first-order logic with a transitive closure operator (when restricted to finite structures), that is again undecidable [Grädel et al. 1998]. The latter result is also implied by Grädel & Otto's [1998] strong undecidability result of $FO^2$ with several built-in equivalence relations. In contrast, Otto [1998] has shown very recently that $FO^2$ with a *single* built-in equivalence relation is still decidable.

**What is the guarded fragment?** In order to capture the nice properties of modal logics, Andréka, van Benthem & Németi [1996] introduced the *guarded fragment* or GF of first-order logic, where all quantifiers are appropriately relativized by *atoms*. This fragment was later generalized by van Benthem [1997] to the *loosely guarded fragment* or LGF, where all quantifiers are appropriately relativized by *conjunctions* of atoms. These fragments are decidable and enjoy several useful syntactic and model theoretic properties that do not, in general, hold for $FO^2$[Andréka et al. 1996, Grädel 1998*b*]. In particular, Grädel [1998*b*] shows that both GF and LGF, unlike $FO^2$, have a certain *tree model property* that generalizes the well-known tree model property for modal logics. Moreover GF has, like $FO^2$, the finite model property. However, the satisfiability problem for LGF restricted to a bounded number of variables or a bounded arity on relation symbols is, unlike for $FO^2$, in *deterministic exponential time* [Grädel 1998*b*].

**The role of the tree model property.** Vardi [1997] argues convincingly that the tree model property is the main reason behind the decidability of various extensions of modal logic, since it provides one with a powerful tool to prove decidability via Rabin's

[1969] theorem. Unfortunately, the same is not true for GF. As Grädel [1998*b*] demonstrates, already very modest extensions of GF lead to undecidability: GF with three variables and transitive relations, and GF with three variables and counting quantifiers, are both undecidable extensions of GF. In the second case the result is optimal with respect to the number of variables, since $\mathrm{FO}^2$ with counting quantifiers is decidable [Grädel, Otto & Rosen 1997, Pacholski, Szwast & Tendera 1997].

**The two-variable guarded fragment.** In this paper we consider certain restrictions and variants of the fragment $\mathrm{GF} \cap \mathrm{FO}^2$ denoted as $\mathrm{GF}^2$(or $\mathrm{GF}^2_-$ if equality is not permitted). When encoding the Kripke semantics of propositional multi-modal logics one ends up in this subclass of the GF. For multi-modal logics with modalities of type K4, S4, and S5, $\mathrm{GF}^2_-$ with transitive relations appears as a natural choice for a representation language. Multi-modal logics of the above types are used to formalize epistemic logics [Fagin, Halpern, Moses & Vardi 1995]. We show that $\mathrm{GF}^2_-$ with transitive relations is undecidable. Moreover, this is the case even when *all* non-unary relations are transitive binary relations. Hence this class is too big to capture these multi-modal logics adequately. On the other hand, when encoding propositional modal logics, the non-unary relations only appear as guards, such guarded formulas are said to be *monadic*.

Our second result is that monadic $\mathrm{GF}^2$ with binary relations that are transitive, symmetric and/or reflexive, is decidable. The latter result will be proved by an encoding of this class in *SkS* (similar to how this can be done for CTL) by which also the tree model property is demonstrated. A potential interest of the decidability result lies also in the context of knowledge representation, due to the relation to description logics [Grädel 1998*a*] and conceptual graphs [Baader, Molitor & Tobies 1998].

The constructions in our undecidability proof were strongly influenced by Grädel's [1998*b*] techniques and may be seen as generalizations of the them. Independently, similar ideas are used by Grädel & Otto [1998] to prove the undecidability of the whole class $\mathrm{FO}^2$ with equality and additional equivalence relations. In our constructions equality is omitted. The new insight is that it suffices to use an equivalence relation instead. In the specific structures that we define, this equivalence will always become a partial congruence, although in general the substitutivity laws of a congruence cannot be expressed as a guarded formula. With this idea, also the corresponding proofs in [Grädel & Otto 1998] could be modified to extend their results also to $\mathrm{FO}^2$ without equality. (That presence or absence of equality may make a difference for decidability is exemplified with the Gödel class, as mentioned before.)

**A remark about LGF.** We also show that LGF without equality becomes undecidable as soon as a *single* relation is allowed to be transitive. The proof uses a reduction from the intersection emptiness problem for context-free languages.

## 2 Undecidability Results

The *guarded fragment* (GF) of first-order logic with equality (we use $\approx$ to denote formal equality) and constants, but no function symbols of arity greater than 0, is defined as the least set of formulas such that (i) $\top$ and $\bot$ are in GF; (ii) any atom is in GF; (iii) GF is closed under the boolean connectives; (iv) if $A$ is an atom and $\phi$ is in GF such that all free variables in $\phi$ occur as arguments in $A$, and if $\bar{x}$ is a list of variables then $\forall\bar{x}(\neg A \vee \phi)$ (equivalently, $\forall\bar{x}(A \Rightarrow \phi)$) and $\exists\bar{x}(A \wedge \phi)$) are in GF.[1] The atoms $A$ which relativize a quantified formula are called *guards*. A formula in GF is called a *guarded formula*. $\mathrm{GF}^n$ is the subset of formulas in GF which contain occurrences of at most $n$ distinct variables. For $\mathrm{GF}^2$ one may assume that every predicate symbol is either unary or binary. $\mathrm{GF}_-$ is GF restricted to formulas without equality. We let $\mathrm{Trans}[R_1, \ldots, R_n]$ stand for the condition that each $R_i$ is a transitive binary relation. The formula in the following example is a classical one used to demonstrate the existence of first-order formulas with only infinite models. Here it shows that transitivity cannot be expressed in GF and, therefore, has to be stipulated on the meta-level, because GF has the finite model property.

**Example 1** Consider the formula $\varphi$ in $\mathrm{GF}^2_-$ expressing that a binary relation $<$ is *non-empty*, *serial*, and *irreflexive*: $\exists xy\,(x < y) \wedge \forall xy\,(x < y \Rightarrow \exists x\,(y < x)) \wedge \neg\exists x\,(x < x)$. Clearly, $\varphi \wedge \mathrm{Trans}[<]$ has only infinite models. Se Figure 1.

We prove that the satisfiability problem for $\mathrm{GF}^2_- + \mathrm{Trans}[R_1, \ldots, R_5]$ is undecidable (Theorem 1). More specifically, it follows from the construction that *all* non-unary relations can be transitive binary relations (Theorem 2). The problem with omitting equality is that the laws of substitutivity for equality cannot generally be specified in the guarded fragment: formulas

[1] Special cases of guarded quantification occur when $\phi = \top$ or $\phi = \bot$, respectively; such trivial bodies of quantification are usually omitted.

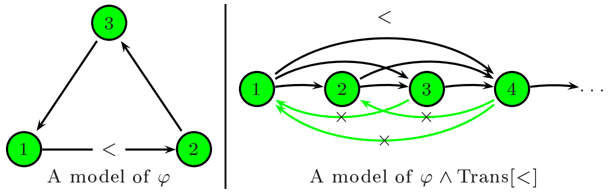


**Figure 1:** Given $\varphi$ as in Example 1. To the right, circles are not possible because $<$ is transitive and irreflexive.

such as $\forall x, y, z(x \approx y \Rightarrow (R(x,z) \Rightarrow R(y,z)))$ are not in GF.

The main idea of the proof is as follows. We construct a formula GRID in the two-variable guarded fragment that describes a two-dimensional grid. (See Figure 2.) We then reduce Minsky machines $M$ (two-counter machines) to formulas $\varphi_M$ in the two-variable guarded fragment that describe "walking" in that grid. The conjunction of GRID, $\varphi_M$, and transitivity of five binary relations is unsatisfiable if and only if $M$ halts.

### 2.1 The GRID formula

We construct a closed formula GRID in the guarded fragment with two variables, four transitive relations $W_0, W_1, B_0, B_1$, a transitive relation $\sim$ called *similarity*, four additional binary relations $\uparrow^0, \uparrow^1, \xrightarrow{0}, \xrightarrow{1}$, called *arc relations*, and some unary relations. When equality is in the language then it can be used instead of the similarity symbol. We use infix notation for the similarity symbol and the arc relation symbols.

There is a unary predicate Node. In any structure in the language of GRID, we are only interested in the elements in Node, such elements are called *nodes*. We will use the following lemma, that follows by easy induction on guarded formulas.

**Lemma 1** *Let $\varphi$ be a closed guarded formula such that all elements that satisfy guards are nodes. Then, for all structures $A$, $A$ satisfies $\varphi$ if and only if the restriction of $A$ to nodes satisfies $\varphi$.*

In the end, we are only interested in models of GRID, and in GRID all formulas are guarded in such a way that the elements that satisfy the guards must be nodes. GRID is a conjunction of formulas (1–17).

The set of nodes is non-empty, and $\sim$ is reflexive and symmetric on nodes:

$$\begin{aligned}&\exists x \texttt{Node}(x) \wedge \\ &\forall x(\texttt{Node}(x) \Rightarrow x \sim x) \wedge \forall xy(x \sim y \Rightarrow y \sim x)\end{aligned} \tag{1}$$

Hence, $\sim$ is, due to the transitivity, an equivalence relation on nodes. Given an equivalence relation $E$ and an $n$-ary relation $R$ on a set $A$, $E$ is a *congruence relation* for $R$ on $A$, if $R(b_1, \ldots, b_n)$ is true whenever $R(a_1, \ldots, a_n)$ and $E(a_i, b_i)$ hold for $1 \leq i \leq n$. We will show that, similarity is a congruence relation on nodes. This will allow us to treat similarity as equality and simplify any further proofs.

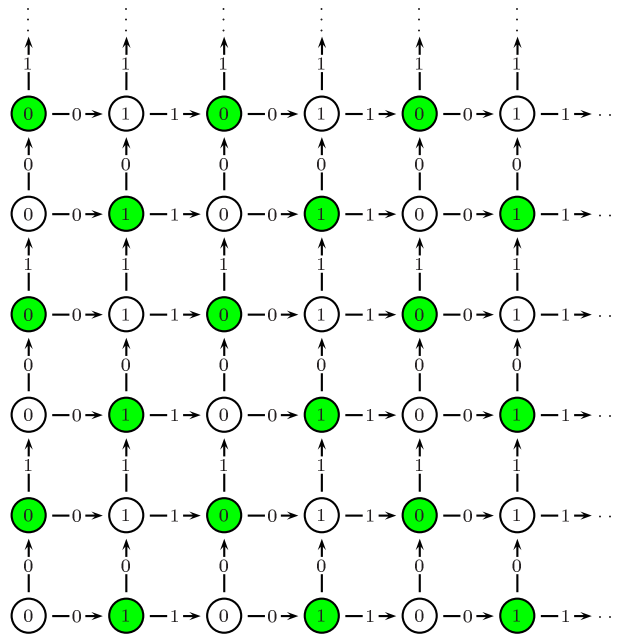

**Figure 2:** The grid structure. Diagonal nodes have the same color. In the horizontal direction the labels of nodes alternate between 0 and 1. In the vertical direction the colors of nodes alternate between black and white.

The intended meaning of the following formulas is best understood by examining Figure 2. When no confusion can arise, we use the relaxed notation

$$\forall(A_1 \vee \cdots \vee A_n \Rightarrow \varphi)$$

for the logically equivalent (guarded) formula

$$\forall(A_1 \Rightarrow \varphi) \wedge \cdots \wedge \forall(A_n \Rightarrow \varphi).$$

*Bottom* nodes have no vertical predecessors and all horizontal successors of bottom nodes are also bottom nodes, similarly for *left* nodes, for $i = 0, 1$:

$$\begin{aligned}&\forall x(\texttt{Bottom}(x) \Rightarrow (\neg\exists y(y \uparrow^0 x) \wedge \neg\exists y(y \uparrow^1 x))) \\ &\wedge \forall xy(x \xrightarrow{i} y \Rightarrow (\texttt{Bottom}(x) \Rightarrow \texttt{Bottom}(y))) \\ &\wedge \forall x(\texttt{Left}(x) \Rightarrow (\neg\exists y(y \xrightarrow{0} x) \wedge \neg\exists y(y \xrightarrow{1} x))) \\ &\wedge \forall xy(x \uparrow^i y \Rightarrow (\texttt{Left}(x) \Rightarrow \texttt{Left}(y)))\end{aligned} \tag{2}$$

All nodes are divided into black and white nodes with labels 0 and 1, and the following properties hold:

$$\begin{aligned}
&\forall x(\texttt{Node}(x) \Leftrightarrow (\texttt{White}(x) \vee \texttt{Black}(x))) \wedge \\
&\forall x(\texttt{White}(x) \Leftrightarrow (\texttt{White}_0(x) \vee \texttt{White}_1(x))) \wedge \\
&\forall x(\texttt{Black}(x) \Leftrightarrow (\texttt{Black}_0(x) \vee \texttt{Black}_1(x))) \wedge \\
&\forall x(\texttt{White}_0(x) \Rightarrow (\neg\texttt{White}_1(x) \wedge \neg\texttt{Black}(x))) \wedge \\
&\forall x(\texttt{White}_1(x) \Rightarrow (\neg\texttt{White}_0(x) \wedge \neg\texttt{Black}(x))) \wedge \\
&\forall x(\texttt{Black}_0(x) \Rightarrow (\neg\texttt{Black}_1(x) \wedge \neg\texttt{White}(x))) \wedge \\
&\forall x(\texttt{Black}_1(x) \Rightarrow (\neg\texttt{Black}_0(x) \wedge \neg\texttt{White}(x))) \wedge \\
&\exists x(\texttt{Origo}(x)) \wedge \\
&\forall x(\texttt{Origo}(x) \Rightarrow (\texttt{Left}(x) \wedge \texttt{Bottom}(x))) \wedge \\
&\forall x(\texttt{Bottom}(x) \Rightarrow (\texttt{Left}(x) \Rightarrow \texttt{Origo}(x))) \wedge \\
&\forall x(\texttt{Origo}(x) \Rightarrow \texttt{White}_0(x))
\end{aligned} \tag{3}$$

The colors and labels of nodes alternate between white and black, and 0 and 1 in both horizontal and vertical directions as follows. For $l \in \{0,1\}$, let $\bar{l} = 0$ if $l = 1$ and let $\bar{l} = 1$ if $l = 0$:

$$\forall xy(x \xrightarrow{l} y \Rightarrow ((\texttt{White}_l(x) \wedge \texttt{Black}_{\bar{l}}(y)) \vee (\texttt{Black}_l(x) \wedge \texttt{White}_{\bar{l}}(y)))) \tag{4}$$

$$\forall xy(x \uparrow^l y \Rightarrow ((\texttt{White}_l(x) \wedge \texttt{Black}_l(y)) \vee (\texttt{Black}_{\bar{l}}(x) \wedge \texttt{White}_{\bar{l}}(y)))) \tag{5}$$

Similar nodes have the same color and label:

$$\begin{aligned}
\forall xy(x \sim y \Rightarrow ((&\texttt{Black}_0(x) \wedge \texttt{Black}_0(y)) \vee \\
(&\texttt{Black}_1(x) \wedge \texttt{Black}_1(y)) \vee \\
(&\texttt{White}_0(x) \wedge \texttt{White}_0(y)) \vee \\
(&\texttt{White}_1(x) \wedge \texttt{White}_1(y))))
\end{aligned} \tag{6}$$

The labeling and the coloring is such that every node with a certain label and color has the following arcs connected to it:

$$\begin{aligned}
\forall x(\texttt{White}_0(x) \Rightarrow (&\exists y(x \xrightarrow{0} y) \wedge \exists y(x \uparrow^0 y) \wedge \\
&(\texttt{Bottom}(x) \vee \exists y(y \uparrow^1 x)) \wedge \\
&(\texttt{Left}(x) \vee \exists y(y \xrightarrow{1} x)))
\end{aligned} \tag{7}$$

$$\begin{aligned}
\forall x(\texttt{White}_1(x) \Rightarrow (&\exists y(x \xrightarrow{1} y) \wedge \exists y(x \uparrow^1 y) \wedge \\
&\exists y(y \uparrow^0 x) \wedge \exists y(y \xrightarrow{0} x)))
\end{aligned} \tag{8}$$

$$\begin{aligned}
\forall x(\texttt{Black}_0(x) \Rightarrow (&\exists y(x \xrightarrow{0} y) \wedge \exists y(x \uparrow^1 y) \wedge \\
&\exists y(y \uparrow^0 x) \wedge \\
&(\texttt{Left}(x) \vee \exists y(y \xrightarrow{1} x))))
\end{aligned} \tag{9}$$

$$\begin{aligned}
\forall x(\texttt{Black}_1(x) \Rightarrow (&\exists y(x \xrightarrow{1} y) \wedge \exists y(x \uparrow^0 y) \wedge \\
&(\texttt{Bottom}(x) \vee \exists y(y \uparrow^1 x)) \wedge \\
&\exists y(y \xrightarrow{0} x)))
\end{aligned} \tag{10}$$

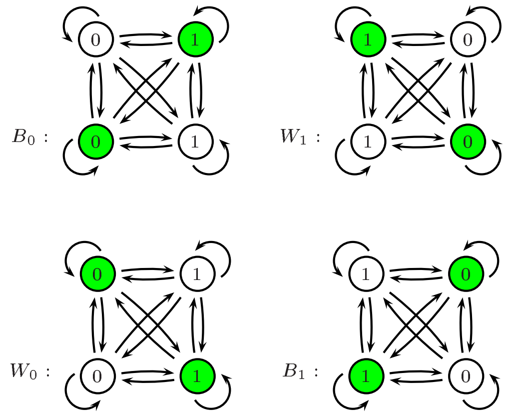


**Figure 3:** The relations $W_0$, $W_1$ $B_0$, and $B_1$.

Note that all nodes with label $l$ have an outgoing horizontal $l$-arc. We say that the arc relations *induce a diagonal* if, whenever $a \rightarrow b \uparrow c$ and $a \uparrow d \rightarrow c'$ then $c \sim c'$, where $\rightarrow$ is either $\xrightarrow{0}$ or $\xrightarrow{1}$ and $\uparrow$ is either $\uparrow^0$ or $\uparrow^1$.

We say that an arc relation $R$ is *functional in both arguments up to similarity* if the following conditions hold for all nodes $a$, $a'$, $b$, and $b'$:

- if $a \sim a'$, $R(a,b)$, and $R(a',b')$ then $b \sim b'$, and
- if $b \sim b'$, $R(a,b)$, and $R(a',b')$ then $a \sim a'$.

For each of the four transitive relations $W_0, W_1, B_0, B_1$ we have the following formulas, the purpose of which is to ensure that: similarity is a congruence for the arc relations on nodes (Lemma 2); the arc relations are functional up to similarity in both arguments (Lemma 3); the arc relations induce a diagonal (Lemma 4). For $l = 0, 1$:

$$\begin{aligned}
\forall xy(W_l(x,y) \Leftarrow (&x \sim y \vee x \xrightarrow{l} y \vee y \xrightarrow{l} x \vee \\
&x \uparrow^l y \vee y \uparrow^l x))
\end{aligned} \tag{11}$$

$$\begin{aligned}
\forall xy(W_l(x,y) \Rightarrow (&x \sim y \vee x \xrightarrow{l} y \vee y \xrightarrow{l} x \vee \\
&x \uparrow^l y \vee y \uparrow^l x \vee \\
&(\texttt{White}_0(x) \wedge \texttt{White}_1(y)) \vee \\
&(\texttt{White}_1(x) \wedge \texttt{White}_0(y)) \vee \\
&(\texttt{Black}_0(x) \wedge \texttt{Black}_1(y)) \vee \\
&(\texttt{Black}_1(x) \wedge \texttt{Black}_0(y))))
\end{aligned} \tag{12}$$

Intuitively, $W_l$ is an equivalence relation between all nodes that are connected in a rectangle where all arcs have label $l$. The lower left corner of such a rectangle is always a *white* node with *label* $l$. (See Figure 3.)

The formulas for $B_0$ and $B_1$ have a similar structure. For $l = 0, 1$, let $\bar{l} = 0$ if $l = 1$ and let $\bar{l} = 1$ if

$l = 0$:

$$\forall xy(B_l(x,y) \Leftarrow (x \sim y \vee x \xrightarrow{l} y \vee y \xrightarrow{l} x \vee x \uparrow^{\bar{l}} y \vee y \uparrow^{\bar{l}} x)) \quad (13)$$

$$\forall xy(B_l(x,y) \Rightarrow (x \sim y \vee x \xrightarrow{l} y \vee y \xrightarrow{l} x \vee x \uparrow^{\bar{l}} y \vee y \uparrow^{\bar{l}} x \vee (\texttt{White}_0(x) \wedge \texttt{White}_1(y)) \vee (\texttt{White}_1(x) \wedge \texttt{White}_0(y)) \vee (\texttt{Black}_0(x) \wedge \texttt{Black}_1(y)) \vee (\texttt{Black}_1(x) \wedge \texttt{Black}_0(y)))) \quad (14)$$

Intuitively, the nodes that are equivalent in $B_l$ correspond to corners of rectangles with lower left corner being a *black* node with *label* $l$. Finally, for each unary predicate $P$ and binary predicate $R$ above, we have the following formulas:

$$\forall x(P(x) \Rightarrow \texttt{Node}(x)) \quad (15)$$

$$\forall xy(R(x,y) \Rightarrow (\texttt{Node}(x) \wedge \texttt{Node}(y))) \quad (16)$$

Thus all elements that satisfy any guard are nodes. In addition, we add the following formula for all unary predicates $P$, to enforce that $\sim$ is a congruence for $P$ on nodes.

$$\forall xy(x \sim y \Rightarrow (P(x) \Rightarrow P(y))) \quad (17)$$

We now prove the following lemmas corresponding to the three properties mentioned above.

**Lemma 2** *$\sim$ is a congruence relation on nodes in all models of* $\text{Trans}[W_0, W_1, B_0, B_1, \sim] \wedge \texttt{GRID}$.

**Proof.** Consider a model of GRID (in the language of GRID). We must prove that $\sim$ is a congruence on nodes for all relations in that model. This is trivially so for all unary relations by (17). For each binary relation $R$ we must prove:

> For all nodes $a, a', b, b'$, if $R(a', b')$, $a \sim a'$ and $b \sim b'$ then $R(a, b)$.

For the binary relations $W_0$, $W_1$, $B_0$ and $B_1$ this holds by transitivity of these relations and the fact that they include similarity by (11) and (13). We prove the statement for $\xrightarrow{0}$ only. The proofs for the relations $\xrightarrow{1}$, $\uparrow^0$ and $\uparrow^1$ are symmetrical.

Assume $a \sim a' \xrightarrow{0} b' \sim b$. We prove that $a \xrightarrow{0} b$. From (11) follows that $W_0(a, a')$, $W_0(a', b')$, and $W_0(b', b)$, and thus $W_0(a, b)$ by transitivity. From (6) follows that the colors and labels of $a$ and $a'$ coincide, and the same holds for $b$ and $b'$. From (4) follows then that, either (i) $a$ is white and 0 and $b$ is black and 1, or (ii) $a$ is black and 0 and $b$ is white and 1.

In either case $a \not\sim b$ by the disjointness of white and black nodes and (6). From (12) follows that either: $a \xrightarrow{0} b$, $b \xrightarrow{0} a$, $a \uparrow^0 b$, or $b \uparrow^0 a$. From (4) follows that if $b \xrightarrow{0} a$ then $b$ has label 0. From (5) follows that if either $a \uparrow^0 b$ or $b \uparrow^0 a$ then $a$ and $b$ have the same label. But these cases would contradict both (i) and (ii). Hence, $a \xrightarrow{0} b$. ⊠

**Lemma 3** *The arc relations are functional in both arguments up to similarity, in all models of* $\texttt{GRID} \wedge \text{Trans}[W_0, W_1, B_0, B_1, \sim]$.

**Proof.** Consider a model of GRID. By Lemma 1 we may assume that all elements are nodes. Then, by Lemma 2, we may assume that all similar elements are identical. Consider the arc relation $\xrightarrow{0}$ again. The proof for the other arc relations is symmetrical. First we prove that for all nodes $a$, $b$ and $c$:

> If $a \xrightarrow{0} b$ and $a \xrightarrow{0} c$ then $b = c$.

Assume that $a \xrightarrow{0} b$ and $a \xrightarrow{0} c$. By (4), $b$ and $c$ have the same color and label. By (11) and transitivity of $W_0$, $W_0(b, c)$. Hence, by (12) $b = c$. Note that none of the other cases are possible because $b$ and $c$ have the same color and the same label. Functionality in the other direction is proved analogously. ⊠

**Lemma 4** *The arc relations induce a diagonal in all models of* $\texttt{GRID} \wedge \text{Trans}[W_0, W_1, B_0, B_1, \sim]$.

**Proof.** Consider a model of GRID. Assume, by using Lemma 1 and 2, that all elements are nodes and similarity is identity. Let $a$ be a white node with label 0. Then we have, by (7–10) and Lemma 3, unique nodes $b, b', c, c'$ such that $a \uparrow^0 b \xrightarrow{0} c$ and $a \xrightarrow{0} b' \uparrow^0 c'$. By (4) and (5) $c$ and $c'$ are white nodes with label 1. By (11) and transitivity of $W_0$, $W_0(c, c')$ holds. Hence, by (12), $c = c'$. The proofs of the other three cases are analogous. ⊠

## 2.2 Reduction from Minsky Machines

Given a Minsky (two-counter) machine $M$ with an empty input string, we construct a formula $\varphi_M$ in the guarded fragment with two variables, using the arc predicates and some unary predicates, such that $\texttt{GRID} \wedge \varphi_M$ is unsatisfiable if and only if $M$ halts. The execution of a Minsky machine can be viewed as walking in the grid. The starting point is the origo, and for example, incrementing the first counter by one means taking a step to the right, and decrementing the second counter by one means taking a step downwards. Checking whether one of the counters is 0 or not amounts to checking whether or not the current position is on one of the borders.

For each state $q$ of $M$ we have a new unary predicate $P_q$. The formula $\varphi_M$ is a conjunction of formulas (18–21) (and some additional ones for symmetrical cases) and formula (17) for all $P_q$ (to ensure that similarity is a congruence for all $P_q$).

The initial state of $M$ is $q_0$ and the final state of $M$ is $q_f$. Initially, the position of $M$ is origo:

$$\forall x(\texttt{Origo}(x) \Rightarrow P_{q_0}(x)) \tag{18}$$

For each transition $\delta(q,m,n) = (p,m+1,n)$ of $M$, i.e., in state $q$, $M$ increments the first counter and enters state $p$, there is a formula for $l \in \{0,1\}$:

$$\forall xy(y \xrightarrow{l} x \Rightarrow (P_q(y) \Rightarrow P_p(x))) \tag{19}$$

For each transition $\delta(q,0,n) = (p,0,n)$, i.e., in state $q$ $M$ checks whether the first counter is zero and enters state $p$ if so, there is a formula:

$$\forall x(P_q(x) \Rightarrow (\texttt{Left}(x) \Rightarrow P_p(x))) \tag{20}$$

For checking non-zero, $\texttt{Left}(x)$ in (20) is simply replaced by $\neg\texttt{Left}(x)$. The corresponding formulas with respect to the second counter use $\texttt{Bottom}$ and $\uparrow^l$. Finally, we add the formula that the final state is not reachable.

$$\neg\exists x(P_{q_f}(x)) \tag{21}$$

We can now prove the following lemma.

**Lemma 5** *$M$ does not halt if and only if* $\texttt{GRID} \wedge \varphi_M \wedge \mathrm{Trans}[W_0, W_1, B_0, B_1, \sim]$ *is satisfiable.*

**Proof.** Assume $M$ does not halt. Consider a structure with universe $\omega\times\omega$, where $(0,0)$ is the origo, horizontal arcs connect $(m,n)$ with $(m+1,n)$, and vertical arcs connect $(m,n)$ with $(m,n+1)$ for all $m,n \in \omega$, and similarity is equality. Obviously, such a structure can be expanded to a model of $\texttt{GRID}$. Expand it further to a structure $A$, by letting the $P_q$'s be the minimal subsets of $\omega\times\omega$ that satisfy the formulas (18–20). Now $P_{q_f}$ is empty, because $M$ does not halt. Hence, $A$ is a model of $\texttt{GRID} \wedge \varphi_M$.

Conversely, assume that $\texttt{GRID} \wedge \varphi_M$ has a model $A$. By Lemma 1 and Lemma 2 we may assume that all elements are nodes and that similarity is equality. By Lemma 3 and Lemma 4, we may assume that $\omega \times \omega$ is a subset of the universe of $A$, where $(0,0)$ is an origo and where $a \xrightarrow{0} b$ or $a \xrightarrow{1} b$ if and only if $a = (m,n)$ and $b = (m+1,n)$, and $a \uparrow^0 b$ or $a \uparrow^1 b$ if and only if $a = (m,n)$ and $b = (m,n+1)$. So, the restriction of $A$ to $\omega \times \omega$ is a substructure of $A$ that satisfies $\texttt{GRID}$ and is thus also a model of $\varphi_M$ (because $\varphi_M$ is equivalent to a universal sentence). Hence, $M$ does not halt. ⊠

As a consequence, we obtain the following result, improving the undecidability result by Grädel [1998*b*], of $\mathrm{GF}^3 + \mathrm{Trans}[R_1, R_2]$, with respect to the number of variables and by omitting equality.

**Theorem 1** *The satisfiability problem for* $\mathrm{GF}^2_- + \mathrm{Trans}[R_1, \ldots, R_5]$ *is undecidable.*

All the arc relations are trivially transitive, consider for example $\xrightarrow{0}$: there are no nodes $a$, $b$, and $c$, such that $a \xrightarrow{0} b \xrightarrow{0} c$. We therefore get the following result. We write $\mathrm{Trans}[all]$ to denote the statement that all non-unary relations are transitive binary relations.

**Theorem 2** *The satisfiability problem for* $\mathrm{GF}^2_- + \mathrm{Trans}[all]$ *is undecidable.*

The undecidability results for the above classes of formulas may be improved to *strong* undecidability results, by encoding certain domino problems (instead of Minsky machines) as in [Grädel 1998*b*], implying that even the *finite satisfiability* problem for these formula classes is undecidable. The main reason why we have chosen to use Minsky machines, although at the price of not obtaining this stronger result, is the more elementary nature, and the conceptual simplicity of Minsky machines.

## 2.3 The Loosely Guarded Fragment with One Transitive Relation

In the *loosely guarded fragment* or LGF, the concept of a guard for relativizing quantification is relaxed to a conjunction of atoms which contains all the free variables $\bar{x}$ of the body of the quantification such that each pair of variables in $\bar{x}$ occurs together among the arguments of one of the atoms in the guard.[2] That is, a formula such as $\forall xyz\,(A(x,y) \wedge B(y,z) \wedge S(x,z) \Rightarrow C(x,z))$ is loosely guarded while the transitivity clause $\forall xyz\,(A(x,y) \wedge A(y,z) \Rightarrow A(x,z))$ is not—the pair $x,z$ does not occur together in one of the negative literals. The loosely guarded fragment with equality is decidable, even by syntactic methods based on superposition [Ganzinger & De Nivelle 1999].

For the LGF the presence of just a single transitive relation causes undecidability. We show this by

[2]This definition of LGF admits less formulas but is essentially the same as the the definition in [van Benthem 1997].

reduction from the intersection emptiness problem for context-free languages [Hopcroft & Ullman 1979].

Consider two context-free grammars in Chomsky normal form, with disjoint sets of nonterminals, start symbols $S_1$ and $S_2$, respectively, and common terminal symbols $\mathtt{a}$ and $\mathtt{b}$. The rules of the grammars are of one of the three forms $A{::=}BC$, $A{::=}\mathtt{a}$ or $A{::=}\mathtt{b}$, respectively, with nonterminals $A$, $B$, and $C$. We construct the following formula in LGF where the indices of the conjunctions range over all rules of the two grammars and $\mathtt{Suffix}$ is intended to be a transitive relation denoting the suffix property between strings:

$$\begin{array}{l}\forall xy(\mathtt{Suffix}(x,y) \Rightarrow (\mathtt{String}(x) \wedge \mathtt{String}(y)))\\ \wedge\, \forall x(\mathtt{String}(x) \Rightarrow (\,\mathtt{Suffix}(x,x) \wedge \\ \qquad \exists x_{\mathtt{a}}(\,\mathtt{Suffix}(x,x_{\mathtt{a}}) \wedge \bigwedge_{A::=\mathtt{a}} A(x_{\mathtt{a}},x)) \wedge \\ \qquad \exists x_{\mathtt{b}}(\,\mathtt{Suffix}(x,x_{\mathtt{b}}) \wedge \bigwedge_{A::=\mathtt{b}} A(x_{\mathtt{b}},x))))\\ \bigwedge_{A::=BC} \forall xyz\,((B(x,y) \wedge C(y,z) \\ \qquad \wedge\, \mathtt{Suffix}(z,x)) \Rightarrow A(x,z))\\ \wedge\, \exists x_\epsilon\,(\mathtt{String}(x_\epsilon) \wedge \neg\exists y(S_1(y,x_\epsilon) \wedge S_2(y,x_\epsilon)))\end{array}$$

Clauses $\forall x,y,z(B(x,y) \wedge C(y,z) \wedge \mathtt{Suffix}(z,x) \Rightarrow A(x,z))$ represent the rule $A{::=}BC$ in an encoding with difference lists: the string $x \setminus z$ is derivable from $C$, if there is a string $y$ such that $x \setminus y$ is derivable from $A$ and $y \setminus z$ is derivable from $B$. To make these clauses loosely guarded, the additional (logically redundant) guard $\mathtt{Suffix}(z,x)$ is added, requiring that $z$ be a suffix of $x$. After Skolemization, the formula has a Herbrand model (over a constant $\epsilon$ and two unary functions $a$ and $b$ for $x_\epsilon$, $x_{\mathtt{a}}$, and $x_{\mathtt{b}}$, respectively) if and only if the intersection of the languages generated by the two grammars is empty.

**Theorem 3** *The LGF without equality is undecidable if one binary relation is transitive.*

## 3 Decidability Results

Recall that a guarded formula is called *monadic*, when every occurrence of every non-unary atom in it is a guard. When encoding the Kripke semantics of multi-modal propositional logics with modalities of the type K4 in first-order logic, one ends up in monadic $\mathrm{GF}^2_-$ with transitive relations. The formula in Example 1 is in monadic $\mathrm{GF}^2_-$, which shows that monadic $\mathrm{GF}^2_-$ is a nontrivial extension of the *modal fragment*[3], because the modal fragment retains the finite model property under extensions like transitivity. This raises the question as to whether monadic $\mathrm{GF}^2$ with transitive relations is decidable. This question is answered positively in this section, by proving a more general result (Theorem 4).

For the decidability proof of monadic $\mathrm{GF}^2$ with transitive relations $R_1,\ldots,R_n$, we consider satisfiability of closed formulas of the form $\varphi_{\mathrm{GF}} \wedge \varphi_{\mathrm{CC}}$ where $\varphi_{\mathrm{GF}}$ is in monadic $\mathrm{GF}^2$ and $\varphi_{\mathrm{CC}}$ is a universal formula consisting of the congruence axioms for $\approx$ and the transitivity axioms for $R_1,\ldots,R_n$. We use the fact that $\varphi_{\mathrm{GF}} \wedge \varphi_{\mathrm{CC}}$ is satisfiable in FOL with equality if and only if $\varphi_{\mathrm{GF}} \wedge \varphi_{\mathrm{CC}}$ is satisfiable in FOL without equality which in turn is the case if and only if its Skolemized form $N \wedge \varphi_{\mathrm{CC}}$ has a Herbrand model. The clausal normal form $N$ of $\varphi_{\mathrm{GF}}$ can be constructed in such a way that the clauses in $N$ are *monadic*: the arity of all function symbols as well as the number of distinct variables in any positive literal is $\leq 1$. From Example 1 we obtain

$$\{0 < 1, \quad \neg(x < y) \vee y < f(y), \quad \neg x < x\}, \qquad (22)$$

where 0 and 1 are new constants, and $f$ is a new unary function symbol, as a clausal normal form.

In our proof we will replace satisfiability of $N \wedge \varphi_{\mathrm{CC}}$ by satisfiability of $N$ in Herbrand interpretations with certain closure constraints that are derived from $\varphi_{\mathrm{CC}}$. A *closure operator* for $n$-ary relations over a domain $A$ is a function $C$ on the power set of $A^n$, such that, for all $R, R' \subseteq A^n$,

1. $R \subseteq C(R)$ ($C$ is *increasing*),
2. if $R \subseteq R'$ then $C(R) \subseteq C(R')$ ($C$ is *monotone*),
3. $C(R) = C(C(R))$ ($C$ is *idempotent*).

Let $E$ be an equivalence relation on $A$ and ler $R$ be a relation on $A$. The *$E$-closure* of $R$, denoted by $E(R)$, is the least $R'$ that includes $R$ such that $E$ is a congruence relation for $R'$ (on $A$). Clearly, the $E$-closure operator (also denoted by $E$) is indeed a closure operator. We are particularly interested in closure operators $C$, such that for all equivalence relations $E$, the composition $E \circ C \circ E$, denoted by $C^{(E)}$, is also a closure operator, hence in particular idempotent. Closure operators $C$ which enjoy this property are said to be *compatible with equivalences*.

[3]The image of multi-modal propositional formulas $\varphi$ under the translation $\varphi^x$: for a propositional constant $P$, $P^x$ is $P(x)$, $(\varphi \wedge \psi)^x$ is $\varphi^x \wedge \psi^x$ (similarly for other connectives), and $(\Box_i \varphi)^x$ is $\forall y(R_i(x,y) \Rightarrow \varphi^y)$.

From now on we assume that every relation symbol $R$ (other than $\approx$) is associated with a closure operator $C_R$ that is compatible with equivalences. More specifically, given $N \wedge \varphi_{\mathrm{CC}}$, if $R$ is one of the transitive $R_i$'s then $C_R$ is the transitive closure operator, otherwise $C_R$ is the trivial closure operator *ID* which is the identity on every relation. These closure operators are, in fact, compatible with equivalences. We say that a Herbrand structure $A$ *satisfies* the *closure constraints* derived from the $C_R$, if $\approx^A$ is an equivalence relation and $C_R^{(\approx^A)}(R^A) = R^A$ for every other relation symbol $R$. Clearly for $N \wedge \varphi_{\mathrm{CC}}$, the closure constraints are satisfied in $A$ if and only if $\varphi_{CC}$ is true in $A$.

Our main technical result is that the satisfiability problem of monadic clauses in Herbrand structures with closure constraints is decidable for certain types of closure constraints. The decidability proof is by reduction to *SkS* and, intuitively, the admissible closure constraints are those that can be expressed through monadic second-order formulas in *SkS* including transitivity and Euclideanness[4]. In the following let $\Sigma$ be a fixed finite signature with function symbols of arity at most 1.

### 3.1 The theory *SkS*

The *tree* here is defined as the term algebra of $\Sigma$ with empty basis, i.e., whose universe is the set of all ground $\Sigma$-terms with each function symbol having the Herbrand interpretation. We write $\mathcal{T}$ or $\mathcal{T}_\Sigma$ both for the tree and its universe. The elements of the tree are called *nodes*.

The formal equality symbol in *SkS* will be denoted by $\doteq$. The set of *monadic second-order* or *mso* formulas of $\Sigma$ includes all atomic formulas $s \doteq t$ and $X(s)$, where $s$ and $t$ are terms and $X$ is a unary *set variable*. The set of *mso* formulas is closed under the logical connectives, the first-order quantifiers over individual variables ($\exists x$ and $\forall x$), and the second-order quantifiers over the set variables ($\exists X$ and $\forall X$). An atom $s \doteq t$ is true in the tree if and only if $s$ and $t$ denote the same node, i.e., $s$ and $t$ are identical terms. The truth value of an arbitrary formula with parameters is defined as usual, e.g., $\forall X \varphi$ is true in the tree if and only if $\varphi$ is true in the tree for all sets $X$ of nodes. Let $(z_i)_{i \geq 1}$ be a fixed enumerable sequence of first-order variables. Given an mso formula $\varphi(z_1, \dots, z_n)$, we let $[\![\varphi(z_1, \dots, z_n)]\!]$ denote the set of all tuples of nodes $(a_1, \dots, a_n)$ such that $\varphi(a_1, \dots, a_n)$ holds in the tree. Hence, every mso formula $\varphi(z_1, \dots, z_n)$ *defines* an $n$-ary relation $[\![\varphi]\!]$ over the nodes. The formula $\varphi$ may include parameters that are free set variables (but, without loss of generality, no free individual variables besides the $z_i$'s), so that the interpretation of the parameters and, hence, the relation $[\![\varphi]\!]$, is dependent on the context. A relation that can be defined by an mso formula is said to be *mso*. Given an mso formula $\psi$ that defines an equivalence relation, it is easy to see that the following mso formula defines the $[\![\psi]\!]$-closure of $[\![\varphi]\!]$:

$$\exists x_1 \cdots x_n ((\bigwedge_{i=1}^{n} \psi(x_i, z_i)) \wedge \varphi(x_1, \dots, x_n))$$

Note that this holds *uniformly* (for all interpretations of the parameters).

The theory *SkS* is the *monadic second-order theory of the tree*, i.e., the set of all mso sentences that are true in the tree. The decidability of *SkS* is known as *Rabin's Tree Theorem* [Rabin 1969].

### 3.2 Reduction to *SkS*

We are interested in closure properties that can be expressed as mso formulas. We write $\varphi[\cdot_1, \cdots, \cdot_k]$ to denote a *formula context* (i.e., a formula where some subformulas are missing and occur as placeholders $\cdot_i$ for some $i$) and $\varphi[\varphi_1, \cdots, \varphi_k]$ denotes the formula that is obtained by simultaneously replacing all occurrences of $\cdot_i$ in $\varphi[\cdot_1, \cdots, \cdot_k]$ by $\varphi_i$.

Given a closure operator $C$ over $n$-ary relations, we say that $C$ is *mso* if there exists an mso formula context $\overline{C}[\cdot]$ such that, for all mso formulas $\varphi$, $[\![\overline{C}[\varphi]]\!] = C([\![\varphi]\!])$ holds uniformly. The formula context $\overline{C}[\cdot]$ is said to *define* $C$. For example, the trivial closure operator *ID* is defined by the empty context $\overline{ID}[\cdot] = \cdot$. Note that if a closure operator $C_R$ is mso, so is $C_R^{(E)}$, for any mso equivalence $E$. The following lemma shows the well-known facts how to define closure operators for the usual closure properties.

**Lemma 6** *The following closure properties are mso: transitivity, reflexivity + transitivity, reflexivity + symmetry + transitivity, and Euclideanness*

**Proof.** Let $\varphi(z_1, z_2)$ be an mso formula that defines a binary relation. Consider the mso formula $\varphi^*(z_1, z_2)$:

$$\forall X (X(z_1) \wedge \forall xy (X(x) \wedge \varphi(x, y) \Rightarrow X(y)) \Rightarrow X(z_2))$$

It is easy to see that $\varphi^*$ defines the transitive and reflexive closure of $[\![\varphi]\!]$. A formula $\varphi^+$ that defines just the transitive closure of $[\![\varphi]\!]$ is obtained easily by using

[4] A binary relation $R$ is *Euclidean* if $\forall xyz(R(x,y) \wedge R(x,z) \Rightarrow R(y,z))$. In epistemic logics, Euclideanness of the accessibility relation corresponds to *negative introspection* that is usually stated as the modal axiom $\neg\Box\phi \Rightarrow \Box\neg\Box\phi$ (if you *don't know* $\phi$ then you *know* that you *don't know* $\phi$). See [Fagin et al. 1995].

$\varphi$ and $\varphi^*$. The Euclidean closure of $[\![\varphi]\!]$ is defined by the formula:

$$\begin{aligned}&\varphi(z_1, z_2) \vee \\ &(\exists z \varphi(z, z_1) \wedge \\ &\ \forall X(X(z_1) \wedge \text{“}X \text{ is e-closed”} \Rightarrow X(z_2))),\end{aligned}$$

where “$X$ is e-closed” says that two nodes are in $X$ whenever they can be reached from a common node via one or more $[\![\varphi]\!]$-steps:

$$\forall xy(X(x) \wedge \exists z(\varphi^+(z,x) \wedge \varphi^+(z,y)) \Rightarrow X(y)).$$

A formula that defines the reflexive + symmetric + transitive closure of $[\![\varphi]\!]$ is a simple modification of the formula $\varphi^*$. ⊠

We will write $RST$ for the reflexive, symmetric, and transitive closure operator and $\overline{RST}[\cdot]$ for a defining mso formula context. The main result of this section is the following theorem.

**Theorem 4** *The satisfiability problem for finite sets of monadic clauses over Herbrand structures with closure constraints where the closure operators are mso definable is decidable.*

**Proof.** Let $N$ be a finite set of monadic clauses and consider the class of Herbrand structures for the language of $N$. We will effectively construct a closed mso formula $MSO[N]$ that is true in the tree if and only if $N$ has a Herbrand model that satisfies the closure constraints.

For each predicate $P$ in $N$ (including $\approx$), say of arity $n$, we first collect all the positive occurrences of $P$ into a formula $\varphi_P$ as follows. Let $P(\vec{t_1}), \ldots, P(\vec{t_m})$ (where $\vec{t_i} = t_{i1}, \ldots, t_{in}$) be a sequence of all the positive $P$-literals in $N$. We may assume that $m \geq 1$. We write $\vec{t_i}[s]$ to denote the result of replacing the variable (if any) in $\vec{t_i}$ by the term (or node) $s$. For each atom $\alpha$ above, let $X_\alpha$ be a new set variable. Let $\varphi_P(z_1, \ldots, z_n)$ stand for the mso formula

$$\bigvee_{i=1}^{m} \exists z(X_{P(\vec{t_i})}(z) \wedge z_1 \doteq t_{i1}[z] \wedge \cdots \wedge z_n \doteq t_{in}[z])$$

where $z$ is a new first-order variable.

Let $\psi_\approx$ be $\overline{RST}[\varphi_\approx]$, hence, $[\![\psi_\approx]\!]$ is the equivalence closure of $[\![\varphi_\approx]\!]$ for any interpretation of the set variables. (Note that if $[\![\varphi_\approx]\!]$ is empty, e.g., when there are no positive occurrences of $\approx$ in $N$, then, by reflexivity, $[\![\psi_\approx]\!]$ is simply the identity relation.)

For every other predicate symbol $P$, by exploiting the mso definability of $C_P$, and hence of $C_P^{(E)}$, we first construct an mso formula context $\overline{C_{\approx,P}}[\cdot_1, \cdot_2]$ such that, for any interpretation of the free set variables in $\psi_\approx$, $\overline{C_{\approx,P}}[\psi_\approx, \cdot_2]$ defines the closure operator $C_P^{([\![\psi_\approx]\!])}$. Let $\psi_P$ denote the mso formula $\overline{C_{\approx,P}}[\psi_\approx, \varphi_P]$. Hence,

$$[\![\overline{C_{\approx,P}}[\psi_\approx, \varphi_P]]\!] = C_P^{([\![\psi_\approx]\!])}([\![\varphi_P]\!]).$$

For each clause $\chi = \bigvee_{i \in I} \alpha_i$ in $N$, let

$$MSO[\chi] = \bigvee_{i \in I} MSO[\alpha_i],$$

where

$$MSO[\alpha] = \begin{cases} X_\alpha(x), & \text{if } \alpha \text{ is a non-ground atom containing } x; \\ \exists z X_\alpha(z), & \text{if } \alpha \text{ is a ground atom;} \\ \neg\psi_P(\vec{t}), & \text{if } \alpha \text{ is a literal } \neg P(\vec{t}). \end{cases}$$

Finally, let

$$MSO[N] = \exists \vec{X} \forall \vec{x} \bigwedge_{\chi \in N} MSO[\chi],$$

where $\vec{X}$ contains all the free set variables in the conjunction and $\vec{x}$ contains all the free individual variables in the conjunction. In the following we prove that $MSO[N]$ is true in the tree if and only if $N$ has a Herbrand model satisfying the closure constraints.

**($\Leftarrow$)** Assume that $N$ has a Herbrand model $A$ satisfying the closure constraints. First, we define witnesses for the set variables in $\vec{X}$. For each ground positive literal $\alpha$ in $N$, let $X_\alpha$ be non-empty if and only if $\alpha$ holds in $A$. For each non-ground positive literal $\alpha = P(\vec{t})$ in $N$, let

$$X_\alpha = \{a \in \mathcal{T} : P(\vec{t}[a]) \text{ is true in } A\}.$$

From this definition and the definition of $\varphi_P$ it follows immediately that

$$[\![\varphi_P]\!] \subseteq P^A.$$

Secondly, consider a clause $\chi(\vec{x})$ in $N$ and a sequence $\vec{a}$ of nodes. We know that $\chi(\vec{a})$ holds in $A$. So, one literal $\alpha(\vec{a})$ of $\chi(\vec{a})$ is true in $A$. We prove that $MSO[\chi](\vec{a})$ is true in $\mathcal{T}$ by showing that $MSO[\alpha](\vec{a})$ is true. There are three cases: If $\alpha$ is a non-ground atom $P(\vec{t})$, then $\alpha$ includes a variable $x_i$ and $MSO[\alpha] = X_\alpha(x_i)$. Hence, $X_\alpha(a_i)$ is true in $\mathcal{T}$ by the definition of $X_\alpha$.

If $\alpha$ is a ground atom $P(\vec{t})$, then $X_\alpha$ is non-empty, and so $MSO[\alpha]$ is true in $\mathcal{T}$ by definition of $X_\alpha$.

Finally, if $\alpha$ is a negative literal $\neg P(\vec{t})$, then $MSO[\alpha] = \neg\psi_P(\vec{t})$. In order to show that $MSO[\alpha](\vec{a})$ is true in the tree, it is enough to show that $[\![\psi_P]\!] \subseteq P^A$. There are two subcases.

(i) Assume that $P = \approx$. So $[\![\psi_\approx]\!] = [\![\overline{RST}[\varphi_\approx]]\!] = RST([\![\varphi_\approx]\!])$. It follows from $[\![\varphi_\approx]\!] \subseteq \approx^A$ and the monotonicity of $RST$ that $[\![\psi_\approx]\!] \subseteq RST(\approx^A)$. But $RST(\approx^A) = \approx^A$.

(ii) Assume that $P \neq \approx$ and let $E = [\![\psi_\approx]\!]$. Hence, $[\![\psi_P]\!] = [\![\overline{C_{\approx,P}}[\psi_\approx, \varphi_P]]\!] = C_P^{(E)}([\![\varphi_P]\!])$. From the previous case we know that $E \subseteq \approx^A$, and thus, for all relations $R$, $E(R) \subseteq \approx^A(R)$. So, by $[\![\varphi_P]\!] \subseteq P^A$ and monotonicity of the closure operators,

$$C_P^{(E)}([\![\varphi_P]\!]) = E(C_P(E([\![\varphi_P]\!]))) \subseteq \approx^A(C_P(\approx^A(P^A))) = C_P^{(\approx^A)}(P^A) = P^A.$$

**(⇒)** Assume that $MSO[N]$ is true in the tree. Consider fixed witnesses for the set variables. We construct a Herbrand model $A$ that satisfies $N$ and the closure constraints. For every relation symbol $P$ in $N$, let $P^A = [\![\psi_P]\!]$. Let also $E = [\![\psi_\approx]\!]$.

To begin with, we show that the closure constraints are satisfied. First, consider $\approx$:

$$\approx^A = [\![\psi_\approx]\!] = [\![\overline{RST}[\varphi_\approx]]\!] = RST([\![\varphi_\approx]\!]) = RST(RST([\![\varphi_\approx]\!])) = RST(\approx^A).$$

Second, consider any $P$ other than $\approx$:

$$P^A = [\![\psi_P]\!] = [\![\overline{C_{\approx,P}}[\psi_\approx, \varphi_P]]\!] = C_P^{(E)}([\![\varphi_P]\!]) = C_P^{(\approx^A)}([\![\varphi_P]\!]) = C_P^{(\approx^A)}(C_P^{(\approx^A)}([\![\varphi_P]\!])) = C_P^{(\approx^A)}(P^A),$$

where we used the idempotency of $C_P^{(\approx^A)}$. It remains to show that $A$ satisfies all clauses in $N$. Let $\chi(x_1, \ldots, x_n)$ be a clause in $N$ and let $\vec{a} = a_1, \ldots, a_n$ be a sequence of nodes. We must show that $\chi(\vec{a})$ holds in $A$. We know that $MSO[\chi](\vec{a})$, and thus a disjunct $MSO[\alpha](\vec{a})$ of $MSO[\chi](\vec{a})$, is true in the tree. There are three cases:

Let $\vec{x} = x_1, \ldots, x_n$ and suppose that $\alpha(\vec{x})$ is a non-ground atom $P(\vec{t}[x_i])$ with the variable $x_i$, i.e., $\alpha(\vec{a}) = P(\vec{t}[a_i])$ and $MSO[\alpha](\vec{a}) = X_\alpha(a_i)$. Since $X_\alpha(a_i)$ holds in $\mathcal{T}$, it follows from the definition of $\varphi_P$ that $\varphi_P(\vec{t}[a_i])$ is true in $\mathcal{T}$. But $[\![\varphi_P]\!] \subseteq C([\![\varphi_P]\!])$ (where $C = RST$, if $P$ is $\approx$; $C = C_P^{(E)}$, otherwise), and $C([\![\varphi_P]\!]) = [\![\psi_P]\!] = P^A$. Hence $P(\vec{t}[a_i])$ holds in $A$.

Suppose that $\alpha$ is a ground atom. This case is similar to the previous one.

Finally, if $\alpha(\vec{a})$ is a negative literal $\neg P(\vec{t})$, then $MSO[\alpha](\vec{a}) = \neg\psi_P(\vec{t})$. Since $[\![\psi_P]\!] = P^A$, $\alpha(\vec{a})$ holds in $A$.

Hence, $\chi(\vec{a})$ is true in $A$, as was to be shown. ⊠

The following example illustrates the constructions in the proof of Theorem 4.

**Example 2** Consider the clause set (22). Then

$$\varphi_<(z_1, z_2) = \exists z(X_{y<s(y)}(z) \wedge z_1 \doteq z \wedge z_2 \doteq s(z)) \vee \exists z(X_{0<1}(z) \wedge z_1 \doteq 0 \wedge z_2 \doteq 1).$$

Let $TC$ be the transitive closure operator. In this case $[\![\psi_\approx]\!]$ is the identity relation and $TC = TC^{([\![\psi_\approx]\!])}$. By Theorem 4, the clause set (22) + Trans[<] is satisfiable if and only if the following formula is true in the tree:

$$\exists X_{0<1}\, \exists X_{y<s(y)}\, (\exists z(X_{0<1}(z)) \wedge \forall xy(\neg\overline{TC}[\varphi_<](x,y) \vee X_{y<s(y)}(y)) \wedge \forall x(\neg\overline{TC}[\varphi_<](x,x))).$$

**Theorem 5** *Satisfiability of monadic* GF$^2$ *with binary relations that are, possibly, transitive, reflexive + transitive, reflexive + symmetric + transitive, or Euclidean, is decidable.*

**Proof.** By using the fact that the corresponding closure constraints can be specified by a universal first-order formula, satisfiability of formulas in the given class reduces effectively to satisfiability of monadic clauses without equality in Herbrand structures with appropriate closure constraints, that are, by Lemma 6, mso definable. Hence, the claim follows from Theorem 4. ⊠

Note that, also many *non-monadic guarded* and even *non-guarded* formulas translate into monadic clauses via standard Skolemization, e.g., all guarded formulas in GF$^2$ where all positive occurrences of atoms that are not guards have at most one distinct variable, and all universal, purely negative disjunctions, such as $\forall xyz((R(x,y) \wedge R(y,z)) \Rightarrow \neg x \approx z)$.

## 4 Conclusions

In this paper we studied the guarded fragment restricted to two variables, GF$^2$. We showed that already GF$^2_-$ is undecidable when extended with transitive relations, improving a recent result of Grädel [1998*b*]. We also identified a so-called monadic subfragment of GF$^2$ (where all non-guard atoms are unary), that retains the robustness of modal logics under various extensions (such as transitivity), while being a nontrivial extension of the modal fragment. An open question at this time is the decidability of the whole GF with transitive relations where transitive relations are only admitted in guards, but where non-transitive relations and equality are allowed to occur everywhere. There are very few known decidable extensions of GF, one exception is the recent decidability result of the extension of GF with least and greatest fixed-points by Grädel & Walukiewicz [1999].

Recently, Hans de Nivelle showed[5] that S4 reduces to monadic $GF^2_-$. His reduction exploits the fact that, guarded formulas of the form $\forall xy R(x,y) \Rightarrow (P(x) \Rightarrow P(y))$ can be used to encode transitivity of $R$. The idea is similar to the construction of $\varphi^*$ in Lemma 6. Such results are relevant in the context of epistemic logics [Fagin et al. 1995] and in the context of knowledge representation, due to the connections to description logics [Grädel 1998*a*] and conceptual graphs [Baader et al. 1998]. Recently, Ganzinger & De Nivelle [1999] have designed a superposition theorem prover for GF and LGF. An open problem is the computational complexity of the monadic $GF^2$ with transitive relations.

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