A Simple Inductive Synthesis Methodology and its Applications

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Abstract

Given a high-level specification and a low-level programming language, our goal is to automatically synthesize an efficient program that meets the specification. In this paper, we present a new algorithmic methodology for inductive synthesis that allows us to do this.

We use Second Order logic as our generic high level specification logic. For our low-level languages we choose small application-specific logics that can be immediately translated into code that runs in expected linear time in the worst case.

We explain our methodology and provide examples of the synthesis of several graph classifiers, e.g, linear-time tests of whether the input graph is connected, acyclic, etc. In another set of applications we automatically derive many finite differencing expressions equivalent to ones that Paige built by hand in his thesis [Pai81]. Finally we describe directions for automatically combining such automatically generated building blocks to synthesize efficient code implementing more complicated specifications.

The methods in this paper have been implemented in Python using the SMT solver Z3 [dMB].

Categories and Subject Descriptors I.2.2 [*Artificial Intelligence*]: Automatic Programming—Program Synthesis; D.1.2 [*Programming Techniques*]: Automatic Programming; D.3.1 [*Programming Languages*]: Formal Definition and Theory—Semantics

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1. Introduction

We describe an algorithmic methodology that takes a high level specification and a low-level target language, and searches for an efficient implementation of the specification. If there is no such implementation then our methodology reports failure.

The input is a specification in an expressive logic. A naive syntax-directed translation of this specification into machine code usually produces an inefficient and sometimes exponential-time implementation. Instead, we automatically convert the input specification into an equivalent low-level specification in the target language. Using a simple syntax-directed translation, the resulting low-level specification is then converted into imperative code that is guaranteed run in expected linear time in the worst case.

The high level specifications are written in subsets of second-order logic (SO), second-order existential logic (SO$\exists$) which by Fagin's Theorem expresses exactly those properties checkable in NP, and first-order logic plus transitive closure (FO(TC)) which expresses exactly the properties checkable in NSPACE$[\log n]$ [Imm99].

We describe two simple target languages, both of which can express only linear-time properties, one for expressing graph properties, and another for expressing properties of sets.

Of course, our algorithm cannot succeed if there is no program in the target language that implements the specification in question. However, when there is such an implementation, we do succeed in a large number of cases.

In the first problem domain we automatically generate linear-time graph classifiers for many simple properties such as connectivity, acyclicity, etc. In the second domain, we automatically generate constant-time finite differencing expressions for many of the examples that Paige worked out by hand in his Ph.D. thesis [Pai81].

Our methods are fairly simple and general. We use the SMT solver Z3 to generate structures satisfying the specifications. Using these models we learn candidate implementations in the low-level languages that are correct on the generated structures. Then we use the solver repeatedly to see if there exist any other examples that satisfy the original specifications but for which the candidate implementation fails. When there are no further counterexamples, we are done.

We feel that it is surprising how well this simple method works to find asymptotically optimal algorithms which are implied by the high-level specifications, but definitely not obvious from the specifications. Indeed a naive implementation of the high-level specifications would lead to very inefficient algorithms.

While many people have tried to do automatic synthesis, much of our inspiration comes from the work of Bob Paige and Jack Schwartz on automatically generating efficient data structures and algorithms for specifications written in the very high-level programming language SETL [Pai81]. An important analogy of our work is the automatic optimization of SQL, in which the specification in SQL (essentially first-order logic plus counting, FO(COUNT) [Imm99, Thm 14.9]) is transformed into a logically equivalent specification, also in SQL but with better run time. In our setting, since the specification language is SO – a language exponentially more powerful than FO – the possible transformations include a much wider range of possibilities and difficulties.

This paper is organized as follows: §2 provides background and definitions. §3 explains our synthesis methodology at a high level. In §4 we explain the details of an implementation of the methodology. §5 shows our first applications: learning efficient code to check whether input graphs have certain properties. In §6 we automatically generate numerous examples of finite differencing code equivalent to code that Paige worked out by hand in his Ph.D. thesis [Pai81]. In §7 we briefly discuss how we generate program code from the synthesized low-level specifications. In §8 we discuss some related work. In §9 we conclude and suggest some future directions of this work.

2. Background

We use standard notation from mathematical logic. For background in descriptive complexity we recommend [Imm99]. All our logical structures are finite and ordered. For example, an ordered graph, G, is a finite logical structure: $G = ([n], E)$, whose universe of vertices is the set $|G| = [n] = \{0, 1, \dots n-1\}$, and whose edge relation is a subset of the set of ordered pairs of vertices: $E \subseteq |G|^2$. In this case the vocabulary of G consists of a single binary relation symbol, $\sigma = \{E^2\}$, which we sometime describe as an input relation symbol.

We use STRUC$[\sigma]$ to denote the set of all finite, ordered structures of vocabulary σ, and STRUC$_{\leq k}[\sigma]$ denotes the subset of these structures with universe size at most k. If $\sigma' \supset \sigma$ is a larger vocabulary, and if $\mathcal{A} \in$ STRUC$[\sigma]$ and $\mathcal{A}' \in$ STRUC$[\sigma']$ is identical to $\mathcal{A}$ except that it also interprets the symbols of $\sigma' - \sigma$ then we say that $\mathcal{A}'$ is an *expansion* of $\mathcal{A}$ to σ' and we write $\mathcal{A} < \mathcal{A}'$. If $\mathcal{A} \in$ STRUC$[\sigma]$ and α, φ are formulas of vocabulary σ, then we write $\mathcal{A} \models \varphi$ to mean that $\mathcal{A}$ satisfies φ, and $\alpha \vdash \varphi$ to mean that there is a proof of φ from assumption α. We write $\alpha \equiv \varphi$ to mean that α is equivalent to φ, i.e., $\alpha \vdash \varphi$ and $\varphi \vdash \alpha$.

The logical languages we consider include first-order logic, FO; first-order logic plus transitive closure, FO(TC) (the closure of first-order logic under the transitive closure operation, i.e, if we can express the binary relation R, then we can also express its transitive closure, R^+, and its reflexive, transitive closure, $R^\star$); first-order logic plus a least-fixed-point operator, FO(LFP) (The least-fixed-point operator formalizes the process of making inductive definitions, so FO(LFP) is the closure of first-order logic under the ability to define new relations by induction.); second-order existential logic, SO∃; and full second-order logic, SO. We assume that these languages have access to the numeric ordering relation ($\leq$) and the numeric constant symbols, $0, 1, \dots, \max$.

It is well known that natural logical languages capture natural complexity classes, for example, FO $=$ CRAM$[1]$, FO(TC) $=$ NSPACE$[\log n]$, FO(LFP) $=$ P, SO∃ $=$ NP, and SO $=$ PH. Here CRAM$[1]$ is the set of problems checkable in constant time by a parallel random access machine with polynomially much hardware, PH is the polynomial-time hierarchy.

Thus, SO is an extremely rich, very expressive algorithmic language, which we use as the main input to our tool. In the future we plan to develop an equally expressive, but more user-friendly specification language.

3. A Methodology for Inductive Synthesis

We start with a specification $\varphi \in$ SO and a target language L. We are hoping to derive an $\alpha \in L$ such that $\alpha \equiv \varphi$. Thus α will be an efficient implementation of the specification, φ. We have an input vocabulary, σ, which contains all of the numeric and input predicate, constant, and function symbols. For example, the numeric symbols might be $0, 1, \max, \leq^2$. The input symbols might be E^2, r, for a rooted graph with constant symbol, r, denoting a specified root node. The output vocabulary σ' consists of σ together with the output symbols. For example, for topological sort, σ' would include the output function symbol, g^1, that denotes a topological ordering, and for the minimum spanning tree algorithm, σ', would include the relation symbol, T^2, denoting the tree edges. In the finite differencing examples, we usually have input relations symbols such as E^2, S^1, and their new values one step later as the output, i.e., E'^2, S'^1. Note that the superscripts denoting arity are shown in the vocabularies, but are omitted when these symbols are used in formulas.

Our synthesis algorithm works in two settings both of which come up often in practice. Sometimes we are asked to compute a well-defined single valued function, e.g., connected components, strongly connected components, etc., in this deterministic case, the desired answer is unique. However, sometimes the problem corresponds to a relation and any answer — or the answer that there is no solution — is what is desired, e.g., minimum spanning tree, topological sort, depth-first search, max network flow, and, of course, SAT. In this nondeterministic case, there may be more than one correct answer. The two cases are distinguished as follows:

- φ is a *deterministic specification*. In this setting, for all $\mathcal{A} \in \text{STRUC}[\sigma]$ there exists a unique expansion $\mathcal{A} < \mathcal{A}' \in \text{STRUC}[\sigma']$ such that $\mathcal{A}' \models \varphi$. In this case we wish to synthesize a formula $\alpha \in L$ such that $\alpha \equiv \varphi$.
- φ is a *nondeterministic specification*. In this setting, for all $\mathcal{A} \in \text{STRUC}[\sigma]$ there exists at least one expansion $\mathcal{A} < \mathcal{A}' \in \text{STRUC}[\sigma']$ such that $\mathcal{A}' \models \varphi$. In this case we wish to synthesize a formula $\alpha \in L$ such that $\alpha \vdash \varphi$ and for all $\mathcal{A} \in \text{STRUC}[\sigma]$ there exists at least one expansion $\mathcal{A} < \mathcal{A}' \in \text{STRUC}[\sigma']$ such that $\mathcal{A}' \models \alpha$.

While our methodology can handle both cases, all the examples in this paper are deterministic specifications. Thus we make the exposition simpler by assuming we are always given a deterministic specification $\varphi \in \text{SO}$. We first generate a set of instance structures $\mathcal{M} = \{\mathcal{A}_1, \ldots \mathcal{A}_k\} \subset \text{STRUC}_{\leq n}[\sigma']$ such that $\mathcal{M} \models \varphi$. Here k, n, the initial number of structures, and the upper bound on the number of elements in the universe of each structure, are parameters.

In this paper we make the simplifying assumption that our target language L consists of conjunctions from the set of *base formulas*, B. We first compute a set of *good formulas*, $G \subseteq B$, having the property that $\mathcal{M} \models G$, i.e., every instance structure satisfies every good formula.

Next we use a greedy algorithm to compute a minimal cover w.r.t. $\mathcal{M}$, $C \subseteq G$. C is a *cover* w.r.t. $\mathcal{M}$ iff $\mathcal{M} \models C$ and for all $\mathcal{A} \in \mathcal{M}$, C determines all the output bits of $\mathcal{A}$. In symbols,

$$\mathcal{M} \models C \text{ and } \forall \mathcal{A} \in \mathcal{M} \left(\bigwedge_{c \in C} c \right) \wedge \Delta_\sigma(\mathcal{A}) \vdash \Delta_{\sigma'}(\mathcal{A}) \quad (1)$$

Here, $\Delta_\tau(\mathcal{A})$, is the diagram of $\mathcal{A}$, i.e., the conjunction of all ground literals (from the vocabulary τ together with constants for the elements of the universe of $\mathcal{A}$) that are satisfied by $\mathcal{A}$.

The greedy algorithm incrementally chooses C by successively choosing an element of G that determines the maximum number of output bits not yet determined by C, and then adding it to C.

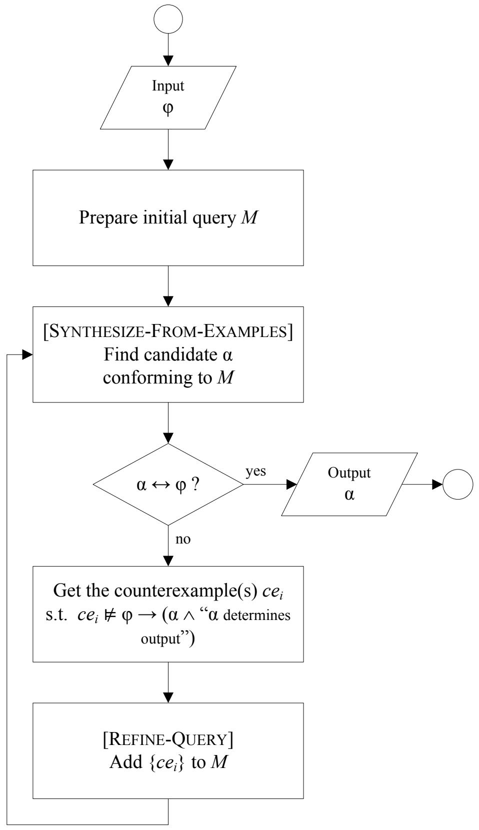


Figure 1. An abstract flowchart of the synthesis algorithm.

Once we have such a cover, C, it gives us a candidate $\alpha = \bigwedge_{c \in C} c$. To determine whether α is a correct candidate, we ask two questions of the SMT solver.

1. $(\exists \mathcal{A} \in \text{STRUC}_{\leq n}[\sigma'])(\mathcal{A} \models \varphi \wedge \neg\alpha)$
2. $(\exists \mathcal{A} \in \text{STRUC}_{\leq n}[\sigma'])(\mathcal{A} \models \varphi \wedge (\alpha \wedge \Delta_\sigma(\mathcal{A}) \not\vdash \Delta_{\sigma'}(\mathcal{A}))$

That is, (1) Is there a small instance for which α is not good? and (2) Is there a small instance for which α does not determine the answer? If the answer to either of these questions is, "yes", then we add the instance structure to $\mathcal{M}$, and repeat the above construction.

4. An Implementation of the Methodology

We now explain our more detailed algorithm that implements the above methodology. Fig. 1 shows a flowchart of the algorithm, and pseudocode is shown in Fig. 2. We now fill in the details of each step. The main phases of the algorithm are:

```
Inductive-Synthesis$_L(\varphi)$
  $\mathcal{M}$ := GENERATE-INSTANCES($\varphi$);
  do {
    $t$ := Synthesize-From-Instances$_L(\mathcal{M})$;
  }
  while (Refine-Query($\varphi$, t, $\mathcal{M}$))
```

Figure 2. The main counterexample-guided refinement loop for synthesizing efficient programs.

1. Generate $\mathcal{M} \subset \text{STRUC}_{\leq n}[\sigma']$, a set of instance structures that all satisfy the specification φ.
2. Find a cover C w.r.t. $\mathcal{M}$ (Eqn. 1).
3. Let $\alpha = \bigwedge_{c \in C} c$. Test whether there is a new instance structure, i.e., $\mathcal{A} \in \text{STRUC}_{\leq n}[\sigma']$ and $\mathcal{A} \models \varphi$, such that α is not good for $\mathcal{A}$, or α does not determine the answer for $\mathcal{A}$. If so, add $\mathcal{A}$ to $\mathcal{M}$ and repeat, otherwise output α.

4.1 Generate the Instances

In both the first and the last step we are required to search for models of the specification φ. For all examples in this paper, the specifications are fully characterized by finite structures, and in fact fairly small such structures. We will fix a parameter, n, throughout this paper to bound the size of these structures. For all the examples we have tested so far, $n = 10$ has been sufficient. Once we have a formula α that passes the test in phase (3), we have found no further counterexamples of size ≤ 20.

We use the subset SO$\exists$ of SO to express our specifications. By Fagin's Theorem (SO$\exists$ = NP) it follows that we can phrase the searches for structures in (1) and (3) as single calls to a SAT solver [Imm99, Thms 7.8, 7.16]. The target languages, L, that we use are always a small subset of SO$\exists \cap$ SO$\forall$ and thus queries that involve φ and α, or $\neg\alpha$, can still be translated to a single instance of SAT, see §A for details.

4.2 Synthesize the Cover

Given the set of instances, $\mathcal{M}$, we must build a cover, C, satisfying Eqn. 1. In the two problem domains explored in this paper, the target languages, L_1 in §5 and L_2 in §6, are sets of conjunctions of *base formulas*, $B_1 \subset L_1$, $B_2 \subset L_2$.

Recall that a formula, β, is *good* w.r.t. $\mathcal{M}$ iff $\mathcal{M} \models \beta$, i.e., it satisfies all the current instances. Let $G_{\mathcal{M}}$ denote the set of good base formulas and assume for the sake of discussion that $G_{\mathcal{M}}$ is finite.

Now, if there is any cover, then $\gamma = \bigwedge_{\beta \in G_{\mathcal{M}}} \beta$ is a cover because it is the strongest good formula. However, note that γ would not be an appropriate candidate for α because typically $G_{\mathcal{M}}$ and thus γ will be huge. Furthermore, γ would probably include too much information about the particular chosen instances.

```
Synthesize-From-Instances$_L(\mathcal{M})$
  Good := $\{\beta \mid \beta$ a base formula of $L, \forall \mathcal{A} \in \mathcal{M}\ \mathcal{A} \models \beta\}$
  Find a minimal subset
  $C \subset$ Good such that $\bigwedge_{c \in C} c$ determines all output bits;
  return $C$
```

Figure 3. Compute Minimal Cover.

Instead it makes sense to use the principle of Occam's razor, and search for the smallest cover, i.e., the smallest good formula that determines the output for all the given instances.

Recall from §2 that the universe of each structure of size n is $[n] = \{0, \ldots, n-1\}$ and that we have the numeric constants, $0, 1, \ldots, \max$, available. Assume for simplicity in the following discussion that the output vocabulary $\sigma' - \sigma$ consists of a single unary function symbol, f. (Additional function symbols would be treated similarly, and relations would be treated as boolean functions.)

Let $\mathcal{A} \in \text{STRUC}[\sigma']$, $i \in |\mathcal{A}|$ and let $\alpha \in L$. We say that α *determines* $f(i)$ for $\mathcal{A}$ if there is a value $j \in |\mathcal{A}|$ such that

$$\alpha \wedge \Delta_\sigma(\mathcal{A}) \vdash f(i) = j$$

The following proposition is just a restatement of the cover property (Eqn. 1):

PROPOSITION 4.1. *If for all $\mathcal{A} \in \mathcal{M}$, $\mathcal{A} \models \alpha$ and for all $i \in |\mathcal{A}|$, α determines $f(i)$ for $\mathcal{A}$, then α is a cover.*

It is relatively straightforward to keep track of which output bits are determined by the current set, C, of base formulas. The process SYNTHESIZE-FROM-INSTANCES is defined in Fig. 3. It uses a greedy algorithm to find a minimal cover, i.e., it chooses a good base formula that adds the largest number of points to the determined set and adds that formula to C until C is a cover. Note that it could be the case that there is no cover. This can only happen if there is no formula $\alpha \in L$ that is equivalent to φ. In this case our procedure will report failure.

4.3 Refine-Query

An α found by the previous phase may still be incorrect since it is only guaranteed to produce correct results for the instances represented by $\mathcal{M}$. To test for the existence of instances outside of $\mathcal{M}$ for which α violates the specification φ, we use the subroutine GENERATE-INSTANCE again, but instead of just trying to find instances satisfying the specification, we attempt to find models of φ that do not satisfy α, or for which α does not determine the answer. This is implemented in the procedure `Refine-Query` shown in Fig. 4.

5. Deriving Graph Classifiers

In this section, we apply the methodology of §3 to derive expected linear-time algorithms for checking properties of directed graphs. Such algorithms have many applications. For

```
Refine-Query$_L$($\varphi$, $\alpha$, $\mathcal{M}$)
    example := GENERATE-INSTANCE($\alpha$ not good or does
    not determine answer)
    if (example $\neq$ null) add example to $\mathcal{M}$
    return (example $\neq$ null)
```

Figure 4. Procedure to Generate New Instances.

Abbrev.	Meaning
self-loop-free N	$\forall u(\neg N(u,u))$
root r via N	$\forall u(N^\star(r,u))$
functional N	$\forall u,v,x\big(N(x,u) \wedge N(x,v) \to u = v\big)$
1:1 N	$\forall u,v,x\big(N(u,x) \wedge N(v,x) \to u = v\big)$

Table 2. Common Abbreviations. The first three are used as integrity constraints. The last one is used in some of the specifications.

example, they can be used to dynamically check runtime assertions for programs that manipulate dynamicaly allocated data structures.

We ask the reader to familiarize herself with the seven example input specifications to our algorithm shown in Table 1. All these specifications are in first-order logic plus transitive closure (FO(TC)), a subset of SO.

5.1 Integrity Constraints and Abbreviations

In the specifications shown in Table 1, certain integrity constraints are assumed in addition to the written formulas. When we specify "*root r via N*" we assume that every vertex is reachable from the root by following edges labeled "N". When a relation, N, is specified as *functional*, we assume that it has outdegree at most one. Furthermore, unless otherwise stated, we make the default assumption that no relation has self-loops. In addition to the integrity constraints, we also use the abbreviaton "1:1 N to meant that N is one-to-one.

These abbreviations and their meanings are given in Table 2. Note that they may all be checked in linear time. We assume that the integrity constraints are part of the input specification and the synthesized specification.

Since the specifications in Tables 1 and 2 make use of transitive closure and doubly nested quantifiers, some of their naive implementations would run in cubic time. (The situation becomes even more interesting when in later sections we use second-order quantification because then the naive implementation would have exponential runtime.)

Table 1 also shows the linear-time specifications in L_1 that our tool automatically synthesized. We next define the language L_1. To help the reader get a feeling for L_1 we give an example first. The base formula $\forall v(\#s_N(v) \leq 1)$ express the fact the the relation N is functional. The function symbol $s_N(v)$ denotes the set of vertices that are successors of v via an edge labeled N. Thus the formula says that for every vertex, v, the cardinality of that set is at most *one*. Note that the formula is checkable in linear time by doing a depth first search and recording the number of edges labeled N that leave each vertex.

5.2 Target Language L_1

Let σ be a vocabulary that includes the constant symbol, r, which is assumed to be present and a root throughout this section. The target language $L_1(\sigma)$ — or just L_1 when σ is understood — is based on single traversals of the graph, classifying the paths along the way using regular expressions. L_1 is a two-sorted first-order logic, with sorts nodes and sets of nodes. Refer to the formal definition of L_1 in Table 3.

As function symbols, L_1 has set constructors s_ρ and p_ρ, which are maps from nodes to sets of nodes, based on regular expressions (ρ) for classifying paths.

In order to make sure that all L_1 statements can be implemented with linear time complexity, and in order to limit the search space, several restrictions apply:

- L_1 has only one variable, v, ranging over nodes.
- When the starting point is a constant, ρ is restricted to the following set of regular expressions:

  $$R(\sigma) = \big\{A^\star B^\star, A^\star B^\star C, A^\star B, A^\star BC^\star \ \big|_{A^2,B^2,C^2\in\sigma}\big\}$$

  (Later on, we may abbreviate $A^\star A$ as A^+.)
- When the starting point is a variable, ρ is restricted to:

  $$S(\sigma) = \big\{A \ \big| \ A^2 \in \sigma\big\}$$

The second restriction is somewhat arbitrary and was chosen to reduce the size of the search space, while still considering the most common cases.

In order to understand why the last restriction is important, consider the following formula:

$$\forall v\big|p_{A^\star}(v)\big| = 1$$

Evaluating $p_{A^\star}(v)$ for all v essentially means constructing the entire relation $A^\star$. The cost of this procedure is $O(n^2 \lg n)$.

In order to define the semantics of L_1 we need some notation involving regular expressions.

DEFINITION 5.1. *For a graph $G \in$ STRUC$[\sigma]$ and a regular expression $e \in R(\sigma)$, we write $x \xrightarrow{e} y$ to mean that there is a path from vertex x to vertex y such that the resulting sequence of edge labels is an element of $L(e)$.*

EXAMPLE 5.2. *Let $G = ([4], A^G, B^G, r^G)$ be a structure of vocabulary $\sigma = (A^2, B^2, r)$ such that $A^G = \{(0,1),(1,2),(3,0)\}$, $B^G = \{(0,3),(2,3)\}$, and $s^G = 0$. Then*

$$G \models 0 \xrightarrow{A^\star} 0 \ \wedge \ r \xrightarrow{A^\star} 2 \ \wedge \ \neg r \xrightarrow{A^\star} 3 \ .$$

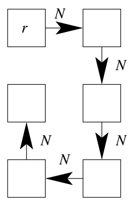
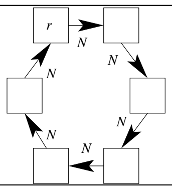
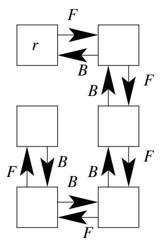
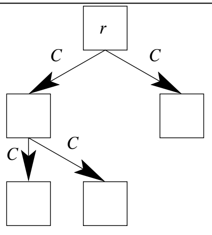
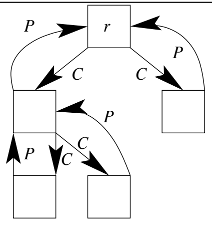
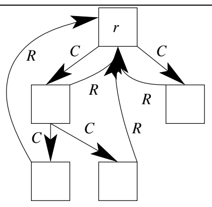
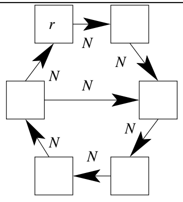
Name	**Example**	**Input Specification** **(+ Integrity Constaints)**	**Synthesized** **Formula**	**Inst.**	**Time**
SLL Singly Linked List		1:1 $N \quad \wedge$ $\forall u(\neg N^{+}(u,u))$ $\left(\begin{array}{c}\text{root } r \text{ via } N\\ \text{functional } N\end{array}\right)$	$\#p_N(r) = 0 \wedge$ $\forall v(\#p_N(v) \leq 1)$	4	14 sec
CYCLE Cyclic Linked List		$\forall u, v(N^{\star}(u,v))$ $\left(\begin{array}{c}\text{root } r \text{ via } N\\ \text{functional } N\end{array}\right)$	$\#p_N(r) = 1$	4	11 sec
DLL Doubly Linked List		1:1 $F \quad \wedge \quad$ 1:1 $B \quad \wedge$ $\forall u, v\big((F(u,v) \leftrightarrow B(v,u))$ $\wedge \quad \neg F^{+}(u,u)\big)$ $\left(\begin{array}{c}\text{root } r \text{ via } F\\ \text{functional } F, B\end{array}\right)$	$\#p_F(r) = 0 \wedge$ $\forall v(s_F(v) = p_B(v))$	18	149 sec
TREE Directed Tree		1:1 $C \quad \wedge$ $\forall u(\neg C(u,r))$ $(\ \text{root } r \text{ via } C\ )$	$\#p_C(r) = 0 \wedge$ $\forall v(\#p_C(v) \leq 1)$	8	21 sec
TREEPP Tree with Parent Ptr.		1:1 $C \quad \wedge$ $\forall u, v\big((C(u,v) \leftrightarrow P(v,u))$ $\wedge \quad \neg C(u,r)\big)$ $\left(\begin{array}{c}\text{root } r \text{ via } C\\ \text{functional } P\end{array}\right)$	$\#s_P(r) = 0 \wedge$ $\forall v(s_P(v) = p_C(v))$	26	181 sec
TREERP Tree with Root Ptr.		1:1 $C \quad \wedge$ $\forall u, v\big(\neg C(u,s) \wedge \neg R(r,u)$ $\wedge\ (u \neq s \rightarrow R(u,r))\big)$ $\left(\begin{array}{c}\text{root } r \text{ via } C\\ \text{functional } R\end{array}\right)$	$\#p_C(r) = 0 \wedge$ $p_R(r) = s_{C^{+}}(r) \wedge$ $\forall v(\#p_C(v) \leq 1)$	48	359 sec
SC Strongly Connected		$\forall u, v\big(E^{\star}(u,v)\big)$ $(\ \text{root } r \text{ via } E\ )$	$p_{E^{\star}}(r) = s_{E^{\star}}(r)$	4	9 sec

Table 1. Graph-classifiers with their input specifications, synthesized formulas, number of instances needed and time to do synthesis. A global integrity constraint is that no relation has self-loops.

$\langle\text{stmt}\rangle$	$::=$	$\langle\text{clause}\rangle \wedge \cdots \wedge \langle\text{clause}\rangle \leftrightarrow d$	
$\langle\text{clause}\rangle$	$::=$	$\langle\text{atom}\rangle \mid \forall v\ \langle\text{atom}\rangle \mid$ $\forall v\ (v \neq r \rightarrow \langle\text{atom}\rangle)$	
$\langle\text{atom}\rangle$	$::=$	$\langle\text{int}\rangle = \langle\text{const}\rangle \mid$ $\langle\text{int}\rangle \leq \langle\text{const}\rangle \mid \langle\text{set}\rangle = \langle\text{set}\rangle$	
$\langle\text{int}\rangle$	$::=$	$\langle\text{const}\rangle \mid \#\langle\text{set}\rangle$	
$\langle\text{const}\rangle$	$::=$	$0 \mid 1$	
$\langle\text{set}\rangle$	$::=$	$\{r\} \mid s_e(r) \mid p_e(r) \mid$	$e \in R(\sigma)$
		$s_\ell(v) \mid p_\ell(v)$	$\ell \in S(\sigma)$

Table 3. Grammar for $L_1(\sigma)$.

The functions of $L_1(\sigma)$ mapping nodes to sets of nodes are defined as follows,

DEFINITION 5.3. *For each regular expression $e \in R(\sigma)$ the function symbols s_e, p_e are available in $L_1(\sigma)$ and have meaning,*

$$s_e(i) \;=\; \{j \mid i \xrightarrow{e} j\} \qquad\qquad p_e(i) \;=\; \{j \mid j \xrightarrow{e} i\}$$

In words, $s_e(i)$ is the set of e-successors of i, i.e., those points reachable from i via a path of label $L(e)$; and $p_e(i)$ is the set of e-predecessors of i.

EXAMPLE 5.4. *Using the graph G from Example 5.2,*

- $s_{A^\star}(r) \;=\; \{0, 1, 2\}$
- $s_{A^+}(r) = \{1, 2\}$
- $s_{A^\star B}(r) = \{3\}$
- $p_B(3) = \{0, 2\}$

The symbol "d" is a boolean-valued constant that is true iff the graph satisfies the specified property. Thus instances satisfying the specification will have d true while instances not satisfying the specification will have d false. The term, "$\#A$" denotes the cardinality of set A.

Although $\langle stmt\rangle$ has non-bounded length, it is possible to search through all statements in L_1 using the following procedure:

1. From the set of instances, M, disinguish the "positive" ones, $P = \{\mathcal{A} \mid \mathcal{A} \models \varphi\}$, from the "negative" ones, $N = \{\mathcal{A} \mid \mathcal{A} \models \neg\varphi\}$
2. Enumerate all $\langle\text{clause}\rangle$s, picking up the formulas γ such that $\forall\mathcal{A} \in P\ \mathcal{A} \models \gamma$.
3. From the set of $\gamma_1 \cdots \gamma_k$ found in the previous step, find a minimal conjunction $\psi = \gamma_{i_1} \wedge \cdots \wedge \gamma_{i_t}$ such that $\forall\mathcal{A} \in N\ \mathcal{A} \models \neg\psi$.

This makes sure that for every instance $\mathcal{A} \in M$, the formula ψ identifies with the specification: $\mathcal{A} \models \psi \iff \mathcal{A} \models \varphi$. The generated low-level specification, α, will therefore be $d \iff \psi$.

THEOREM 5.5. *Every element of the language L_1 runs in expected linear time in the worst case.*

Proof: This follows from the fact that each set $s_e(r)$ or $p_e(r)$ can be computed in linear time via a depth first search starting at r. Furthermore, each set of sets $s_\ell(v)$ or $p_\ell(v)$, $v \in [n]$ has a total number of elements bounded by the number of edges in the input graph, and can be computed in linear time by examining each edge of label ℓ once. Finally, by maintaining the sets via hash tables, we can test equality of sets in expected linear time. $\square$

5.3 Result Summary

The current implementation is Python-based and does not meet high standards for efficiency, but still, the running time is in the order of minutes rather than hours. The number of instances shows how many examples were needed before the correct formula was synthesized. We suspect that the reason so many examples were needed in the case of TREERP and TREEPP is that on small examples there were many alternate characterizations of the extra pointers, P, R.

We are confident that this algorithm will be similarly successful in quickly and automatically deriving linear-time tests for many other simple properties of graphs and data structures.

6. Finite Differencing

In his Ph.D. thesis, Bob Paige considered the common programming situation in which an expression,

$$C = f(x_1, \ldots, x_k) \tag{2}$$

is repeatedly evaluated in a block of code after some of the variables x_i may have been slightly modified [Pai81]. It may be the case that a slight change to the variable x_i, $x_i = e$, results in a slight change from C to C' so that it is much easier to compute the change incrementally than to recompute $C' = f(x_1, \ldots, x_i', \ldots, x_k)$ from scratch. If so, Paige says informally that C is "continuous" with respect to this change, and he calls the code that incrementally computes C' the "formal derivative" of C with respect to the change.

Paige calls the part of the code that goes before the change $x_i = e$ the pre-derivative ($\partial^-(C, x_i = e)$) and the part of the code that goes after the change he calls the post-derivative ($\partial^+(C, x_i = e)$). However, for simplicity we will assume that we have access to the before and after values, x_i and x_i'. Thus we will refer to the code to compute C' and thus reestablish the invariant Eqn 2 as the *derivative* of C w.r.t. the change $x_i = e$, ($\partial(C, x_i = e)$).

See Table 4 for fourteen examples of formal derivatives, the code we automatically synthesized to evaluate them, and the required time. In each case the code — which is constant time and thus asymptotically optimal — is equivalent to the derivatives that Paige computed by hand and listed in tables in his thesis.

As in §5, our algorithm follows the methodology described in §3 and §4. The program was written in Python making use of the built in implementation for sets.

6.1 Target Language, L_2

For each input vocabulary, σ, the target language, $L_2(\sigma)$ is similar to $L_1(\sigma)$ but somewhat simpler. We again have a single domain variable, v. For each relation symbol R (or function symbol f) from σ that may change, the vocabulary, σ' of $L_2(\sigma)$ contains both R and R' (or f and f') denoting the value before and after the change. When the target formula is the relation C, we model this in L_2 as the boolean function c. When the target formula is an integer, or boolean variable, c, we model it as a domain valued, or boolean valued function also called c.

DEFINITION 6.1. *The base formulas $B_2(\sigma)$ consist of all universally quantified literals or implications of literals from σ', $B_2(\sigma) = \{\forall v(\ell_1), \forall v(\ell_1 \to \ell_2) \mid \ell_1, \ell_2$ literals from $\sigma'\}$.*

From the formulas in $B_2(\sigma)$ we derive a subset $\widehat{B_2}(\sigma)$ of desirable formulas, using a simple filter that ensures that all chosen formulas will yield a $O(1)$-run-time procedure. (The formulas in $\widehat{B_2}(\sigma)$ are those such that any literal asserting a changed value, e.g., c'(t), is restricted so that t is a constant, or if it is variable, v, then it is restricted by an assumption clause, $\ell_1(v)$, that can hold for at most one value of v.)

L_2 consists of arbitrary conjunctions of base formulas from $\widehat{B_2}(\sigma)$.

6.2 Results

Besides being able to regenerate, for some base cases, results that were previously done manually, the same method proves useful in some more advanced settings when using a complex expression as the specification, as seen in Table 5. Moreover, such expressions cannot be synthesized simply by composing base rules in a deductive manner. To illustrate this, the first statement in the table,

$$\forall u\ f(u) = \Big|\{v \mid E(v,u)\}\Big|$$

is a composition of the base constructs:

1. $c = |S|$
2. $S = \{v \in V \mid \varphi(v)\}$
3. $\forall u\ \varphi(u)$

Without any insight on the correlations among the particles, a composition of the above 3 programs will have to deduce a loop iterating over V, and, for each $u \in V$, runs the corresponding differencing program for $S_u = \{v \mid E(v,u)\}$. This results in linear run-time for the entire program. However, in this special case, when a single edge is added to E, only a single node is in fact affected with respect to f, and so this update can be computed in constant time

We are pleased that for many examples we can automatically generate formal derivatives that run in constant time and are thus asymptotically optimal. This is a validation of our methodology and will be useful in future work on synthesis. The automatic derivation of asymptotically optimal formal derivatives can be a useful building block for the automatic synthesis of efficient data structures and algorithms for a richer class of algorithmic specifications.

7. Code Generation

Actual, procedural program code can be generated directly from the formulas synthesized by the framework. We do not go into the detail of how exactly this is done, but rather just mention that it is done by a series of syntactic transformations, and refer the reader to Table 6 for some entry-level examples.

8. Related Work

Counterexample Guided Inductive Synthesis Inductive synthesis refers to the process of generating a system from input-output examples. This process involves using each new input-output example to refine the hypothesis about what the correct system should be until convergence is reached. Inductive synthesis had its origin in the pioneering work by Gold on language learning [Gol67] and by Shapiro on algorithmic debugging and its application to automated program construction [Sha83]. The inductive approach [Mug92, FP94] for synthesizing a program involves *debugging* the program with respect to positive and negative examples until the correct program is synthesized. The negative examples can be counterexamples discovered while trying to prove a program's correctness. Counterexamples have been used in incremental synthesis of programs [SLTB$^+$06] and discrete event systems [BMM04]. The technique presented in this paper is also a form of inductive synthesis, but applied in a novel program synthesis setting.

SAT/SMT Based Program Synthesis Recent advances in SAT/SMT technologies have revived interest in program synthesis techniques. Recent work shows how to user SMT constraint based program verification techniques for synthesizing programs from first order logic specifications [SGF10]. In contrast, we allow for more expressive second order logic constraints and use a very different methodology based on inductive synthesis. Sketching[SLTB$^+$06] uses SAT solvers to implement the inductive program synthesis technique, but requires the programmer to write a partial program whose holes are limited to taking values over finite domains. Though our inductive synthesis based technique also uses SAT solvers, it searches for full programs (as opposed to values for holes) over the space of their representations as logical formulas (as opposed to small finite set of values for holes).

Expression	**Change**	**Synthesized Derivative**	**Handwritten Code**	**Time**
$C = T + S$	$T \mathrel{+}= \{a\}$	$v = a \rightarrow c'(v) = 1$ $v \neq a \rightarrow c'(v) = c(v))$	$C \mathrel{+}= \{a\}$	121.88 sec
$C = T + S$	$S \mathrel{+}= \{a\}$	$v = a \rightarrow c'(v) = 1$ $v \neq a \rightarrow c'(v) = c(v)$	$C \mathrel{+}= \{a\}$	121.94 sec
$C = T + S$	$T \mathrel{-}= \{a\}$	$v \neq a \rightarrow c'(v) = c(v)$ $\neg T(a) \rightarrow c'(a) = 0$ $T(a) \rightarrow c'(a) = c(a)$	`if` $a \notin S : C \mathrel{-}= \{a\}$	87.95 sec
$C = T + S$	$S \mathrel{-}= \{a\}$	$v \neq a \rightarrow c'(v) = c(v)$ $\neg S(a) \rightarrow c'(a) = 0$ $S(a) \rightarrow c'(a) = c(a)$	`if` $a \notin T : C \mathrel{-}= \{a\}$	96.5 sec
$C = T - S$	$T \mathrel{+}= \{a\}$	$v \neq a \rightarrow c'(v) = c(v)$ $\neg S(a) \rightarrow c'(a) = 1$ $S(a) \rightarrow c'(a) = 0$	`if` $a \notin S : C \mathrel{+}= \{a\}$	69.85 sec
$C = T - S$	$T \mathrel{-}= \{a\}$	$c(v) = 0 \rightarrow c'(v) = 0$ $v \neq a \rightarrow c'(v) = c(v)$ $v = a \rightarrow c'(a) = 0$	$C \mathrel{-}= \{a\}$	115.57 sec
$C = T - S$	$S \mathrel{+}= \{a\}$	$c(v) = 0 \rightarrow c'(v) = 0$ $v \neq a \rightarrow c'(v) = c(v)$ $v = a \rightarrow c'(a) = 0$	$C \mathrel{-}= \{a\}$	117.0 sec
$C = T - S$	$S \mathrel{-}= \{a\}$	$v \neq a \rightarrow c'(v) = c(v)$ $\neg T(a) \rightarrow c'(a) = 0$ $T(a) \rightarrow c'(a) = 1$	`if` $a \in T : C \mathrel{+}= \{a\}$	115.57 sec
$C = f(S)$	$S \mathrel{+}= \{a\}$	$v \neq f(a) \rightarrow c'(v) = c(v)$ $v = a \rightarrow c'(f(a)) = 1$	$C \mathrel{+}= \{f(a)\}$	100.91 sec
$C = f^{-1}(S)$	$f(a) = b$	$v \neq a \rightarrow c'(v) = c(v)$ $S(b) \rightarrow c'(a) = 1$ $\neg S(b) \rightarrow c'(a) = 0$	`if` $b \notin S : C \mathrel{-}= \{a\}$ `else` $: C \mathrel{+}= \{a\}$	71.35 sec
$c_S = \#S$	$S \mathrel{+}= \{a\}$	$S(a) \rightarrow c'_S = c_S$ $\neg S(a) \rightarrow c_S + 1 = c'_S$	`if` $a \notin S : c_S \mathrel{+}= 1$	70.28 sec
$c_S = \#S$	$S \mathrel{-}= \{a\}$	$\neg S(a) \rightarrow c'_S = c_S$ $S(a) \rightarrow c'_S + 1 = c_S$	`if` $a \in S : c_S \mathrel{-}= 1$	61.29 sec
$c = (\#S == 0)$	$S \mathrel{+}= \{a\}$	$v = a \rightarrow c' = 0$	$c =$ `false`	4.46 sec
$c = (\#S == 0)$	$S \mathrel{-}= \{a\}$	$c_S \neq 1 \rightarrow c' = c$ $c'_S = c_S \rightarrow c = c'$ $c'_S = 0 \rightarrow c' = 1$	`if` $a \in S : c_S \mathrel{-}= 1$ $c = (c_S == 0)$	7.59 sec

Table 4. Finite differencing problems with their automatically synthesized, asymptotically optimal formal derivatives in L_2, assumed to be universally quantified conjuncts; When the expression is a set C, c and c' are the characteristic functions of C before and after the change, respectively.

Finite Differencing This paper was inspired in part by Bob Paige's work on transformational program. He used finite differencing to try to automatically derive efficient data structures and algorithms for high level specifications in SETL [Pai81, CP87]. Annie Liu is doing some notable work pushing this methodology forward [LT95, LS09].

9. Conclusion

We have shown that a simple, general inductive synthesis algorithm can automatically translate high-level algorithmic specifications into asymptotically optimal code. There is much further work to do. In particular,

- This methodology as presented is widely applicable and should be used and tested in many settings to see how far it can go in this simple form.
- Once these limits are better understood, there is room to test richer learning algorithms on richer target languages.
- Most exciting to us is the idea that building blocks such as automatic finite differencing can let us take second order existential specifications and derive efficient algorithms to maintain their implied invariants as we start with the empty graph and add edges incrementally until we have

Expression	Change	Synthesized Derivative	Time
$\forall u\ f(u) = \lvert\{v \mid E(v,u)\}\rvert$	$E+ = \{(u,v)\}$	$x \neq v \to f'(x) = x$ $\neg E(u,v) \to f'(v) = f(v) + 1$ $E(u,v) \to f'(v) = f(v)$	55.60 sec
$b = \sum_{x \in T} g(x)$	$T \mathrel{+}= \{a\}$	$\neg T(a) \to b' = b + g(a)$ $T(a) \to b' = b$	33.35 sec
$b = \max K$	$K \mathrel{+}= \{c\}$	$\neg(c < b) \to b' = c$ $c < b \to b' = b$	16.35 sec
ar: $N \to V$, 1:1 $\forall i\ f(\text{ar}(i)) = i$	exchange ar$(b) \leftrightarrow$ ar(c)	$c = b \to f'(\text{ar}(b)) = b$ $c \neq b \to f'(\text{ar}(b)) = c$ $c \neq b \to f'(\text{ar}(c)) = b$ $\neg(x = \text{ar}(c) \vee x = \text{ar}(b)) \to f'(x) = f(x)$	1075 sec

Table 5. Examples of more advanced finite differencing programs that were synthesized using the same procedure.

Synthesized Derivative	Functional Representation	Automatically Generated Code	Handwritten Code
$v = a \to c'(v) = 1$ $v \neq a \to c'(v) = c(v))$	$c'(v) = \begin{cases} 1 & v = a \\ c(v) & v \neq a \end{cases}$	$c(a) := 1$	$C \mathrel{+}= \{a\}$
$v \neq a \to c'(v) = c(v)$ $\neg T(a) \to c'(a) = 0$ $T(a) \to c'(a) = c(a)$	$c'(v) = \begin{cases} 0 & v = a \wedge a \notin T \\ c(a) & v = a \wedge a \in T \\ c(v) & v \neq a \end{cases}$	`if` $a \notin T : c(a) := 0$	`if` $a \notin T : C \mathrel{-}= \{a\}$
$v \neq a \to c'(v) = c(v)$ $\neg S(a) \to c'(a) = 1$ $S(a) \to c'(a) = 0$	$c'(v) = \begin{cases} 1 & v = a \wedge a \notin S \\ 0 & v = a \wedge a \in S \\ c(v) & v \neq a \end{cases}$	`if` $a \notin S : c(a) := 1$ `else` $: c(a) := 0$	`if` $a \notin S : C \mathrel{+}= \{a\}$
$c(v) = 0 \to c'(v) = 0$ $v \neq a \to c'(v) = c(v)$ $v = a \to c'(a) = 0$	$c'(v) = \begin{cases} 0 & v = a \\ 0 & c(v) = 0 \\ c(v) & v \neq a \end{cases}$	$c(a) := 0$	$C \mathrel{-}= \{a\}$
$v \neq a \to c'(v) = c(v)$ $\neg T(a) \to c'(a) = 0$ $T(a) \to c'(a) = 1$	$c'(v) = \begin{cases} 0 & v = a \wedge a \notin T \\ 1 & v = a \wedge a \in T \\ c(v) & v \neq a \end{cases}$	`if` $a \in T : c(a) := 1$ `else` $: c(a) := 0$	`if` $a \in T : C \mathrel{+}= \{a\}$
$v \neq f(a) \to c'(v) = c(v)$ $v = a \to c'(f(a)) = 1$	$c'(v) = \begin{cases} 1 & v = f(a) \\ c(v) & v \neq f(a) \end{cases}$	$c(f(a)) := 1$	$C \mathrel{+}= \{f(a)\}$
$v \neq a \to c'(v) = c(v)$ $S(b) \to c'(a) = 1$ $\neg S(b) \to c'(a) = 0$	$c'(v) = \begin{cases} 1 & v = a \wedge b \in S \\ 0 & v = a \wedge b \notin S \\ c(v) & v \neq a \end{cases}$	`if` $b \in S : c(a) := 1$ `else` $: c(a) := 0$	`if` $b \notin S : C \mathrel{-}= \{a\}$ `else` $: C \mathrel{+}= \{a\}$
$S(a) \to c'_S = c_S$ $\neg S(a) \to c_S + 1 = c'_S$	$c'_S(v) = \begin{cases} c_S + 1 & a \notin S \\ c_S & a \in S \end{cases}$	`if` $a \notin S : c'_S := c_S + 1$	`if` $a \notin S : c_S \mathrel{+}= 1$
$\neg S(a) \to c'_S = c_S$ $S(a) \to c'_S + 1 = c_S$	$c'_S(v) = \begin{cases} c_S - 1 & a \in S \\ c_S & a \notin S \end{cases}$	`if` $a \in S : c'_S := c_S - 1$	`if` $a \in S : c_S \mathrel{-}= 1$
$v = a \to c' = 0$	$c' = 0$	$c := 0$	$c =$ `false`
$c_S \neq 1 \to c' = c$ $c'_S = c_S \to c = c'$ $c'_S = 0 \to c' = 1$	$c' = \begin{cases} 1 & c'_S = 0 \\ c & c_S \neq 1 \\ c & c_S = c'_S \end{cases}$	`if` $c'_S = 0 : c := 1$	`if` $a \in S : c_S \mathrel{-}= 1$ $c = (c_S == 0)$

Table 6. Code generated from synthesized derivative

the entire input graph. This might also be an approach for synthesizing incremental algorithms.

A. Propositional Encoding of First and Second-Order Formulas

In this appendix, we explain how to use a SAT solver to test whether there exists a model of a given size for a second-order exsitential formula, Ψ. Our construction is immediate from the proof of Cook's Theorem from Fagin's Theorem in [Imm99, Thms 7.8, 7.16].

PROPOSITION A.1. *Given a second-order existenital formula Ψ and a positive integer n, we can construct a boolean formula φ such that $\varphi \in \mathrm{SAT}$ iff Φ has a model of size n.*

Proof: Let's take as an example $\Phi \equiv \exists R^1 \psi$ with first-order part $\psi \equiv \forall x \exists y (E(x,y) \wedge R(y))$. We want to know if there exists a structure $\mathcal{A} = ([n], E^{\mathcal{A}})$ that satisfies Ψ. That is the same question as whether there exists a structure $\mathcal{B} = ([n], E^{\mathcal{B}}, R^{\mathcal{B}})$ that satisfies ψ.

To guess such a structure, $\mathcal{B}$, we must guess the answers to $n^2 + n$ binary questions, i.e, whether $E(0,0)$ holds, whether $E(1,0)$ holds, …, whether $E(n-1,n-1)$ holds, whether $R(0)$, …, whether $R(n-1)$ holds. Thus in the boolean formula φ that we construct there will be n^2+n boolean variables: $E(0,0), E(1,0), \ldots, E(n-1,n-1)$; $R(0), \ldots, R(n-1)$. The boolean formula φ will assert that the chosen structure $\mathcal{B}$ satisfies ψ:

$$\varphi \equiv \bigwedge_{i=0}^{n-1} \bigvee_{j=0}^{n-1} (E(i,j) \wedge R(j)) \ .$$

We believe that the reader will be able to synthesize the general algorithm from this one example[1]. Note that the formula φ can be constructed in time linear in its size. The size of φ is $O(n^{\max(a,r)}|\psi|)$ where a is the maximum arity of the relations existentialy quantified, and r is the depth of nesting of first-order quantifiers in ψ. □

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[1] One of us has been accused of being too terse in his writing. For him, the elegance and simplicity of Proposition A.1 is embedded in the simple example given. Other coauthors said, "What about the cases when a function or a numeric relation occurs in ψ?" Well, each value of a function is determined by $\log n$ bits, so we need $n \log n$ boolean variables to encode a unary function. For numeric relations, if $x < y$ occurs in ψ, then in φ it would be replaced by $i < j$ for each fixed value of i and j. For example, $3 < 7$ would then be replaced by true and $5 < 2$ would be replaced by false.